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Some Results on the p-Weak Approximation Property in Banach Spaces

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Abstract

In this study, some existing results dealing with the weak approximation property of Banach spaces are considered for the p-weak approximation property. Also, an observation on the bounded weak approximation and the p-bounded weak approximation properties is given. Moreover, the proof of the solution of the duality problem for the p-weak approximation property which exists in the literature is given in a shorter way as an alternative.

1. Introduction

The approximation property, which closely related to basis property of Banach spaces, appeared in Banach's book in 1932 [1], and the variants of this property were systematically studied by Grothendieck, in 1955 [2]. A Banach space W has the approximation property (AP) if for every $\varepsilon > 0$ and every compact set M in W, there is a finite-rank operator $R : W \to W$ satisfying $\|Rw - w\| < \varepsilon$, for every $w \in M$ [2]. Let $1 \leq \lambda < \infty$. A Banach space W has the λ-bounded approximation property (λ-BAP) if for every $\varepsilon > 0$ and every compact set M in W, there exists a finite-rank operator $R : W \to W$ satisfying $\|R\| \leq \lambda$ and $\|Rw - w\| < \varepsilon$, for every $w \in M$ (see [3]). W has the bounded approximation property (BAP) if W has the λ-BAP, for some λ (see [3]). W has the metric approximation property (MAP) if W has the 1-BAP (see [3]). Clearly, a Banach space with the BAP has the AP, but the converse is not generally true (see [3]). It is possible to find many studies on the AP and its versions in the literature. For examples, we can mention from [4]-[9].

Grothendieck characterized the concept of compactness in Banach spaces as follows. Let W be a Banach space and let $M \subset W$. M is a relatively compact set if and only if there is a null sequence $(w_n)_n$ in W satisfying $M \subset \{\sum_{n=1}^{\infty} a_n w_n : (a_n)_n \in B_{l_1}\}$ ([2] and see [10, Proposition1.e.2]). Inspired by Grothendieck's characterization, Sinha and Karn [11] introduced the concept of p-compactness in Banach spaces. Let M be a subset of the Banach space W, and let $1 \leq p \leq \infty$. If there exists a p-summable sequence $(w_n)_n$ in W ($\|w_n\| \to 0$ as $p = \infty$) such that $M \subset \{\sum_{n=1}^{\infty} a_n w_n : (a_n)_n \in B_{l_q}\}$ (where $\frac{1}{p} + \frac{1}{q} = 1$), M is said to be relatively p-compact [11]. We remember that the ∞-compact sets are exactly the compact sets, and p-compact sets are r-compact if $1 \leq p < r \leq \infty$ [11].

The concept of a p-compact set leaded to the concept of the p-approximation property (p-AP). Sinha and Karn [11] defined the concept of the p-approximation property by replacing compact sets with p-compact sets in the definition of the AP. In recent years, the plenty of studies which focused on p-compactness, the p-AP, and some versions of the p-AP appeared. Some from these are [12]-[16]. Note that any Banach space has the 2-AP (and thus the p-AP for $1 \leq p \leq 2$) [11, Theorem 6.4].

Inspired from a result characterizing the AP given by Grothendieck, Choi and Kim [4] defined the weak approximation and the bounded weak approximation properties as weaker versions of the AP. Let W be a Banach space. If for every compact

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operator $R : W \to W$, every $\varepsilon > 0$, and every compact subset M of W, there exists a finite-rank operator $R_0 : W \to W$ satisfying $\|Rw - R_0w\| < \varepsilon$ for all $w \in M$, then W has the weak approximation property (WAP) [4]. Also, W has the bounded weak approximation property (BWAP) if for every compact operator $R : W \to W$, there exists a positive number λ_R such that for every compact subset M of W, and every $\varepsilon > 0$, there is a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0\| \le \lambda_R$ and $\|R_0w - Rw\| < \varepsilon$ for all $w \in M$ [4]. It is clear that the BWAP implies the WAP. Also, it is showed in [4] that the AP implies the BWAP. In the BWAP, if for every compact operator $R : W \to W$ with $\|R\| \le 1$, $\lambda_R = 1$, then W has the metric weak approximation property (MWAP) [6].

