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S-n-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and S a multiplicatively closed subset of R. This paper aims to introduce the concept of S-n-ideals as a generalization of n-ideals. An ideal I of R disjoint with S is called an S-n-ideal if there exists $s \in S$ such that whenever $ab \in I$ for $a, b \in R$, then $sa \in \sqrt{0}$ or $sb \in I$. The relationships among S-n-ideals, n-ideals, S-prime and S-primary ideals are clarified. Besides several properties, characterizations and examples of this concept, S-n-ideals under various contexts of constructions including direct products, localizations and homomorphic images are given. For some particular S and $m \in \mathbb{N}$, all S-n-ideals of the ring $\mathbb{Z}_m$ are completely determined. Furthermore, S-n-ideals of the idealization ring and amalgamated algebra are investigated.

1. Introduction

Throughout this paper, we assume that all rings are commutative with non-zero identity. For a ring R, we will denote by $U(R)$, $reg(R)$ and $Z(R)$, the set of unit elements, regular elements and zero-divisor elements of R, respectively. For an ideal I of R, the radical of I denoted by $\sqrt{I}$ is the ideal $\{a \in R : a^n \in I$ for some positive integer $n\}$ of R. In particular, $\sqrt{0}$ denotes the set of all nilpotent elements of R. We recall that a proper ideal I of a ring R is called prime (primary) if for $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$ ($b \in \sqrt{I}$). Several generalizations of prime and primary ideals were introduced and studied, (see for example [2]- [4], [6], [17]).

Let S be a multiplicatively closed subset of a ring R and I an ideal of R disjoint with S. Recently, Hamed and Malek [12] used a new approach to generalize prime ideals by defining S-prime ideals. I is called an S-prime ideal of R if there exists

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an $s \in S$ such that for all $a, b \in R$ whenever $ab \in I$, then $sa \in I$ or $sb \in I$. Then analogously, Visweswaran [16] introduced the notion of S-primary ideals. I is called an S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in \sqrt{I}$. Many other generalizations of S-prime and S-primary ideals have been studied. For example, in [1], the authors defined I to be a weakly S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in I$, then $sa \in I$ or $sb \in I$. In 2015, Mohamadian [14] defined a new type of ideals called r-ideals. An ideal I of a ring R is said to be r -ideal, if $ab \in I$ and $a \notin Z(R)$ imply that $b \in I$ for each $a, b \in R$. Generalizing this concept, in 2017 the notion of n-ideals was first introduced and studied [15]. The authors called a proper ideal I of R an n-ideal if $ab \in I$ and $a \notin \sqrt{0}$ imply that $b \in I$ for each $a, b \in R$. Many other generalizations of n-ideals have been introduced recently, see for example [13] and [18]. Motivated and inspired by these studies, in this article, we study the S-version of the class of n-ideals by determining the structure of S-n-ideals of a ring. We call I an S-n-ideal of a ring R if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$. We call this fixed element $s \in S$ an S-element of I. Clearly, for any multiplicatively closed subset S of R, every n-ideal is an S-n-ideal and the classes of n-ideals and S-n-ideals coincide if $S \subseteq U(R)$. However, this generalization of n-ideals is proper as we can see in Example 1. In Section 2, we start by giving an example of an S-n-ideal of a ring R that is not an n-ideal. Then we give many properties of S-n-ideals and show that S-n-ideals enjoy analogs of many of the properties of n-ideals. Also we discuss the relationship among S-n-ideals, n-ideals, S-prime and S-primary ideals, (Propositions 1, 6 and Examples 1, 2). In Theorems 1 and 2, we present some characterizations for S-n-ideals of a general commutative ring. Moreover, we investigate some conditions under which $(I :_R s)$ is an S-n-ideal of R for an S-n-ideal I of R and an S-element s of I, (Propositions 2, 3 and Example 3). For a particular case that $S \subseteq reg(R)$, we justify some other results. For example, in this case, we prove that a maximal S-n-ideal of R is S-prime, (Proposition 6). In addition, we show in Proposition 4 that every proper ideal of a ring R is an S-n-ideal if and only if R is a UN-ring (a ring for which every nonunit element is a product of a unit and a nilpotent). Let $n \in \mathbb{N}$, say, $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and $r_i \geq 1$ for all i. Then for all $2 \leq i \leq k-1$, $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k} = \left\{ \bar{p}_1^{m_1} \bar{p}_2^{m_2} ... \bar{p}_{i-1}^{m_{i-1}} \bar{p}_{i+1}^{m_{i+1}} ... \bar{p}_{k-1}^{m_{k-1}} : m_j \in \mathbb{N} \cup \{0\} \right\}$ is a multiplicatively closed subset of $\mathbb{Z}_n$. In Theorem 4, we determine all $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$-$n$-ideals of $\mathbb{Z}_n$ for all i. In particular, we determine all S_p-n-ideals of $\mathbb{Z}_n$ where $S_p = \left\{ 1, \bar{p}, \bar{p}^2, \bar{p}^3, ... \right\}$ for any prime integer p dividing n, (Theorem 3). Furthermore, we study the stability of S-n-ideals with respect to various ring theoretic constructions such as localization, factor rings and direct product of rings, (Propositions 11, 12 and 14). Let R be a ring and M be an R-module. For a multiplicatively closed subset S of R, the set $S(+)M = \{(s, m) : s \in S, m \in M\}$ is clearly a multiplicatively closed subset of the idealization ring $R(+)M$. In Section 3, first, we clarify the relation between the S-n-ideals of a ring R and the $S(+)M$-n-ideals $R(+)M$, (Proposition 17). For rings R and R', an ideal J of R' and a ring homomorphism $f : R \to R'$, the amalgamation of R and R' along J with respect to f is the subring $R \bowtie^f J = \{(r, f(r) + j) : r \in R,\ j \in J\}$ of $R \times R'$. Clearly, the set $S \bowtie^f J = \{(s, f(s) + j) : s \in S,\ j \in J\}$ is a multiplicatively closed subset of $R \bowtie^f J$ whenever S is a multiplicatively closed subset of R. We finally determine when the ideals $I \bowtie^f J = \{(i, f(i) + j) : i \in I,\ j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R,\ j \in J,\ f(a) + j \in K\}$ of $R \bowtie^f J$ are $(S \bowtie^f J)$-n-ideals, (Theorems 5 and 6).

2. Properties of S-n-ideals

Definition 1. *Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R disjoint with S. We call I an S-n-ideal of R if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$. This fixed element $s \in S$ is called an S-element of I.*

Let I be an ideal of a ring R. If I is an n-ideal of R, then clearly I is an S-n-ideal for any multiplicatively closed subset of R disjoint with I. However, it is clear that the classes of n-ideals and S-n-ideals coincide if $S \subseteq U(R)$. Moreover, obviously any S-n-ideal is an S-primary ideal and the two concepts coincide if the ideal is contained in $\sqrt{0}$. However, the converses of these implications are not true in general as we can see in the following examples.

Example 1. *Let $R = \mathbb{Z}_{12}$, $S = \{\overline{1}, \overline{3}, \overline{9}\}$ and consider the ideal $I =< \overline{4} >$. Choose $s = \overline{3} \in S$ and let $a, b \in R$ with $ab \in I$ but $3b \notin I$. Now, $ab \in< \overline{2} >$ implies $a \in< \overline{2} >$ or $b \in< \overline{2} >$. Assume that $a \notin< \overline{2} >$ and $b \in< \overline{2} >$. Since $a \notin< \overline{2} >$, then $a \in \{\overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$ and since $3b \notin I$, we have $b \in \{\overline{2}, \overline{6}, \overline{10}\}$. Thus, in each case $ab \notin I$, a contradiction. Hence, we must have $a \in< \overline{2} >$ and so $\overline{3}a \in< \overline{6} >= \sqrt{0}$. On the other hand, I is not an n-ideal as $\overline{2} \cdot \overline{2} \in I$ but neither $\overline{2} \in \sqrt{0}$ nor $\overline{2} \in I$.*

A (prime) primary ideal of a ring R that is not an n-ideal is a direct example of an (S-prime) S-primary ideal that is not an S-n-ideal where $S = \{1\}$. For a less trivial example, we have the following.

