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Numerical solution of differential difference equations by Laguerre collocation method

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Keywords
Laguerre polynomials and series,
Laguerre polynomial solutions,
differential-difference equations,
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ABSTRACT
This paper presents a numerical method for the approximate solution of mth-order linear differential difference equations with variable coefficients under the mixed conditions in terms of Laguerre polynomials. The technique we have used is an improved Laguerre collocation method. In addition, examples that illustrate the pertinent features of the method are presented and the results of study are discussed.

Diferansiyel fark denklemlerinin Laguerre sıralama yöntemi ile nümerik çözümleri

Anahtar Kelimeler
Laguerre polinomları ve serileri,
Laguerre polinom çözümleri,
diferansiyel fark denklemleri,
Laguerre sıralama yöntemi

ÖZET
Bu çalışmada m.mertebeden değişken katsayılı lineer diferansiyel fark denklemlerinin karışık koşullar altında Laguerre polinomları ile numerik çözümleri verilmiştir. Burada önerilen yöntem Laguerre sıralama yönteminin genelleştirilmiş halidir. Yöntemin hassasiyetini belirtmek için örnekler verilmiş ve bulunan sonuçlar tartışılmıştır.

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1. Introduction

In recent years, the differential-difference equations, treated as models of some physical phenomena, have been received considerable attention. When a mathematical model is developed for a physical system, it is usually assumed that all of the independent variables, such as space and time, are continuous. Usually, this assumption leads to a realistic and justified approximation of the real variables of the system. However, some of the physical systems for which, this continuous variable assumptions cannot be made. Since then, differential difference equations have played an important role in modeling problems that appear in various branches of science, e.g., mechanical engineering, condensed matter, biophysics, mathematical statistics and control theory. Differential–difference equations occur whenever discrete phenomena are studied or differential equations are discretized. In this paper, we are concerned with the use of Laguerre polynomials to solve difference difference equations. In recent years, the studies of differential difference equations are developed very rapidly and intensively. It is well known that linear differential difference equations have been considered by many authors[1-11]. The past couple decades have seen a dramatic increase in the application of difference models to problems in biology, physics and engineering[12-15]. In the field of differential difference equation the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more. Based on the obtained method, we shall give sufficient approximate solution of the linear differential difference Eq.(1). The results can extend and improve the recent works. An example is given to demonstrate the effectiveness of the results. In recent years, Taylor and Chebyshev approximation methods have been given to find polynomial solutions of differential equations by Sezer et al. [16-22].

In this study, the basic ideas of the above studies are developed and applied to the mth-order linear differential-difference equation with variable coefficients

$$\sum_{k=0}^{m} \mathrm{P}_k(t) y^{(k)}(t) + \sum_{j=1}^{n} \mathrm{Q}_j(t) y(t+j) = f(t), \quad k, j \geq 0, k, j \in N \tag{1}$$

with the conditions

$$\sum_{k=0}^{m-1} [a_{ik} y^{(k)}(0) + b_{ik} y^{(k)}(b)] = \mu_i \qquad 0 \leq t \leq b, \ i = 0,1,2,...,m-1 \tag{2}$$

where $P_k(t)$ and $f(t)$ are analytical functions; a_{ik}, b_{ik} *and* μ_i are real or complex constants. The aim of this study is to get solution as truncated Laguerre series defined by

$$y(t)=\sum_{n=0}^{m} a_n \mathrm{L}_n(t)\ ,\qquad \mathrm{L}_n(t)=\sum_{r=0}^{m}\frac{(-1)^r}{r!}\binom{n}{r}t^r\ ,\qquad 0\le t\le 1 \qquad (3)$$

where $\mathrm{L}_n(t)$ denotes the Laguerre polynomials, a_n $(0\le n\le N)$ are unknown Laguerre polynomial coefficients and N is chosen any positive integer such that $N\ge m$.