As the weaker versions of the WAP and the BWAP, Li and Fang in [14] introduced the concepts of the p-weak approximation property (p-WAP) and the p-bounded weak approximation property (p-BWAP), respectively. A Banach space W has the p-weak approximation property (p-WAP) if for every compact operator $R : W \to W$, every p-compact subset M of W and, every $\varepsilon > 0$, there exists a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0w - Rw\| < \varepsilon$ for all $w \in M$ [14]. W has the p-bounded weak approximation property (p-BWAP) if for every compact operator $R : W \to W$, there exists a positive number λ_R such that for every p-compact subset M of W and every $\varepsilon > 0$, there is a finite-rank operator $R_0 : W \to W$ satisfying $\|R_0\| \le \lambda_R$ and $\|R_0w - Rw\| < \varepsilon$ for all $w \in M$ [14]. It is clear that the WAP implies the p-WAP and the BWAP implies the p-BWAP.

The aim of this study is to obtain for the p-WAP the some results which given on the WAP in [6]-[8], by using the proof techniques in these results. Firstly, through a characterization given on the BWAP in [4, Lemma 3.7], it has been observed that the concepts of the p-BWAP and the BWAP are equivalent to each other. After, as a modification for the p-WAP of [6, Theorem 1.4 (a)], it is shown that the p-WAP of a Banach space W passes to its closed subspace N whenever $N^{\perp}$ is a complemented subspace of the dual space W^* and W^* has the v_p^* density, and also shown that the metric weak* density property in [6, Theorem 1.4 (b)] can be changed with the metric v_p^* density property. The proof of the solution of the duality problem for the p-WAP (respectively, p-BWAP) proved by Li and Fang [14] has been proved in a shorter way as an alternative. Moreover, as modifications of [7, Theorem 3.5] and [8, Theorem 1.3], respectively, it has been observed that the direct sum of two Banach spaces with the p-WAP and the p-AP has the p-WAP, and every ideal in a Banach space W has the p-WAP if and only if W has the p-WAP.

2. Notation and preliminaries

The symbols W and Z will denote Banach spaces. Let K be a subset of W. The symbol I_K represents the identity mapping on K, and for any topology τ on W, $\overline{K}^{\tau}$ denotes the τ-closure of K in W. If the τ is a norm topology, then we write $\overline{K}$. The symbol B_W denotes the closed unit ball of W. For $1 \le p < \infty$, the symbol $l_p(W)$ (respectively, $l_\infty(W)$) denotes the Banach space of all p-summable sequences (respectively, bounded sequences) in W, and $c_0(W)$ denotes the Banach space of all null sequences in W, respectively. $L(W,Z)$ denotes the Banach space of all linear bounded operators from W to Z with usual operator norm $\|,\|$. In this case $F = \mathbb{C}$, we write W^* instead of $L(W,\mathbb{C})$. A linear operator R from W to Z is called compact if $\overline{R(B_W)}$ is a compact subset of Z. The symbols $F(W,Z)$ and $K(W,Z)$ denote subspaces of finite rank and compact operators of $L(W,Z)$, respectively. Let $\lambda > 0$. $K^{\lambda}(W,W)$ (respectively, $F^{\lambda}(W,W)$) denotes the collection of compact (respectively, finite rank) operators $R : W \to W$ with $\|R\| \le \lambda$. $K^{\lambda}_{z^*}(W^*,W^*)$ (respectively, $F^{\lambda}_{z^*}(W^*,W^*)$) denotes the collection of compact (respectively, finite rank) and *weak**-to-*weak** continuous operators $R : W^* \to W^*$ with $\|R\| \le \lambda$. For a set $K \subset W$, the annihilator of K in W^* will be denoted by $K^{\perp}$. That is, $K^{\perp} = \{w^* \in W^* : w^*(w) = 0 \text{ for each } w \in K\}$. The notations τ and τ_p will denote the topologies on $L(W,Z)$, which of uniform convergence on the compact sets and p-compact sets in W, respectively. Through the paper, for p with $1 < p < \infty$, the q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