Example 2. *Let $R = \mathbb{Z}[X]$ and let $I = \langle 4x \rangle$. consider the multiplicatively closed subset $S = \{4^m : m \in \mathbb{N} \cup \{0\}\}$ of R. Then I is an S-prime (and so S-primary) ideal of R, [16, Example 2.3]. However, I is not an S-n-ideal since for all $s = 4^m \in S$, we have $(2x)(2) \in I$ but $s(2x) \notin \sqrt{0_{\mathbb{Z}[x]}}$ and $s(2) \notin I$.*

Proposition 1. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S.*

(1) If I is an S-n-ideal, then $sI \subseteq \sqrt{0}$ for some $s \in S$. If moreover, $S \subseteq reg(R)$, then $I \subseteq \sqrt{0}$.
(2) $\sqrt{0}$ is an S-n-ideal of R if and only if $\sqrt{0}$ is an S-prime ideal of R.
(3) Let $S \subseteq reg(R)$. Then 0 is an S-n-ideal of R if and only if 0 is an n-ideal.

Proof. (1) Let $a \in I$. Since $I \cap S = \emptyset$, $s \cdot 1 \notin I$ for all $s \in S$. Hence, $a \cdot 1 \in I$ implies that there exists an $s \in S$ such that $sa \in \sqrt{0}$. Thus, $sI \subseteq \sqrt{0}$ as desired. Moreover, if $S \subseteq reg(R)$, then clearly $I \subseteq \sqrt{0}$.

(2) Clear.

(3) Suppose s is an S-element of 0 and $ab = 0$ for some $a, b \in R$. Then $sa \in \sqrt{0}$ or $sb = 0$ which implies $s^n a^n = 0$ for some positive integer n or $sb = 0$. Since $S \subseteq reg(R)$, we have $a^n = 0$ or $b = 0$, as needed. □

Next, we characterize S-n-ideals of rings by the following.

Theorem 1. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. The following statements are equivalent.*

(1) I is an S-n-ideal of R.
(2) There exists an $s \in S$ such that for any two ideals J, K of R, if $JK \subseteq I$, then $sJ \subseteq \sqrt{0}$ or $sK \subseteq I$.

Proof. (1)⇒(2). Suppose I is an S-n-ideal of R. Assume on the contrary that for each $s \in S$, there exist two ideals J', K' of R such that $J'K' \subseteq I$ but $sJ' \nsubseteq \sqrt{0}$ and $sK' \nsubseteq I$. Then, for each $s \in S$, we can find two elements $a \in J'$ and $b \in K'$ such that $ab \in I$ but neither $sa \in \sqrt{0}$ nor $sb \in I$. By this contradiction, we are done.

(2)⇒(1). Let $a, b \in R$ with $ab \in I$. Taking $J =< a >$ and $K =< b >$ in (2), we get the result. □

Theorem 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If $\sqrt{0}$ is an S-n-ideal of R, then the following are equivalent.*

(1) I is an S-n-ideal of R.
(2) There exists $s \in S$ such that for ideals $I_1, I_2, ..., I_n$ of R, if $I_1 I_2 \cdots I_n \subseteq I$, then $sI_j \subseteq \sqrt{0}$ or $sI_k \subseteq I$ for some $j, k \in \{1, ..., n\}$.
(3) There exists $s \in S$ such that for elements $a_1, a_2, ..., a_n$ of R, if $a_1 a_2 \cdots a_n \in I$, then $sa_j \in \sqrt{0}$ or $sa_k \in I$ for some $j, k \in \{1, ..., n\}$.

Proof. (1)⇒(2). Let $s_1 \in S$ be an S-element of I. To prove the claim, we use mathematical induction on n. If $n = 2$, then the result is clear by Theorem 1. Suppose $n \geq 3$ and the claim holds for $n - 1$. Let $I_1, I_2,...,$ I_n be ideals of R with $I_1 I_2 \cdots I_n \subseteq I$. Then by Theorem 1, we conclude that either $s_1 I_1 \subseteq \sqrt{0}$ or $s_1 I_2 \cdots I_n \subseteq I$. Assume $(s_1 I_2) \cdots I_n \subseteq I$. By the induction hypothesis, we have either, say, $s_1^2 I_2 \subseteq \sqrt{0}$ or $s_1 I_k \subseteq I$ for some $k \in \{3, ..., n\}$. Assume $s_1^2 I_2 \subseteq \sqrt{0}$ and choose an S-element $s_2 \in S$ of $\sqrt{0}$. If $s_2(s_1^2 R) \subseteq \sqrt{0} \cap S$, we get a contradiction. Thus, $s_2 I_2 \subseteq \sqrt{0}$. By choosing $s = s_1 s_2$, we get $sI_j \subseteq \sqrt{0}$ or $sI_k \subseteq I$ for some $j, k \in \{1, ..., n\}$, as needed.

(2)⇒(3). This is a particular case of (2) by taking $I_j :=< a_j >$ for all $j \in \{1, ..., n\}$.

(3)⇒(1). Clear by choosing $n = 2$ in (3). □

Proposition 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then*

(1) If $(I : s)$ is an n-ideal of R for some $s \in S$, then I is an S-n-ideal.
(2) If I is an S-n-ideal and $(\sqrt{0} : s)$ is an n-ideal where $s \in S$ is an S-element of I, then $(I : s)$ is an n-ideal of R.
(3) If I is an S-n-ideal and $S \subseteq reg(R)$, then $(I : s)$ is an n-ideal of R for any S-element s of I.

Proof. (1) Suppose that $(I : s)$ is an n-ideal of R for some $s \in S$. We show that s is an S-element of I. Let $a, b \in R$ with $ab \in I$ and $sa \notin \sqrt{0}$. Then $ab \in (I : s)$ and $a \notin \sqrt{0}$ imply that $b \in (I : s)$.Thus, $sb \in I$ and I is an S-n-ideal.

(2) Suppose $a, b \in R$ with $ab \in (I : s)$. Then $a(sb) \in I$ which implies $sa \in \sqrt{0}$ or $s^2 b \in I$. Suppose $sa \in \sqrt{0}$. Since $(\sqrt{0} : s)$ is an n-ideal, $(\sqrt{0} : s) = \sqrt{0}$ by [15, Proposition 2.3] and so $a \in \sqrt{0}$. Now, suppose $s^2 b \in I$. If $sb \notin I$, then since I is an S-n-ideal, $s^3 \in \sqrt{0}$ and so $s \in \sqrt{0}$ which contradicts the assumption that $(\sqrt{0} : s)$ is proper. Thus, $sb \in I$ and $b \in (I : s)$ as needed.