Here $\mathrm{P_k}(t)$, $\mathrm{Q_j}(t)$ and $f(t)$ are functions defined on 0≤t≤b, the real coefficients a_{ik} and μ_i are appropriate constants.

The rest of this paper is organized as follows. We describe the formulation of Laguerre polynomials required for our subsequent development in section 2. Higher-order linear differential difference equation with variable coefficients and fundamental relations are presented in Section 3. The new scheme are based on Laguerre collocation method. The method of finding approximate solution is described in Section 4. To support our findings, we present result of numerical experiments in Section 5. Section 6 concludes this article with a brief summary. Finally some references are introduced at the end.

2. Properties of the Laguerre polynomials

A total orthonormal sequence in $\mathrm{L}^2(-\infty,b]$ or $\mathrm{L}^2[a,+\infty)$ can be obtained from such a sequence in $\mathrm{L}^2[0,+\infty)$ by transformations t=b-s and t=s+a, respectively. We consider $\mathrm{L}^2[0,+\infty)$. Applying the Gram-Schmidt process to the sequence defined by

$e^{-t/2},\quad te^{-t/2},\quad t^2e^{-t/2},\ldots$

We can obtain an orthonormal sequence (e_n). It can be shown that (e_n) is total in $\mathrm{L}^2[0,+\infty)$ and is given by

$e_n(t)=e^{-t/2}\mathrm{L}_n(t)$, n=0,1,2,…

where the Laguerre polynomial of order n is defined by

$$\mathrm{L}_0(t)=1,\qquad \mathrm{L}_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),$$

n=1,2,3,… (4)

That is

$$\mathrm{L}_n(t)=\sum_{r=0}^{n}\frac{(-1)^r}{r!}\binom{n}{r}t^r \qquad (5)$$

Explicit expressions for the first few Laguerre polynomials are

$\mathrm{L}_0(t)=1,\qquad \mathrm{L}_1(t)=1-t,$

$\mathrm{L}_2(t)=1-2t+\frac{1}{2}t^2,$

$\mathrm{L}_3(t)=1-3t+\frac{3}{2}t^2-\frac{1}{6}t^3$

The Laguerre polynomials $\mathrm{L}_n(t)$ are solutions of the Laguerre differential equation

$t\mathrm{L}_n''(t)+(1-t)\mathrm{L}_n'(t)+n\mathrm{L}_n(t)=0$ [24] (6)

In the present application, an approximate solution in terms of linear combination of Laguerre polynomial is assumed of the following form:

$$y(t)=\sum_{n=0}^{N} a_n\mathrm{L}_n(t),\quad 0\le n\le N$$

3.Fundamental relations

Let us consider the mth-order linear differential-difference equation with variable coefficients (1) and find the matrix forms of each term in the equation. First we can convert the solution $y(t)$ defined by a truncated Laguerre series (3) and its derivative $y^{(k)}(t)$ to matrix forms

$$y(t)=\mathbf{L}(t)\mathbf{A},\quad y^{(k)}(t)=\mathbf{L}^{(k)}(t)\mathbf{A} \qquad (7)$$

where

$$\mathbf{L}(t) = [\mathbf{L_0}(t) \quad \mathbf{L_1}(t) \quad \mathbf{L_2}(t) \quad \dots \quad \mathbf{L_N}(t)]$$
$$\mathbf{A} = [a_0 \ a_1 \dots a_N]^T$$
$$\mathbf{X}(t) = [1 \ t \dots t^N] \qquad (8)$$

By using the expression (5) and taking n=0,1,…,N we find the corresponding matrix relation as follows

$$(\mathbf{L}(t))^T = \mathbf{H}(\mathbf{X}(t))^T \qquad (9)$$

where

$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix} \qquad (10)$$

Then, by taking into account(9) we obtain

$$\mathbf{L}(t) = \mathbf{X}(t)\mathbf{H}^T \qquad (11)$$

and

$$(\mathbf{L}(t))^{(k)} = \mathbf{X}^{(k)}(t)(\mathbf{H})^T \ , \ k = 0,1,2,...$$