3. Some results for the p-weak approximation property

In this section, we will give an observation on the p-BWAP, some results on the p-WAP, and an alternative proof of solution of the duality problem for the p-WAP (respectively, p-BWAP). Firstly, we remember that the definitions of the v_p and v_p^* topologies given in [15] as the modifications of the v and *weak** topologies in [5, 6], respectively.

Definition 3.1. *([15], see [5, 6]) Let $1 < p < \infty$. Let X_1 be space of all linear functionals ϑ on $L(W,W)$ as in the form below*

$$\vartheta(S) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_i^k (w_k^*)(Sw_i)$$

in which $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty} \|z_k\|_q \|w_k^*\| < \infty$.

Let X_2 be space of all linear functionals ϕ on $L(W^*,W^*)$ as in the form below

$$\phi(R) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \beta_i^k (Rw_k^*)(w_i)$$

in which $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $t_k = (\beta_i^k)_{i=1}^{\infty} \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^{\infty} \|t_k\|_q \|w_k^*\| < \infty$.

Let v_p be the topology induced by X_1 on $L(W,W)$, and let v_p^* be the topology induced by X_2 on $L(W^*,W^*)$. From elementary facts, the v_p and v_p^* are locally convex topologies (see [5, 6, 17, 18]). Also, by using [13, Theorem 2.5], we get $(L(W,W),\tau_p)^* = X_1 = (L(W,W),v_p)^*$.
An operator S and a net $(S_\alpha)_\alpha$ in $L(W,W)$,

$$S_\alpha \xrightarrow{v_p} S \text{ if and only if } \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(w_k^*)(S_\alpha w_i) \to \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(w_k^*)(Sw_i)$$

for every $(w_i)_{i=1}^\infty \in l_p(W)$, $(w_k^*)_{k=1}^\infty \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^\infty \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^\infty \|z_k\|_q \|w_k^*\| < \infty$ ([15], see [5, 6]).
Similarly, for an operator R and a net $(R_\alpha)_\alpha$ in $L(W^*,W^*)$,

$$R_\alpha \xrightarrow{v_p^*} R \text{ if and only if } \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha w_k^*)(w_i) \to \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(Rw_k^*)(w_i)$$

for every $(w_i)_{i=1}^\infty \in l_p(W)$, $(w_k^*)_{k=1}^\infty \subset W^*$ and $z_k = (\beta_i^k)_{i=1}^\infty \in l_q$ for $\forall k \in \mathbb{N}$ with $\sum_{k=1}^\infty \|z_k\|_q \|w_k^*\| < \infty$ ([15], see [5, 6]).

Remark 3.2. *([15], see [5]) For any $1 < p < \infty$, by [13, Theorem 2.5], we can easily see that the τ_p-topology on the space $L(W,W)$ is stronger than the v_p-topology. Also, the v_p^*-topology on the space $L(W^*,W^*)$ is weaker than the v_p-topology. The v_p and v_p^* topologies coincide if W is a reflexive Banach space. Also, we remember that for an operator S and a net $(S_\alpha)_\alpha$ in $L(W,W)$*

$$S_\alpha \xrightarrow{v_p} S \text{ if and only if } S_\alpha^* \xrightarrow{v_p^*} S^*.$$

Remark 3.3. *([15], see [5, 6]) We have the following for a Banach space W.*

- *Let $2 < p < \infty$. W has the p-AP if and only if $I_W \in \overline{F(W,W)}^{v_p}$.*
- *Let $1 < p < \infty$. W has the λ-BAP if and only if $I_W \in \overline{F^\lambda(W,W)}^{v_p}$.*
- *Let $2 < p < \infty$. W has the p-WAP if and only if $K(W,W) \subset \overline{F(W,W)}^{v_p}$.*

Now we recall that the properties v_p^*D and Bv_p^*D given in [15] for compact operators on the dual space W^*.