(3) Suppose $S \subseteq reg(R)$ and I is an S-n-ideal. Let $a, b \in R$ with $ab \in (I : s)$ so that $a(sb) \in I$. If $sa \in \sqrt{0}$, then $s^m a^m = 0$ for some integer m. Since $S \subseteq reg(R)$, we get $a^m = 0$ and so $a \in \sqrt{0}$. If $s^2 b \in I$, then similar to the proof of (2) we conclude that $b \in (I : s)$. □

Note that the conditions that $(\sqrt{0} : s)$ is an n-ideal in (2) and $S \subseteq reg(R)$ in (3) of Proposition 2 are crucial. Indeed, consider $R = \mathbb{Z}_{12}$, $S = \{\overline{1}, \overline{3}, \overline{9}\}$. We showed in Example 1 that $I =< \overline{4} >$ is an S-n-ideal which is not an n-ideal, and so $(I : \overline{3}) = I$ is not an n-ideal. Here, observe that $S \nsubseteq reg(R)$ and $(\sqrt{0} : 3) =< \overline{2} >$ is not an n-ideal of $\mathbb{Z}_{12}$.

Proposition 3. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R and I be an S-prime ideal of R. Then I is an S-n-ideal if and only if $(I : s) = \sqrt{0}$ for some $s \in S$.*

Proof. Suppose I is an S-n-ideal of R and s_1 be an S-element of I. Then $(I : s_1)$ is an n-ideal of R by Proposition 2. Moreover, $(I : ts_1)$ is an n-ideal for all $t \in S$. Indeed, if $ab \in (I : ts_1)$ for $a, b \in R$, then $abts_1 \in I$ and so either $s_1^2 a \in \sqrt{0}$ or $s_1 tb \in I$. If $s_1^2 a \in \sqrt{0}$, then $a \in \sqrt{0}$ as $S \subseteq reg(R)$. Otherwise, we have $b \in (I : ts_1)$ as needed. Since I is an S-prime ideal of R, $(I : s_2)$ is a prime ideal of R where $s_2 \in S$ such that whenever $ab \in I$ for $a, b \in R$, either $s_2 a \in I$ or $s_2 b \in I$, [12, Proposition 1]. Similar to the above argument, we can also conclude that $(I : ts_2)$ is a prime ideal for all $t \in S$. Now, choose $s = s_1 s_2$. Then $(I : s)$ is both a prime and an n-ideal of R and so $(I : s) = \sqrt{0}$ by [15, Proposition 2.8]. Conversely, suppose $(I : s) = \sqrt{0}$ for some $s \in S$. Since I is an S-prime ideal, $(I : s')$ is a prime ideal of R for some $s' \in S$. Moreover, if $a \in (I : s')$, then $as' \in I \subseteq (I : s) \subseteq \sqrt{0}$ and so $a \in \sqrt{0}$ as $S \subseteq reg(R)$. Thus, $(I : s') = \sqrt{0}$ is a prime ideal and so it an n-ideal again by [15, Proposition 2.8]. Therefore, I is an S-n-ideal by Proposition 2. □

In the following example we justify that the condition $S \subseteq reg(R)$ can not be omitted in Proposition 3.

Example 3. *The ideal $I =< \overline{2} >$ of $\mathbb{Z}_{12}$ is prime and so S-prime for $S = \{\overline{1}, \overline{3}, \overline{9}\} \nsubseteq reg(\mathbb{Z}_{12})$. Moreover, one can directly see that $s = 3$ is an S-element of I and so I is also an S-n-ideal of $\mathbb{Z}_{12}$. But $(I : s) = I \neq \sqrt{0}$ for all $s \in S$.*

A ring R is said to be a UN-ring if every nonunit element is a product of a unit and a nilpotent. Next, we obtain a characterization for rings in which every proper ideal is an S-n-ideal where $S \subseteq reg(R)$.

Proposition 4. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R. The following are equivalent.*

(1) Every proper ideal of R is an n-ideal.
(2) Every proper ideal of R is an S-n-ideal.
(3) R is a UN-ring.

Proof. Since (1)⇒(2) is straightforward and (3)⇒(1) is clear by [15, Proposition 2.25], we only need to prove (2)⇒(3).

(2)⇒(3). Let I be a prime ideal of R. Then I is an S-prime and from our assumption, it is also an S-n-ideal. Thus $I \subseteq (I : s) = \sqrt{0}$ is a prime ideal of R by Proposition 3. Thus $\sqrt{0}$ is the unique prime ideal of R and so R is a UN-ring by [7, Proposition 2 (3)]. □

The equivalence of (1) and (2) in Proposition 4 need not be true if $S \nsubseteq reg(R)$.

Example 4. *Consider the ring $\mathbb{Z}_6$ and let $S = \{1, 3\}$. If $I = \langle \bar{0} \rangle$ or $\langle \bar{2} \rangle$, then a simple computations can show that I is an S-n-ideal of $\mathbb{Z}_6$. However, $\mathbb{Z}_6$ has no proper n-ideals, [15, Example 2.2].*

A ring R is said to be von Neumann regular if for all $a \in R$, there exists an element $b \in R$ such that $a = a^2 b$.

Proposition 5. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R.*

(1) Let R be a reduced ring. Then R is an integral domain if and only if there exists an S-prime ideal of R which is also an S-n-ideal
(2) R is a field if and only if R is von Neumann regular and 0 is an S-n-ideal of R.

Proof. (1) Let R be an integral domain. Since $0 = \sqrt{0}$ is prime, it is also an n-ideal again by [15, Corollary 2.9]. Thus $\sqrt{0}$ is both S-prime and S-n-ideal of R, as required. Conversely, suppose I is both S-prime and S-n-ideal of R. Hence, from Proposition 3 we conclude $(I : s) = \sqrt{0}$ which is an n-ideal by Proposition

2. $\sqrt{0} = 0$ is also a prime ideal by [15, Corollary 2.9], and thus R is an integral domain.

(2) Since $S \subseteq reg(R)$, from Proposition 1, 0 is an S-n-ideal of R if and only if 0 is an n-ideal. Thus, the claim is clear by [15, Theorem 2.15]. □

Let $n \in \mathbb{N}$. For any prime p dividing n, we denote the multiplicatively closed subset $\{1, \bar{p}, \bar{p}^2, \bar{p}^3, ...\}$ of $\mathbb{Z}_n$ by S_p. Next, for any p dividing n, we clarify all S_p-n-ideals of $\mathbb{Z}_n$.

Theorem 3. *Let* $n \in \mathbb{N}$.

(1) If $n = p^r$ for some prime integer p and $r \geq 1$, then $\mathbb{Z}_n$ has no S_p-n-ideals.
(2) If $n = p_1^{r_1} p_2^{r_2}$ where p_1 and p_2 are distinct prime integers and $r_1, r_2 \geq 1$, then for all $i = 1, 2$, every ideal of $\mathbb{Z}_n$ disjoint with S_{p_i} is an S_{p_i}-n-ideal.
(3) If $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and $k \geq 3$, then for all $i = 1, 2, ..., k$, $\mathbb{Z}_n$ has no S_{p_i}-n-ideals.

Proof. (1) Clear since $I \cap S_p \neq \phi$ for any ideal I of $\mathbb{Z}_n$.

(2) Let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} \rangle$ be an ideal of $\mathbb{Z}_n$ distinct with S_{p_1}. Then we must have $t_2 \geq 1$. Choose $s = \bar{p}_1^{t_1} \in S_{p_1}$ and let $ab \in I$ for $a, b \in \mathbb{Z}_n$. If $a \in \langle \bar{p}_2 \rangle$, then $sa \in \langle \bar{p}_1 \bar{p}_2 \rangle = \sqrt{0}$. If $a \notin \langle \bar{p}_2 \rangle$, then clearly $b \in \langle \bar{p}_2^{t_2} \rangle$ and so $sb \in I$. Therefore, I is an S_{p_1}-n-ideal of $\mathbb{Z}_n$. By a similar argument, we can show that every ideal of $\mathbb{Z}_n$ distinct with S_{p_2} is an S_{p_2}-n-ideal.