To obtain the matrix $\mathbf{X}^{(k)}(t)$ in terms of the matrix $\mathbf{X}(t)$, we can use the following relation:

$$\mathbf{X}^{(1)}(t) = \mathbf{X}(t)\mathbf{B}^T$$
$$\mathbf{X}^{(2)}(t) = \mathbf{X}^{(1)}(t)\mathbf{B}^T = \mathbf{X}(t)(\mathbf{B}^T)^2$$
$$\mathbf{X}^{(3)}(t) = \mathbf{X}^{(2)}(t)\mathbf{B}^T = \mathbf{X}(t)(\mathbf{B}^T)^3$$
$$\mathbf{X}^{(k)}(t) = \mathbf{X}^{(k-1)}(t)\mathbf{B}^T = \mathbf{X}(t)(\mathbf{B}^T)^k \qquad (12)$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & N & 0 \end{bmatrix} \qquad (13)$$

Consequently, by substituting the matrix forms (11) and (12) into (7) we have the matrix relation

$$y^{(k)} = \mathbf{X}(t)\mathbf{B}^k\mathbf{H}^\mathbf{T}\mathbf{A} \qquad (14)$$

To obtain the matrix $\mathrm{X}(t+j)$ in terms of the matrix $\mathrm{X}(t)$, we can use the following relation:

$$\mathbf{X}(t+j) = \mathbf{X}(t)\mathbf{B}_j \qquad (15)$$

Where

$$\mathbf{X}(t+j) = [1 \ (t+j) \ (t+j)^2 \dots (t+j)^N]$$

$$\mathbf{B}_j^T = \begin{bmatrix} \binom{0}{0}(j)^0 & \binom{1}{0}(j)^1 & \binom{2}{0}(j)^2 & \cdots & \binom{N}{0}(j)^N \\ 0 & \binom{1}{1}(j)^0 & \binom{2}{1}(j)^1 & \cdots & \binom{N}{1}(j)^{N-1} \\ 0 & 0 & \binom{2}{2}(j)^0 & \cdots & \binom{N}{2}(j)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(j)^0 \end{bmatrix} \qquad (16)$$

Consequently, by substituting the matrix forms (11) and (12) into (7), we have the matrix relation of solution

$$y(t+j) = \mathbf{L}(t+j)\mathbf{A} = \mathbf{X}(t+j)\mathbf{H}^T\mathbf{A} \qquad (17)$$

and by means of (15), the matrix relation is

$$y(t+j) = \mathbf{X}(t)\mathbf{B}_j^T\mathbf{H}^T\mathbf{A} \qquad (18)$$

3. Method of Solution

In this section, we consider high order linear differential-difference equation in (1) and approximate to solution by means of finite Laguerre series defined in (3). The aim is to find Laguerre coefficients, that is the matrix A. For this purpose, substituting the matrix relations (14) and (18) into Eq.(1) and then simplifying, we obtain the fundamental matrix equation

$$\sum_{k=0}^{m}\mathbf{P}_k(t)\mathbf{X}(t)(\mathbf{B}^T)^k\mathbf{H}^T\mathbf{A}+\sum_{j=1}^{n}\mathbf{Q}_j(t)\mathbf{X}(t)(\mathbf{B}_j)^T\mathbf{H}^T\mathbf{A}=f(t) \qquad (19)$$

By using in Eq. (19) collocation points t_i defined by

$$t_i=\frac{i}{N},\ i=0,1,...,N \quad (20)$$

we get the system of matrix equations

$$\sum_{k=0}^{m}\mathbf{P}_k(t_i)\mathbf{X}(t_i)(\mathbf{B}^T)^k\mathbf{H}^T\mathbf{A}+\sum_{j=1}^{n}\mathbf{Q}_j(t_i)\mathbf{X}(t_i)(\mathbf{B}_j)^T\mathbf{H}^T\mathbf{A}=f(t_i),\ i=0,1,...N \qquad (21)$$