Definition 3.4. *([15], see [5, 6]) Let W be a Banach space and let $1 < p < \infty$.*
(a) *W^* is said to have the v_p^* density (v_p^*D) if $K(W^*,W^*) \subset \overline{K_{z^*}(W^*,W^*)}^{v_p^*}$.*
(b) *W^* is said to have the bounded v_p^* density (Bv_p^*D) if $K^1(W^*,W^*) \subset \overline{K_{z^*}^\lambda(W^*,W^*)}^{v_p^*}$ for some $\lambda > 0$.*

W^* is said to have the metric v_p^* density (Mv_p^*D) if the Bv_p^*D is satisfied for $\lambda = 1$.

Lemma 3.5. *([15], see [10, Lemma 1.e.17], see [4, Lemma 3.11]) For a Banach space W and $1 < p < \infty$, we have the following.*
(a) *$F(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{\tau_p} \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$.*
(b) *$F^\lambda(W^*,W^*) \subset \overline{F_{z^*}^\lambda(W^*,W^*)}^{\tau_p} \subset \overline{F_{z^*}^\lambda(W^*,W^*)}^{v_p^*}$ for all $\lambda > 0$.*

Lemma 3.6. *([6, Lemma 3.6]) Let W be a Banach space, let N be a closed subspace of W such that let $N^\perp$ be a complemented subspace in W^*. Then, there exists a linear bounded map $U : N^* \to W^*$ satisfying $(Un^*)(n) = n^*(n)$ for $\forall\, n^* \in N^*$ and $n \in N$.*

3.1. Main results

Now, we give the main results of this paper.

Remark 3.7. *For any $1 < p < \infty$, Li and Fang in [14] defined the p-BWAP as weak version of the BWAP. On the other hand, Choi and Kim in [4, Lemma 3.7] showed that compact sets can be replaced by finite sets in the BWAP. Since every finite set is p-compact and every p-compact set is compact, the [4, Lemma 3.7] will also be correct when its part (a) is replaced with the p-BWAP. So, p-compact sets can be replaced with finite sets in p-BWAP. Thus, the concepts of the p-BWAP and the BWAP are equivalent.*

Using Remark 3.7, the relation definitions and (see [18, Lemma 3.5]), the following characterizations are obtained.

Remark 3.8. *(see [4, Lemma 3.7], and see [18, Lemma 3.5]) Let $1 < p < \infty$. We get the followings.*
(a) *A Banach space W has the BWAP if and only if for every $R \in K(W,W)$, there is a $\lambda_R > 0$ such that $R \in \overline{F^{\lambda_R}(W,W)}^{v_p}$.*
(b) *A Banach space W has the MWAP if and only if $K^1(W,W) \subset \overline{F^1(W,W)}^{v_p}$.*

The part (a) of the following theorem is a modification of [6, Theorem 1.4 (a)] for the p-WAP, and the part (b) shows that a similar result will be obtained when the metric weak* density (MW*D) property are replaced with the Mv_p^*D in [6, Theorem 1.4 (b)].