(3) Let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ distinct with S_{p_1}. Then there exists $j \neq 1$ such that $t_j \geq 1$, say, $j = k$. Thus, $\bar{p}_k^{t_k} (\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \in I$ but $s\bar{p}_k^{t_k} \notin \sqrt{0}$ and $s(\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \notin I$ for all $s \in S_{p_1}$. Therefore, I is not an S_{p_1}-n-ideal of $\mathbb{Z}_n$. Similarly, I is not an S_{p_i}-n-ideal of $\mathbb{Z}_n$ for all $i = 1, 2, ..., k$. □

Corollary 1. *Let* $n \in \mathbb{N}$. *Then for any prime* p *dividing* n, *either* $\mathbb{Z}_n$ *has no* S_p-n-*ideals or every ideal of* $\mathbb{Z}_n$ *disjoint with* S_p *is an* S_p-n-*ideal.*

In general if $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $r_i \geq 1$ for all i, then

$$S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k} = \{\bar{p}_1^{m_1} \bar{p}_2^{m_2} ... \bar{p}_{i-1}^{m_{i-1}} \bar{p}_{i+1}^{m_{i+1}} ... \bar{p}_k^{m_k} : m_j \in \mathbb{N} \cup \{0\}\}$$

is also a multiplicatively closed subset of $\mathbb{Z}_n$ for all i. Next, we generalize Theorem 3.

Theorem 4. *Let* $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ *where* $p_1, p_2, ..., p_k$ *are distinct prime integers and* $r_i \geq 1$ *for all* i.

(1) $\mathbb{Z}_n$ has no $S_{p_1 p_2 ... p_k}$-n-ideals.
(2) For $i = 1, 2, ..., k$, every ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$ is an $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$-$n$-ideal.
(3) Let $k \geq 3$. If $m \leq k - 2$, then $\mathbb{Z}_n$ has no $S_{p_1 p_2 ... p_m}$-n-ideals.

Proof. (1) This is clear since $I \cap S_{p_1 p_2 ... p_k} \neq \phi$ for any ideal I of $\mathbb{Z}_n$.

(2) With no loss of generality, we may choose $i = k$. Let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1 p_2 ... p_{k-1}}$. Then we must have $t_k \geq 1$. Choose $s = \bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}} \in S_{p_1 p_2 ... p_{k-1}}$ and let $a, b \in \mathbb{Z}_n$ such that $ab \in I$. If $a \in \langle \bar{p}_k \rangle$, then $sa \in \langle \bar{p}_1 \bar{p}_2 ... \bar{p}_k \rangle = \sqrt{0}$. If $a \notin \langle \bar{p}_k \rangle$, then we must have $b \in \langle \bar{p}_k^{t_k} \rangle$. Thus, $sb \in I$ and I is an $S_{p_1 p_2 ... p_{k-1}}$-n-ideal of $\mathbb{Z}_n$.

(3) Assume $m = k - 2$ and let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1 p_2 ... p_{k-2}}$. Then at least one of t_{k-1} and t_k is nonzero, say, $t_k \gneqq 0$. Hence, $\bar{p}_k^{t_k} (\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \in I$ but clearly $s\bar{p}_k^{t_k} \notin \sqrt{0}$ and $s(\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \notin I$ for all $s \in S_{p_1 p_2 ... p_{k-2}}$. Therefore, $\mathbb{Z}_n$ has no $S_{p_1 p_2 ... p_{k-2}}$-n-ideals. A similar proof can be used if $1 \leq m \lneqq k - 2$. □

An ideal I of a ring R is called a maximal S-n-ideal if there is no S-n-ideal of R that contains I properly. In the following proposition, we observe the relationship between maximal S-n-ideals and S-prime ideals.

Proposition 6. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R. If I is a maximal S-n-ideal of R, then I is S-prime (and so $(I : s) = \sqrt{0}$ for some $s \in S$).*

Proof. Suppose I is a maximal S-n-ideal of R and $s \in S$ is an S-element of I. Then $(I : s)$ is an n-ideal of R by Proposition 2. Moreover, $(I : s)$ is a maximal n-ideal of R. Indeed, if $(I : s) \subsetneq J$ for some n-ideal (and so S-n-ideal) J of R, then $I \subseteq (I : s) \subsetneq J$ which is a contradiction. By [15, Theorem 2.11], $(I : s) = \sqrt{0}$ is a prime ideal of R and so I is an S-prime ideal by [12, Proposition 1]. □

Proposition 7. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If I is an S-n-ideal, and J is an ideal of R with $J \cap S \neq \emptyset$, then IJ and $I \cap J$ are S-n-ideals of R.*

Proof. Let $s' \in J \cap S$. Let $a, b \in R$ with $ab \in IJ$. Since $ab \in I$, we have $sa \in \sqrt{0}$ or $sb \in I$ where s is an S-element of I. Hence, $(s's)a \in J\sqrt{0} \subseteq \sqrt{0}$ or $(s's)b \in IJ$. Thus, IJ is an S-n-ideal of R. The proof that $I \cap J$ is an S-n-ideal is similar. □

Proposition 8. *Let S be a multiplicatively closed subset of a ring R and I_1, I_2,..., I_n be proper ideals of R.*

(1) If I_i is an S-n-ideal of R for all $i = 1, ..., n$, then $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R.

(2) If $\left(\bigcap_{j \in \Omega} I_j \right) \cap S \neq \emptyset$ for $\Omega \subseteq \{1, ..., n\}$ and I_k is an S-n-ideal of R for all $k \in \{1, ..., n\} - \Omega$, then $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R.

Proof. (1) Suppose that for all $i = 1, ..., n$, I_i is an S-n-ideal of R and note that $\left(\bigcap_{i=1}^{n} I_i\right) \cap S = \emptyset$. For all $i = 1, ..., n$, choose $s_i \in S$ such that whenever $a, b \in R$ such that $ab \in I_i$, then $s_i a \in \sqrt{0}$ or $s_i b \in I_i$. Let $a, b \in R$ such that $ab \in \bigcap_{i=1}^{n} I_i$. Then $ab \in I_i$ for all $i = 1, ..., n$. If we let $s = \prod_{i=1}^{n} s_i \in S$, then clearly $sa \in \sqrt{0}$ or $sb \in \bigcap_{i=1}^{n} I_i$ and the result follows.

(2) Choose $s' \in \left(\bigcap_{j\in\Omega} I_j\right) \cap S$. Let $a, b \in R$ with $ab \in \bigcap_{i=1}^{n} I_i$. Then for all $k \in \{1, ..., n\} - \Omega$, $ab \in I_k$ and so $s_k a \in \sqrt{0}$ or $s_k b \in I_j$ for some S-element s_k of I_k. Hence, $(s' \prod_{k\in\{1,...,n\}-\Omega} s_k)a \in \sqrt{0}$ or $(s' \prod_{k\in\{1,...,n\}-\Omega} s_k)b \in \bigcap_{i=1}^{n} I_i$ and so $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R. □

Let S and T be two multiplicatively closed subsets of a ring R with $S \subseteq T$. Let I be an ideal disjoint with T. It is clear that if I is a S-n-ideal, then it is T-n-ideal. The converse is not true since while $I =< \bar{4} >$ is an S-n-ideal of $\mathbb{Z}_{12}$ for $S = \{\bar{1}, \bar{3}, \bar{9}\}$, it is not a T-n-ideal for $T = \{\bar{1}\} \subseteq S$.