or briefly the fundamental matrix equation

$$\sum_{k=0}^{m}\mathbf{P}_k(t)\mathbf{X}(t)(\mathbf{B}^T)^k\mathbf{H}^T\mathbf{A}+\sum_{j=1}^{n}\mathbf{Q}_\mathbf{j}(t)\mathbf{X}(t)(\mathbf{B}_j)^T\mathbf{H}^T\mathbf{A}=\mathbf{F} \qquad (22)$$

where

$$\mathbf{P}_k=\begin{bmatrix}P_k(t_0) & 0 & 0 & \dots & 0\\ 0 & P_k(t_1) & 0 & \dots & 0\\ 0 & 0 & P_k(t_2) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & P_k(t_N)\end{bmatrix} \quad \mathbf{Q}_j=\begin{bmatrix}Q_j(t_0) & 0 & 0 & ... & 0\\ 0 & Q_j(t_1) & 0 & ... & 0\\ 0 & 0 & Q_j(t_2) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & ... & Q_j(t_N)\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}f(t_0)\\ f(t_1)\\ .\\ .\\ .\\ f(t_N)\end{bmatrix} \qquad \mathbf{X}=\begin{bmatrix}X(t_0)\\ X(t_1)\\ .\\ .\\ .\\ X(t_N)\end{bmatrix}=\begin{bmatrix}1 & t_0 & t_0^{\ 2} & . & . & . & t_0^{\ N}\\ 1 & t_1 & t_1^{\ 2} & . & . & . & t_1^{\ N}\\ . & . & . & & & & .\\ . & . & . & & & & .\\ . & . & . & & & & .\\ 1 & t_N & t_N^{\ 2} & . & . & . & t_N^{\ N}\end{bmatrix}$$

Hence, the fundamental matrix equation (22) corresponding to Eq. (1) can be written in the form

$$\mathbf{WA}=\mathbf{F}\ \text{ or }\ [\mathbf{W};\mathbf{F}],\ \ \mathbf{W}=[w_{i,j}],\ i,j=0,1,...,N \qquad (23)$$

where

$$\mathbf{W}=\sum_{k=0}^{m}\mathbf{P}_k(t)\mathbf{X}(t)(\mathbf{B}^T)^k\mathbf{H}^T+\sum_{j=1}^{n}\mathbf{Q}_j(t)\mathbf{X}(t)(\mathbf{B}_j)^T\mathbf{H}^T \qquad (24)$$

Here, Eq. (23) corresponds to a system of $(N+1)$ linear algebraic equations with unknown Laguerre coefficients $a_0, a_1, ..., a_N$. We can obtain the corresponding matrix forms for the conditions (2), by means of the relation (14),

$$\sum_{k=0}^{m-1}[a_{ik}y^{(k)}(0)+b_{ik}y^{(k)}(b)]=\mu_i \qquad 0\le t\le b,\ i=0,1,2,...,m-1$$

On the other hand, the matrix form for conditions can be written as

$$\mathbf{U}_i\mathbf{A}=[\mu_i] \text{ or } [\mathbf{U}_i;\mu_i],\ i=0,1,2,...,m-1 \tag{25}$$

where

$$\mathbf{U}_i=\sum_{k=0}^{m-1}\left[a_{ik}\mathbf{X}(0)+b_{ik}\mathbf{X}(b)\right]\mathbf{H}^T,\quad i=0,1,\ldots,m-1 \tag{26}$$

and

$$\mathbf{U}_i=[u_{i0}\ u_{i1}\ u_{i2}\ldots u_{iN}],\ i=0,1,2,...m-1 \tag{27}$$

To obtain the solution of Eq. (1) under conditions (2), by replacing the row matrices (26) by the last m rows of the matrix (25), we have the new augmented matrix,

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & f(t_0) \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & f(t_1) \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & f(t_{N-m}) \\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \mu_0 \\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \mu_1 \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \mu_{m-1} \end{bmatrix} \tag{28}$$

If $rank\widetilde{\mathbf{W}}=rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]=N+1$, then we can write

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{F}} \tag{29}$$

Thus the matrix A (thereby the coefficients $a_0,a_1,\ldots,a_N$) is uniquely determined. Also the Eq.(1) with conditions (2) has a unique solution. This solution is given by truncated Laguerre series (3).