Theorem 3.9. *(see [6, Theorem 1.4]) Let $2 < p < \infty$. Let W be a Banach space, let N a closed subspace of W such that let $N^{\perp}$ be a complemented subspace in W^*.*

(a) *N has the p-WAP if W has the p-WAP and W^* has the v_p^*D.*
(b) *$K^1(N,N) \subset \overline{F^{\mu}(N,N)}^{\tau_p}$ for some $\mu > 0$ if W has the MWAP and W^* has the Mv_p^*D. In particular, N has the BWAP.*

Proof. **(a)** By using that W has the p-WAP, $K_{z^*}(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$ is obtained. If this inclusion is combined with the property v_p^*D of W^*, then we get $K(W^*,W^*) \subset \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$. Now, let $R \in K(N,N)$. We show that $R \in \overline{F(N,N)}^{v_p}$.
Let $I_N : N \to W$ be the inclusion map, and let the operator $U : N^* \to W^*$ be such as in Lemma 3.6. Since $UR^*I_N^* \in K(W^*,W^*)$, there exists a net $(R_\alpha^*)_\alpha \subset F_{z^*}(W^*,W^*)$ such that $R_\alpha^* \xrightarrow{v_p^*} UR^*I_N^*$. That means,

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(R_\alpha^* w_k^*)(w_i) \xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k(UR^*I_N^* w_k^*)(w_i) \tag{3.1}$$

for every $(w_i)_{i=1}^{\infty} \in l_p(W)$, $(w_k^*)_{k=1}^{\infty} \subset W^*$ and $z_k = (\lambda_i^k)_{i=1}^{\infty} \in l_q$ for each $k \in \mathbb{N}$ satisfying $\sum_{k=1}^{\infty}\|z_k\|_q\|w_k^*\| < \infty$.

Now, we take the sequences $(n_i)_{i=1}^{\infty} \in l_p(N)$, $(n_k^*)_{k=1}^{\infty} \subset N^*$ and, $t_k = (\beta_i^k)_{i=1}^{\infty} \in l_q$ for each $k \in \mathbb{N}$ satisfying $\sum_{k=1}^{\infty}\|t_k\|_q\|n_k^*\| < \infty$. Therefore, we get from (3.1)

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha^* U(n_k^*))(I_N n_i) \xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(I_N n_i) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(n_i). \tag{3.2}$$

Therefore, by (3.2), and the definition of the operator U, we get

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(I_N^*R_\alpha^*Un_k^*)(n_i) &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R_\alpha^*U(n_k^*))(I_N n_i)\\
&\xrightarrow{\alpha} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(UR^*I_N^*U(n_k^*))(n_i)\\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R^*I_N^*U(n_k^*))(n_i)\\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(Un_k^*)(Rn_i)\\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(n_k^*)(Rn_i)\\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\beta_i^k(R^*n_k^*)(n_i).
\end{aligned}$$

Thus, from the definition v_p^*, we have $I_N^*R_\alpha^*U \xrightarrow{v_p^*} R^*$. It follows that $R^* \in \overline{F(N^*,N^*)}^{v_p^*}$. From Lemma 3.5 (a), $R^* \in \overline{F_{z^*}(N^*,N^*)}^{v_p^*}$, and by Remark 3.2, we get $R \in \overline{F(N,N)}^{v_p}$. This proves (a).

(b) Since W has the MWAP, $K^1(W,W) \subset \overline{F^1(W,W)}^{v_p}$. Thus, as in the proof of (a), we get $K_{z^*}^1(W^*,W^*) \subset \overline{F_{z^*}^1(W^*,W^*)}^{v_p^*}$. Now, let R be an operator in $K^1(N,N)$. Then $UR^*I_N^* \in K^{\|U\|}(W^*,W^*)$. Using that W^* has the Mv_p^*D, we get $UR^*I_N^* \in K^{\|U\|}(W^*,W^*) \subset \overline{F_{z^*}^{\|U\|}(W^*,W^*)}^{v_p^*}$. By following similar steps in the proof of the part (a), if Lemma 3.5 (b) is applied, then it is obtained $R^* \in \overline{F_{z^*}^{\|U\|^2}(N^*,N^*)}^{v_p^*}$. By Remark 3.2, $R \in \overline{F^{\|U\|^2}(N,N)}^{v_p}$. Since $(L(W,W),\tau_p)^* = (L(W,W),v_p)^*$, by (see [18, Lemma 3.5]), $R \in \overline{F^{\|U\|^2}(N,N)}^{\tau_p}$, where $\mu =: \|U\|^2$. Thus, the proof is completed. □