Proposition 9. *Let S and T be two multiplicatively closed subsets of a ring R with $S \subseteq T$ such that for each $t \in T$, there is an element $t' \in T$ such that $tt' \in S$. If I is a T-n-ideal of R, then I is an S-n-ideal of R.*

Proof. Suppose $ab \in I$. Then there is a T-element $t \in T$ of I satisfying $ta \in \sqrt{0}$ or $tb \in I$. Hence there exists some $t' \in T$ with $s = tt' \in S$, and thus $sa \in \sqrt{0}$ or $sb \in I$. □

Let S be a multiplicatively closed subset of a ring R. The saturation of S is the set $S^* = \{r \in R : \frac{r}{1}$ is a unit in $S^{-1}R\}$. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$. Moreover, it is well known that $S^* = \{x \in R : xy \in S$ for some $y \in R\}$, see [11]. The set S is called saturated if $S^* = S$.

Proposition 10. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then I is an S-n-ideal of R if and only if I is an S^*-n-ideal of R.*

Proof. Suppose I is an S^*-n-ideal of R. By Proposition 9, it is enough to prove that for each $t \in S^*$, there is an element $t' \in S^*$ such that $tt' \in S$. Let $t \in S^*$ and choose $t' \in R$ such that $ty \in S$. Then $t' \in S^*$ and $tt' \in S$ as required. The converse is obvious. □

Let S and T be multiplicatively closed subsets of a ring R with $S \subseteq T$. Then clearly, $T^{-1}S = \left\{\frac{s}{t} : t \in T,\ s \in S\right\}$ is a multiplicatively closed subset of $T^{-1}R$.

Proposition 11. *Let S, T be multiplicatively closed subsets of a ring R with $S \subseteq T$ and I be an ideal of R disjoint with T. If I is an S-n-ideal of R, then $T^{-1}I$ is an $T^{-1}S$-n-ideal of $T^{-1}R$. Moreover, we have $T^{-1}I \cap R = (I : u)$ for some S-element u of I.*

Proof. Suppose I is an S-n-ideal. Suppose $T^{-1}S \cap T^{-1}I \neq \phi$, say, $\frac{a}{t} \in T^{-1}S \cap T^{-1}I$. Then $a \in S$ and $ta \in I$ for some $t \in T$. Since $S \subseteq T$, then $ta \in T \cap I$, a contradiction. Thus, $T^{-1}I$ is proper in $T^{-1}R$ and $T^{-1}S \cap T^{-1}I = \phi$. Let $s \in S$ be an S-element of I and choose $\frac{s}{1} \in T^{-1}S$. Suppose $a, b \in R$ and $t_1, t_2 \in T$ with $\frac{a}{t_1}\frac{b}{t_2} \in T^{-1}I$ and $\frac{s}{1}\frac{a}{t_1} \notin \sqrt{0_{T^{-1}R}}$. Then $tab \in I$ for some $t \in T$ and $sa \notin \sqrt{0}$. Since I is an S-n-ideal, we must have $stb \in I$. Thus, $\frac{s}{1}\frac{b}{t_2} = \frac{stb}{tt_2} \in T^{-1}I$ as needed. Now, let $r \in T^{-1}I \cap R$ and choose $i \in I$, $t \in T$ such that $\frac{r}{1} = \frac{i}{t}$. Then $vr \in I$ for some $v \in T$. Since I is an S-n-ideal, then there exists $u \in S \subseteq T$ such that $uv \in \sqrt{0}$ or $ur \in I$. But $uv \notin \sqrt{0}$ as $T \cap \sqrt{0} = \phi$ and so $ur \in I$. It follows that $r \in (I : u)$ for some S-element u of I. Since clearly $(I : u) \subseteq T^{-1}I \cap R$ for all $u \in T$, the proof is completed. □

In particular, if $S = T$, then all elements of $T^{-1}S$ are units in $T^{-1}R$. As a special case of of Proposition 11, we have the following.

Corollary 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If I is an S-n-ideal of R, then $S^{-1}I$ is an n-ideal of $S^{-1}R$. Moreover, we have $S^{-1}I \cap R = (I : s)$ for some S-element s of I.*

Proof. Suppose I is an S-n-ideal. Then $S^{-1}I$ is an $S^{-1}S$-n-ideal of $S^{-1}R$ by Proposition 11. Let $a, b \in R$, $s_1, s_2 \in S$ with $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$. Then by assumption, $\frac{s}{t}\frac{a}{s_1} \in \sqrt{0_{S^{-1}R}}$ or $\frac{s}{t}\frac{b}{s_2} \in S^{-1}I$ for some $S^{-1}S$-element $\frac{s}{t}$ of $S^{-1}I$. Since $\frac{s}{t}$ is a unit in $S^{-1}R$, then $S^{-1}I$ is an n-ideal of $S^{-1}R$ as required. The other part follows directly by Proposition 11. □

Corollary 3. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then I is an S-n-ideal of R if and only if $S^{-1}I$ is an n-ideal of $S^{-1}R$, $S^{-1}I \cap R = (I : s)$ and $S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$ for some $s, t \in S$.*

Proof. $\Rightarrow$) Suppose I is an S-n-ideal of R. Then $S^{-1}I$ is an n-ideal of $S^{-1}R$ by Corollary 2. The other part of the implication follows by using a similar approach to that used in the proof of Proposition 11.

$\Leftarrow$) Suppose $S^{-1}I$ is an n-ideal of $S^{-1}R$, $S^{-1}I \cap R = (I : s)$ and $S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$ for some $s, t \in S$. Choose $u = st \in S$ and let $a, b \in R$ such that $ab \in I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ and so $\frac{a}{1} \in \sqrt{S^{-1}0} = S^{-1}\sqrt{0}$ or $\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1} \in \sqrt{S^{-1}0}$, then there is $w \in S$ such that $wa \in \sqrt{0}$. Thus, $a = \frac{wa}{w} \in S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$. Hence, $ta \in \sqrt{0}$ and so $ua = sta \in \sqrt{0}$. If $\frac{b}{1} \in S^{-1}I$, then there is $v \in S$ such that $vb \in I$ and so $b = \frac{vb}{v} \in S^{-1}I \cap R = (I : s)$. Therefore, $ub = tsb \in I$ and I is an S-n-ideal of R. □

Proposition 12. *Let $f : R_1 \to R_2$ be a ring homomorphism and S be a multiplicatively closed subset of R_1. Then the following statements hold.*

(1) If f is an epimorphism and I is an S-n-ideal of R_1 containing $Ker(f)$, then $f(I)$ is an $f(S)$-n-ideal of R_2.
(2) If $Ker(f) \subseteq \sqrt{0_{R_1}}$ and J is an $f(S)$-n-ideal of R_2, then $f^{-1}(J)$ is an S-n-ideal of R_1.

Proof. First we show that $f(I) \cap f(S) = \emptyset$. Otherwise, there is $t \in f(I) \cap f(S)$ which implies $t = f(x) = f(s)$ for some $x \in I$ and $s \in S$. Hence, $x-s \in Ker(f) \subseteq I$ and $s \in I$, a contradiction.

(1) Let $a, b \in R_2$ and $ab \in f(I)$. Since f is onto, $a = f(x)$ and $b = f(y)$ for some $x, y \in R_1$. Since $f(x)f(y) \in f(I)$ and $Ker(f) \subseteq I$, we have $xy \in I$ and so there exists an $s \in S$ such that $sx \in \sqrt{0_{R_1}}$ or $sy \in I$. Thus, $f(s)a \in \sqrt{0_{R_2}}$ or $f(s)b \in f(I)$, as needed.