We can easily check the accuracy of the method. Since the truncated Laguerre series (3) is an approximate solution of Eq.(1), when the solution $y_N(t)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for $t=t_q\in[0,1],\ q=0,1,2,...$

$$\mathbf{E}(t_q)=\left|\sum_{k=0}^{m}P_k(t)y^{(k)}(t)+\sum_{j=1}^{n}Q_j(t)y(t+j)-f(t)\right|\cong 0 \tag{30}$$

and $\mathrm{E}(t_q)\le 10^{-k_q}$ (k_q positive integer). If max $10^{-k_q}=10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $\mathrm{E}(t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, the error can be estimated by the function

$$\mathrm{E}_N(t)=\sum_{k=0}^{m}\mathrm{P}_k(t)y^{(k)}(t)+\sum_{j=1}^{n}\mathrm{Q}_j(t)y(t+j)-f(t) \quad (31)$$

If $\mathrm{E}_N(t)\to 0$, when N is sufficiently large enough, then the error decreases.

4.Illustrative examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple 9. The absolute errors in Tables are the values of $|y(x)-y_N(x)|$ at selected points.

Example1. Let us first consider the second order linear differential difference equation with variable coefficients

$$y''(t)-y'(t)+e^{-t}y(t)+y(t+1)+y(t+2)=1+e^{t+1}+e^{t+2}$$

with

$$y(0)=1, y'(0)=1$$

and seek the solution $y(t)$ as a truncated Laguerre series

$$y(t)=\sum_{n=0}^{N}a_n\mathrm{L}_n(t)$$

So that $P_0(t)=e^{-t}$, $P_1(t)=-1$, $P_2(t)=1$, $Q_1(t)=1$, $Q_2(t)=1$, $f(t)=1+e^{t+1}+e^{t+2}$.

Then, for N=5, the collocation points are

$t_0=0$, $t_1=1/5$, $t_2=2/5$, $t_3=3/5$, $t_4=4/5$, $t_5=1$

and the fundamental matrix equation of the problem is defined by

$$\{\mathbf{P}_0\mathbf{XH}+\mathbf{P}_1\mathbf{XBH}+\mathbf{P}_2\mathbf{XB}^2\mathbf{H}+\mathbf{Q_1}\mathbf{XB}_1\mathbf{H}+\mathbf{Q_2}\mathbf{XB}_2\mathbf{H}\}\mathbf{A}=\mathbf{F}$$

where $\mathbf{P}_0,\mathbf{P}_1,\mathbf{P}_2,\mathbf{Q_1}, \mathbf{Q_2},\mathbf{X}$ are matrices of order (6x6) defined by

$$\mathbf{X}=\begin{bmatrix}1&0&0&0&0&0\\1&\frac{1}{5}&\frac{1}{25}&\frac{1}{125}&\frac{1}{625}&\frac{1}{3125}\\1&\frac{2}{5}&\frac{4}{25}&\frac{8}{125}&\frac{16}{625}&\frac{32}{3125}\\1&\frac{3}{5}&\frac{9}{25}&\frac{27}{125}&\frac{81}{625}&\frac{243}{3125}\\1&\frac{4}{5}&\frac{16}{25}&\frac{64}{125}&\frac{256}{625}&\frac{1024}{3124}\\1&1&1&1&1&1\end{bmatrix},\quad \mathbf{P}_0=\begin{bmatrix}1&0&0&0&0&0\\0&e^{-1/5}&0&0&0&0\\0&0&e^{-2/5}&0&0&0\\0&0&0&e^{-3/5}&0&0\\0&0&0&0&e^{-4/5}&0\\0&0&0&0&0&e^{-1}\end{bmatrix},$$