Remark 3.10. *Let $2 < p < \infty$. Li and Fang [14] proved that if the W^* has p-WAP (respectively, p-BWAP), then W has the p-WAP (respectively, p-BWAP). The proof of this theorem can be shortened by using Remark 3.2 and Lemma 3.5. Actually, suppose that W^* has the p-WAP, and let $R \in K(W,W)$. It follows from Lemma 3.5 (a) that $R^* \in \overline{F_{z^*}(W^*,W^*)}^{v_p^*}$. Thus, there exists a net $(R_\alpha)_\alpha$ in $F(W,W)$ such that $R_\alpha^* \xrightarrow{v_p^*} R^*$. By Remark 3.2, $R_\alpha \xrightarrow{v_p} T$. Thus, $R \in \overline{F(W,W)}^{v_p}$.This shows that W has the p-WAP. Using Lemma 3.5 (b), the shortened proof for the p-BWAP can be made as similar.*

The following theorem is a modification for the p-WAP of [7, Theorem 3.5]. The proof of theorem is omitted since similar to [7, Theorem 3.5].

Theorem 3.11. *(see [7, Theorem 3.5]) The Banach space $W \oplus Z$ has the p-WAP if W has the p-WAP and Z has the p-AP.*

Li and Fang in [14] proved that the complemented subspaces of a Banach space with the p-WAP have the p-WAP. Combining this result with Theorem 3.11, we obtain the following result.

Corollary 3.12. *Let $2 < p < \infty$. Let a closed subspace N of a Banach space W be complemented in W. Then, we have the following:*
(a) *The space N has the p-WAP if W has the p-WAP, [14].*
(b) *The space W/N has the p-WAP if W has the p-WAP.*
(c) *The space W has the p-WAP if N has the p-WAP and W/N has the p-AP.*

Proof. **(a)** This part is proved by [14].
(b) Since N is a complemented subspace of W, it is well known that there is a closed subspace M of W such that M is complementary of N and the spaces W/N and M are isomorphic (see [17]). From (a), since every complemented subspace of W has the p-WAP, M has the p-WAP. Thus, W/N has the p-WAP.
(c) As in (b), there exists a closed subspace M of W such that the spaces W/N and M are isomorphic. Thus, from the hypothesis, M has the p-AP. Since N has the p-WAP and M has the p-AP, from Theorem 3.11 and [17, p. 65], we get that W has the p-WAP. □

By a modification of [8, Theorem 1.3], we obtain the following theorem for the p-WAP of a Banach space W. The proof of this theorem is omitted since similar to [8, Theorem 1.3]. (The locally complemented subspace and ideal concepts in the following theorem can be found in [8].)

Theorem 3.13. *(see [8, Theorem 1.3]) Let $2 < p < \infty$. For a Banach space W the following are equivalent.*
(a) *W has the p-WAP.*
(b) *Every locally complemented subspace of W has the p-WAP.*
(c) *Every ideal in W has the p-WAP.*
(d) *For every closed and separable subspace Z of W, there is a closed and separable subspace $Y \subset W$ containing the subspace Z such that Y has the p-WAP.*

Remark 3.14. *The above theorem also shows that without the property v_p^*D on the space W^* in Theorem 3.9 (a), Theorem 3.9 (a) will still be true (see [19, 20]).*

4. Conclusion

In the paper, it has been observed that the BWAP and the p-BWAP concepts are equivalent to each other. Some results on the p-WAP of Banach spaces have been given. The proof of the solution of the duality problem for the p-WAP (respectively, p-BWAP) which exists in the literature is given in a shorter way as an alternative.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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