(2) Let $a, b \in R_1$ with $ab \in f^{-1}(J)$. Then $f(ab) = f(a)f(b) \in J$ and since J is an $f(S)$-n-ideal of R_2, there exists $f(s) \in f(S)$ such that $f(s)f(a) \in \sqrt{0_{R_2}}$ or $f(s)f(b) \in J$. Thus, $sa \in \sqrt{0_{R_1}}$ (as $Ker(f) \subseteq \sqrt{0_{R_1}}$) or $sb \in f^{-1}(J)$. □

Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If we denote $r + I \in R/I$ by $\bar{r}$, then clearly the set $\bar{S} = \{\bar{s} : s \in S\}$ is a multiplicatively closed subset of R/I. In view of Proposition 12, we conclude the following result for $\overline{S}$-n-ideals of R/I.

Corollary 4. *Let S be a multiplicatively closed subset of a ring R and I, J are two ideals of R with $I \subseteq J$.*

(1) If J is an S-n-ideal of R, then J/I is an $\overline{S}$-n-ideal of R/I. Moreover, the converse is true if $I \subseteq \sqrt{0}$.
(2) If R is a subring of R' and I' is an S-n-ideal of R', then $I' \cap R$ is an S-n-ideal of R.

Proof. (1) Note that $(J/I) \cap \overline{S} = \phi$ if and only if $I \cap S = \phi$. Now, we apply the canonical epimorphism $\pi : R \to R/I$ in Proposition 12.

(2) Apply the natural injection $i : R \to R'$ in Proposition 12 (2). □

We recall that a proper ideal I of a ring R is called superfluous if whenever $I + J = R$ for some ideal J of R, then $J = R$.

Proposition 13. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R.*

(1) If I is an S-n-ideal of R, then it is superfluous.
(2) If I and J are S-n-ideals of R, then $I + J$ is an S-n-ideal.

Proof. (1) Suppose $I + J = R$ for some ideal J of R and let $j \in J$. Then $1 - j \in I \subseteq \sqrt{0} \subseteq J(R)$ by (1) of Proposition 1. Thus, $j \in U(R)$ and $J = R$ as needed.

(2) Suppose I and J are S-n-ideals of R. Since $I, J \subseteq \sqrt{0}$, $I + J \subseteq \sqrt{0}$ and so $(I + J) \cap S = \phi$. Now, $I/(I \cap J)$ is an $\overline{S}_1$-n-ideal of $R/(I \cap J)$ by (1) of Corollary

4 where $\bar{S}_1 = \{s + (I \cap J) : s \in S\}$. If $\bar{S}_2 = \{s + J : s \in S\}$, then clearly $\bar{S}_1 \subseteq \bar{S}_2$ and so $I/(I \cap J)$ is also an $\overline{S}_2$-n-ideal of $R/(I \cap J)$. By the isomorphism $(I + J)/J \cong I/(I \cap J)$, we conclude that $(I + J)/J$ is an $\overline{S}_2$-n-ideal of R/J. Now, the result follows again by (1) of Corollary 4. □

Proposition 14. *Let R and R' be two rings, $I \trianglelefteq R$ and $I' \trianglelefteq R'$. If S and S' are multiplicatively closed subsets of R and R', respectively, then*

(1) $I \times R'$ is an $(S \times S')$-n-ideal of $R \times R'$ if and only if I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$.
(2) $R \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ if and only if I' is an S'-n-ideal of R' and $S \cap \sqrt{0_R} \neq \phi$.

Proof. It is clear that $(I \times R') \cap (S \times S') = \emptyset$ if and only if $I \cap S = \emptyset$ and $(R \times I') \cap (S \times S') = \emptyset$ if and only if $I' \cap S' = \emptyset$.

(1) Let $a, b \in R$ with $ab \in I$. Choose an $(S \times S')$-element (s, s') of $I \times R'$. If $sb \notin I$, then $(a, 1)(b, 1) \in I \times R'$ with $(s, s')(b, 1) \notin I \times R'$. Since $I \times R'$ is an $(S \times S')$-n-ideal, then $(s, s')(a, 1) \in \sqrt{0_{R \times R'}} = \sqrt{0_R} \times \sqrt{0_{R'}}$. Thus, $sa \in \sqrt{0_R}$ and $s' \in S' \cap \sqrt{0_{R'}}$ I. If $sb \in I$, then $(b, 1)(s, s') \in I \times R'$and so $(s, s')(b, 1) \in \sqrt{0_{R \times R'}} = \sqrt{0_R} \times \sqrt{0_{R'}}$ as $(s, s')^2 \notin I \times R'$. In both cases, we conclude that I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$. Conversely, suppose I is an S-n-ideal of R, s is some S-element of I and $s' \in S' \cap \sqrt{0_{R'}}$. Let $(a, a')(b, b') \in I \times R'$ for $(a, a'), (b, b') \in R \times R'$. Then $ab \in I$ which implies $sa \in \sqrt{0_R}$ or $sb \in I$. Hence, we have either $(s, s')(a, a') \in \sqrt{0_R} \times \sqrt{0_{R'}}$ or $(s, s')(b, b') \in I \times R'$. Therefore, (s, s') is an $S \times S'$-element of $I \times R'$ as needed.

(2) Similar to (1). □

The assumptions $S' \cap \sqrt{0_{R'}} \neq \phi$ and $S \cap \sqrt{0_R} \neq \phi$ in Proposition 14 are crucial. Indeed, let $R = R' = \mathbb{Z}_{12}$, $S = S' = \{\overline{1}, \overline{3}, \overline{9}\}$ and $I =< \overline{4} >$. It is shown in Example 1 that I is an S-n-ideal of R while $I \times R'$ is not an $(S \times S')$-n-ideal of $R \times R'$ as $(\overline{2}, \overline{1})(\overline{2}, \overline{1}) \in I \times R'$ but for all $(s, s') \in S \times S$, neither $(s, s')(\overline{2}, \overline{1}) \in I \times R'$ nor $(s, s')(\overline{2}, \overline{1}) \in \sqrt{0_{R \times R'}}$.

Remark 1. *Let S and S' be multiplicatively closed subsets of the rings R and R', respectively. If I and I' are proper ideals of R and R' disjoint with S, S', respectively, then $I \times I'$ is not an $(S \times S')$-n-ideal of $R \times R'$.*

Proof. First, note that $S \cap \sqrt{0_R} = S' \cap \sqrt{0_{R'}} = \emptyset$. Assume on the contrary that $I \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ and (s, s') is an $(S \times S')$-element of $I \times I'$. Since $(1, 0)(0, 1) \in I \times I'$, we conclude either $(s, s')(1, 0) \in \sqrt{0_R} \times \sqrt{0_{R'}}$ or $(s, s')(0, 1) \in I \times I'$ which implies $s \in \sqrt{0_R}$ or $s' \in I'$, a contradiction. □

Proposition 15. *Let R and R' be two rings, S and S' be multiplicatively closed subsets of R and R', respectively. If I and I' are proper ideals of R, R', respectively then $I \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ if one of the following statements holds.*

(1) I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$.
(2) I' is an S'-n-ideal of R' and $S \cap \sqrt{0_R} \neq \phi$.