$$\mathbf{P}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{H}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & 3 & 5 \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{24} & \frac{5}{24} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{120} \end{bmatrix}, \mathbf{B}_1^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_2^T = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 & 32 \\ 0 & 1 & 4 & 12 & 32 & 80 \\ 0 & 0 & 1 & 6 & 24 & 80 \\ 0 & 0 & 0 & 1 & 8 & 40 \\ 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{Q}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If these matrices are substituted in (22), it is obtained linear algebraic system.This system yields the approximate solution of the problem. The result with N=6,8,10 using the Laguerre collocation method discussed in Section 3 and also the exact values of $y = \exp(t)$ are shown in Table1.

Table 1
Error analysis of Example 1 for the t value

t	Exact Solution	Present Method N=6	N_e=6	N=8	N_e=8	N=10	N_e=10
0.0	1.000000	1.000000	0.100E-8	0.999999	0.580E-7	0.999999	0.866E-7
0.1	1.105171	1.105194	0.228E-4	1.105171	0.271E-6	1.105167	0.367E-5
0.2	1.221403	1.221493	0.907E-4	1.221403	0.270E-6	1.221389	0.140E-4
0.3	1.349859	1.350058	0.199E-3	1.349862	0.363E-5	1.349829	0.297E-4
0.4	1.491825	1.492161	0.337E-3	1.491837	0.121E-4	1.491776	0.486E-4
0.5	1.648721	1.649206	0.485E-3	1.648749	0.278E-4	1.648654	0.671E-4
0.6	1.822119	1.822755	0.620E-3	1.822171	0.527E-4	1.822037	0.810E-4
0.7	2.013753	2.014466	0.713E-3	2.013841	0.878E-4	2.013667	0.857E-4
0.8	2.225541	2.226273	0.732E-3	2.225674	0.133E-3	2.225645	0.760E-4
0.9	2.459603	2.460251	0.647E-3	2.459603	0.188E-3	2.459556	0.475E-4
1.0	2.718282	2.718715	0.433E-3	2.718282	0.250E-3	2.718285	0.357E-5

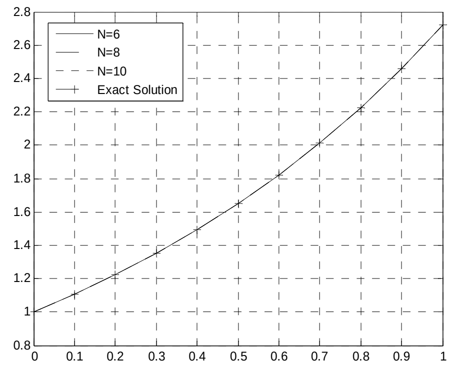

Fig.1.Numerical and exact solution of the Example1 for N=6,8,10

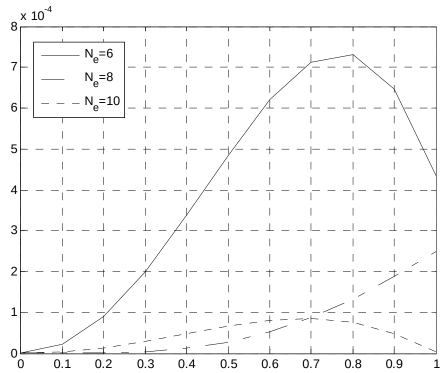

Fig.2.Error function of Example1 for various N.

Fig.1 shows the resulting graph of solution of Example1 for N = 6,8,10 and it is compared with exact solution. In Fig.2 we plot error function for Example1. This plot clearly indicates that when we increase the truncation limit N, we have less error.

Example2. Let us find the Laguerre series solution of first order linear differential difference equation

$$y'(t)+y(t)+y(t+1)-y(t+2)=t^2-t-3$$

with conditions

$y(0)=0$

and the exact solution is $y=t^2-t$. Using the procedure in Section 3, we find the approximate solution of this equation which is the same with the exact solution.