Proof. Clearly $(I\times I')\cap(S\times S')=\emptyset$ if and only if $I\cap S=\emptyset$ or $I'\cap S'=\emptyset$. Suppose I is an S-n-ideal of R and $S'\cap\sqrt{0_{R'}}\neq\phi$. Then $I\cap S=\emptyset$ and $0_{R'}\in I'\cap S'\neq\emptyset$. Choose an S-element s of I and let $(a,a')(b,b')\in I\times I'$ for $(a,a'),(b,b')\in R\times R'$. Then $ab\in I$ which implies $sa\in\sqrt{0_R}$ or $sb\in I$. Hence, we have either $(s,0)(a,a')\in\sqrt{0_R}\times\sqrt{0_{R'}}$ or $(s,0)(b,b')\in I\times I'$. Therefore, $(s,0)$ is an $S\times S'$-element of $I\times I'$. Similarly, if I' is an S'-n-ideal of R' and $S\cap\sqrt{0_R}\neq\phi$, then also $I\times I'$ is an $(S\times S')$-n-ideal of $R\times R'$. □

3. S-n-ideals of Idealizations and Amalgamations

Recall that the idealization of an R-module M denoted by $R(+)M$ is the commutative ring $R\times M$ with coordinate-wise addition and multiplication defined as $(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+r_2m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM\subseteq N$. It is well known that if $I(+)N$ is an ideal of $R(+)M$, then $\sqrt{I(+)N}=\sqrt{I}(+)M$ and in particular, $\sqrt{0_{R(+)M}}=\sqrt{0}(+)M$. If S is a multiplicatively closed subset of R, then clearly the sets $S(+)M=\{(s,m):s\in S,\, m\in M\}$ and $S(+)0=\{(s,0):s\in S\}$ are multiplicatively closed subsets of the ring $R(+)M$.

Next, we determine the relation between S-n-ideals of R and $S(+)M$-n-ideals of the $R(+)M$.

Proposition 16. *Let N be a submodule of an R-module M, S be a multiplicatively closed subset of R and I be an ideal of R where $IM\subseteq N$. If $I(+)N$ is an $S(+)M$-n-ideal of $R(+)M$, then I is an S-n-ideal of R.*

Proof. Clearly, $S\cap I=\phi$. Choose an $S(+)M$-element (s,m) of $I(+)N$ and let $a,b\in R$ such that $ab\in I$. Then $(a,0)(b,0)\in I(+)N$ and so $(s,m)(a,0)\in\sqrt{0}(+)M$ or $(s,m)(b,0)\in I(+)N$. Hence, $sa\in\sqrt{0}$ or $sb\in I$ and I is an S-n-ideal of R □

Proposition 17. *Let S be a multiplicatively closed subset of a ring R, I be an ideal of R disjoint with S and M be an R-module. The following are equivalent.*

(1) I is an S-n-ideal of R.
(2) $I(+)M$ is an $S(+)0$-n-ideal of $R(+)M$.
(3) $I(+)M$ is an $S(+)M$-n-ideal of $R(+)M$.

Proof. (1)⇒(2). Suppose I is an S-n-ideal of R, s is an S-element of I and note that $S(+)0\cap I(+)M=\phi$. Choose $(s,0)\in S(+)0$ and let $(a,m_1),(b,m_2)\in R(+)M$ such that $(a,m_1)(b,m_2)\in I(+)M$. Then $ab\in I$ and so either $sa\in\sqrt{0}$ or $sb\in I$. It follows that $(s,0)(a,m_1)\in\sqrt{0}(+)M=\sqrt{0_{R(+)M}}$ or $(s,0)(b,m_2)\in I(+)M$. Thus, $I(+)M$ is an $S(+)0$-n-ideal of $R(+)M$.

(2)⇒(3). Clear since $S(+)0\subseteq S(+)M$.

(3)⇒(1). Proposition 16. □

Remark 2. *The converse of Proposition 16 is not true in general. For example, if $S=\{1,-1\}$, then 0 is an S-n-ideal of $\mathbb{Z}$ but $0(+)\bar{0}$ is not an $(S(+)\mathbb{Z}_6)$-n-ideal* *of* $\mathbb{Z}(+)\mathbb{Z}_6$*. For example,* $(2,\bar{0})(0,\bar{3}) \in 0(+)\bar{0}$ *but clearly* $(s,m)(2,\bar{0}) \notin \sqrt{0}(+)\mathbb{Z}_6 = \sqrt{0_{\mathbb{Z}(+)\mathbb{Z}_6}}$ *and* $(s,m)(0,\bar{3}) \notin 0(+)\bar{0}$ *for all* $(s,m) \in S(+)\mathbb{Z}_6$*.*

Let R and R' be two rings, J be an ideal of R' and $f : R \to R'$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r)+j) : r \in R, j \in J\}$ is a subring of $R \times R'$ called the amalgamation of R and R' along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r+j) : r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [9], [10].

Let I be an ideal of R and K be an ideal of $f(R)+J$. Then $I \bowtie^f J = \{(i, f(i)+j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a)+j) : a \in R, j \in J, f(a)+j \in K\}$ are ideals of $R \bowtie^f J$, [10]. For a multiplicatively closed subset S of R, one can easily verify that $S \bowtie^f J = \{(s, f(s)+j) : s \in S, j \in J\}$ and $W = \{(s, f(s)) : s \in S\}$ are multiplicatively closed subsets of $R \bowtie^f J$. If $J \subseteq \sqrt{0_{R'}}$, then one can easily see that $\sqrt{0_{R \bowtie^f J}} = \sqrt{0_R} \bowtie^f J$.

Next, we determine when the ideal $I \bowtie^f J$ is $(S \bowtie^f J)$-n-ideal in $R \bowtie^f J$.

Theorem 5. *Consider the amalgamation of rings* R *and* R' *along the ideals* J *of* R' *with respect to a homomorphism* f*. Let* S *be a multiplicatively closed subset of* R *and* I *be an ideal of* R *disjoint with* S*. Consider the following statements:*

(1) $I \bowtie^f J$ *is a* W*-*n*-ideal of* $R \bowtie^f J$*.*

(2) $I \bowtie^f J$ *is a* $(S \bowtie^f J)$*-*n*-ideal of* $R \bowtie^f J$*.*

(3) I *is a* S*-*n*-ideal of* R*.*

Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3). Moreover, if* $J \subseteq \sqrt{0_{R'}}$*, then the statements are equivalent.*

Proof. (1)$\Rightarrow$(2). Clear, as $W \subseteq S \bowtie^f J$.

(2)$\Rightarrow$(3). First note that $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$ if and only if $S \cap I = \emptyset$. Suppose $I \bowtie^f J$ is an $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$. Choose an $(S \bowtie^f J)$-element $(s, f(s))$ of $I \bowtie^f J$. Let $a, b \in R$ such that $ab \in I$ and $sa \notin \sqrt{0_R}$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$ and clearly $(s, f(s))(a, f(a)) \notin \sqrt{0_{R \bowtie^f J}}$. Hence, $(s, f(s))(b, f(b)) \in I \bowtie^f J$ and so $sb \in I$. Thus, s is an S-element of I and I is an S-n-ideal of R.

Now, suppose $J \subseteq \sqrt{0_{R'}}$. We prove (3)$\Rightarrow$(1). Suppose s is an S-element of I and let $(a, f(a)+j_1)(b, f(b)+j_2) = (ab, (f(a)+j_1)(f(b)+j_2)) \in I \bowtie^f J$ for $(a, f(a)+j_1), (b, f(b)+j_1) \in R \bowtie^f J$. If $(s, f(s))(a, f(a)+j_1) \notin \sqrt{0_{R \bowtie^f J}} = \sqrt{0_R} \bowtie^f J$, then $sa \notin \sqrt{0_R}$. Since $ab \in I$, we conclude that $sb \in I$ and so $(s, f(s))(b, f(b)+j_2) \in I \bowtie^f J$. Thus, $(s, f(s))$ is a W-element of $I \bowtie^f J$ and $I \bowtie^f J$ is a W-n-ideal of $R \bowtie^f J$. □

Corollary 5. *Consider the amalgamation of rings* R *and* R' *along the ideal* $J \subseteq \sqrt{0_{R'}}$ *of* R' *with respect to a homomorphism* f*. Let* S *be a multiplicatively closed subset of* R*. The* $(S \bowtie^f J)$*-*n*-ideals of* $R \bowtie^f J$ *containing* $\{0\} \times J$ *are of the form* $I \bowtie^f J$ *where* I *is a* S*–*n*-ideal of* R*.*

Proof. From Theorem 5, $I \bowtie^f J$ is a $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$ for any S-n-ideal I of R. Let K be a $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\varphi : R \bowtie^f J \to R$ defined by $\varphi(a, f(a) + j) = a$ for all $(a, f(a) + j) \in R \bowtie^f J$. Since $Ker(\varphi) = \{0\} \times J \subseteq K$, $I := \varphi(K)$ is a S-n-ideal of R by Proposition 12. Since $\{0\} \times J \subseteq K$, we conclude that $K = I \bowtie^f J$. □

Let T be a multiplicatively closed subset of R'. Then clearly, the set $\bar{T}^f = \{(s, f(s) + j) : s \in R,\ j \in J,\ f(s) + j \in T\}$ is a multiplicatively closed subset of $R \bowtie^f J$.

Theorem 6. *Consider the amalgamation of rings R and R' along the ideals J of R' with respect to an epimorphism f. Let K be an ideal of R' and T be a multiplicatively closed subset of R' disjoint with K. If $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$, then K is a T-n-ideal of R'. The converse is true if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.*

Proof. First, note that $T \cap K = \phi$ if and only if $\bar{T}^f \cap \bar{K}^f = \phi$. Suppose $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$ and $(s, f(s) + j)$ is some $\bar{T}^f$-element of $\bar{K}^f$. Let a',$b' \in R'$ such that $a'b' \in K$ and choose $a, b \in R$ where $f(a) = a'$ and $b = f(b')$. Then $(a, f(a)), (b, f(b)) \in R \bowtie^f J$ with $(a, f(a))(b, f(b)) = (ab, f(ab)) \in \bar{K}^f$. By assumption, we have either $(s, f(s) + j)(a, f(a)) = (sa, (f(s) + j)f(a)) \in \sqrt{0_{R\bowtie^f J}}$ or $(s, f(s) + j)(b, f(b)) = (sb, (f(s) + j)f(b)) \in \bar{K}^f$. Thus, $f(s) + j \in T$ and clearly, $(f(s)+j)f(a) \in \sqrt{0_{R'}}$ or $(f(s)+j)f(b) \in K$. It follows that K is a T-n-ideal of R'. Now, suppose K is a T-n-ideal of R', $t = f(s)$ is a T-element of K, $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, (f(a) + j_1)(f(b) + j_2)) \in \bar{K}^f$ for $(a, f(a) + j_1), (b, f(b) + j_2) \in R \bowtie^f J$. Then $(f(a) + j_1)(f(b) + j_2) \in K$ and so $f(s)(f(a) + j_1) \in \sqrt{0_{R'}}$ or $f(s)(f(b) + j_2) \in K$. Suppose $f(s)(f(a) + j_1) \in \sqrt{0_{R'}}$. Since $J \subseteq \sqrt{0_{R'}}$, then $f(sa) \in \sqrt{0_{R'}}$ and so $(sa)^m \in Ker(f) \subseteq \sqrt{0_R}$ for some integer m. Hence, $sa \in \sqrt{0_R}$ and $(s, f(s))(a, f(a) + j_1) \in \sqrt{0_{R\bowtie^f J}}$. If $f(s)(f(b) + j_2) \in K$, then clearly, $(s, f(s))(b, f(b) + j_2) \in \bar{K}^f$. Therefore, $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$ as needed. □

In particular, $S \times f(S)$ is a multiplicatively closed subset of $R \bowtie^f J$ for any multiplicatively closed subset S of R. Hence, we have the following corollary of Theorem 6.

Corollary 6. *Let R, R', J, S and f be as in Theorem 5. Let K be an ideal of R' and $T = f(S)$. Consider the following statements.*

(1) $\bar{K}^f$ is a $(S \times T)$-n-ideal of $R \bowtie^f J$.

(2) $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$.

(3) K is a T-n-ideal of R.

Then (1) ⇒ (2) ⇒ (3). Moreover, if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$, then the statements are equivalent.

We note that if $J \nsubseteq \sqrt{0_{R'}}$, then the equivalences in Theorems 5 and 6 are not true in general.

Example 5. *Let $R=\mathbb{Z}$, $I=\langle 0\rangle = K$, $J=\langle 3\rangle \nsubseteq \sqrt{0_{\mathbb{Z}}}$ and $S=\{1\}=T$. We have $I\bowtie J=\{(0,3n):n\in\mathbb{Z}\}$, $\bar{K}=\{(3n,0):n\in\mathbb{Z}\}$, $S\bowtie J=\{(1,3n+1):n\in\mathbb{Z}\}$, $\bar{T}=\{(1-3n,1):n\in\mathbb{Z}\}$ and $\sqrt{0_{R\bowtie J}}=\{(0,0)\}$.*

(1) I is a S-n-ideal of R but $I\bowtie J$ is not a $(S\bowtie J)$-n-ideal of $R\bowtie J$. Indeed, we have $(0,3),(1,4)\in R\bowtie J$ with $(0,3)(1,4)=(0,12)\in I\bowtie J$. But $(1,3n+1)(0,3)\notin\sqrt{0_{R\bowtie J}}$ and $(1,3n+1)(1,4)\notin I\bowtie J$ for all $n\in\mathbb{Z}$.

(2) K is a T-n-ideal of R but $\bar{K}$ is not a $\bar{T}$-n-ideal of $R\bowtie J$. For example, $(-3,0),(-4,-1)\in R\bowtie J$ with $(-3,0)(-4,-1)=(12,0)\in\bar{K}$. However, $(1-3n,1)(-3,0)\notin\sqrt{0_{R\bowtie J}}$ and $(1-3n,1)(-4,-1)\notin\bar{K}$ for all $n\in\mathbb{Z}$.

By taking $S=\{1\}$ in Theorem 5 and Corollary 6, we get the following particular case.

Corollary 7. *Let R, R', J, I, K and f be as in Theorems 5 and 6.*

(1) If $I\bowtie^f J$ is an n-ideal of $R\bowtie^f J$, then I is an n-ideal of R. Moreover, the converse is true if $J\subseteq\sqrt{0_{R'}}$.

(2) If $\bar{K}^f$ is an n-ideal of $R\bowtie^f J$, then K is an n-ideal of R'. Moreover, the converse is true if $J\subseteq\sqrt{0_{R'}}$ and $Ker(f)\subseteq\sqrt{0_R}$.

Corollary 8. *Let R, R',I, J, K, S and T be as in Theorems 5 and 6.*

(1) If $I\bowtie J$ is a $(S\bowtie J)$-n-ideal of $R\bowtie J$, then I is a S–n-ideal of R. Moreover, the converse is true if $J\subseteq\sqrt{0_{R'}}$.

(2) If $\bar{K}$ is a $\bar{T}$-n-ideal of $R\bowtie J$, then K is a T-n-ideal of R'. The converse is true if $J\subseteq\sqrt{0_{R'}}$ and $Ker(f)\subseteq\sqrt{0_R}$.

As a generalization of S-n-ideals to modules, in the following we define the notion of S -n-submodules which may inspire the reader for the other work.

Definition 2. *Let S be a multiplicatively closed subset of a ring R, and let M be a unital R-module. A submodule N of M with $(N:_R M)\cap S=\emptyset$ is called an S -n-submodule if there is an $s\in S$ such that $am\in N$ implies $sa\in\sqrt{(0:_R M)}$ or $sm\in N$ for all $a\in R$ and $m\in M$.*

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