Example3. Let us find the Laguerre series solution of the following second order linear difference equation

$$5y''(t)-7ty'(t)+3y(t+2)=-5\sin()-7t\cos()+3\sin(+2)$$

with $y(0)=3,\ y'(0)=0$. The exact solution of this problem is $y(t)=t^2+3$.

Table 3
Error analysis of Example 3 for the t value

t	Exact Solution	Present Method N=7	N_e=7	N=8	N_e=8	N=9	N_e=9
0.0	0.000000	-0.45E-8	0.45000E-8	0.110E-8	0.11000E-8	0.110E-8	0.11000E-8
0.1	0.099833	0.099835	0.17356E-5	0.099833	0.58038E-6	0.099833	0.36465E-6
0.2	0.198669	0.198676	0.70480E-5	0.198717	0.24398E-5	0.198670	0.16127E-5
0.3	0.295520	0.295536	0.15888E-4	0.295552	0.57201E-5	0.295524	0.39567E-5
0.4	0.389418	0.389446	0.27898E-4	0.389428	0.10473E-4	0.389425	0.75683E-5
0.5	0.479425	0.479467	0.42308E-4	0.479442	0.16625E-4	0.479438	0.12543E-4
0.6	0.564642	0.564700	0.57863E-4	0.564666	0.23929E-4	0.564661	0.18864E-4
0.7	0.644217	0.644290	0.72777E-4	0.644249	0.31934E-4	0.644244	0.26361E-4
0.8	0.717356	0.717440	0.84756E-4	0.717396	0.39959E-4	0.717390	0.34669E-4
0.9	0.783326	0.783417	0.91085E-4	0.783374	0.47091E-4	0.783370	0.43208E-4
1.0	0.841470	0.841559	0.88801E-4	0.841523	0.52214E-4	0.841522	0.51173E-4

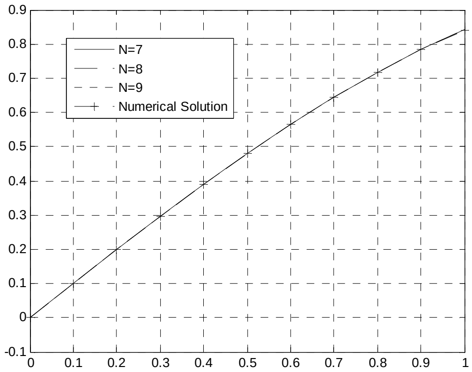

Fig.3.Numerical and exact solution of the Example3 for N=7,8,9

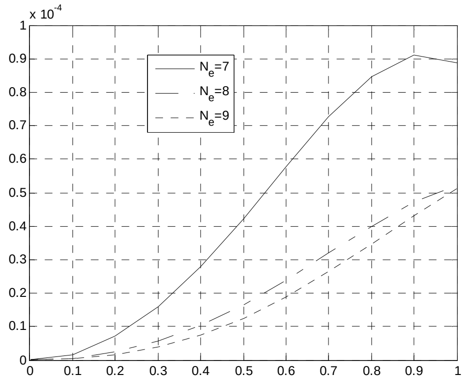

Fig.4.Error function of Example3 for various N.

The solution of the second order linear differential difference equation is obtained for N=7,8,9. For numerical results, see Table 3. We display a plot of Laguerre collocation method and Exact solution for N=7,8,9 in Fig.3. It seems that the solutions almost identical and we compare error functions for various N with each others in Fig. 4.

Example4. Let us find the Laguerre polynomial solution of the following linear differential-difference equation

$$y''(t) + y'(t) - 2y(t) + y(t+1) = e^{t+1}$$

with $y(0) = 1, \quad y'(0) = 1$. The exact solution of this problem is $y(t) = e^t$.

Table4
Error analysis of Example 4 for the t value

t	Exact			Present Method			
	Solution	N=6	N_e=6	N=8	N_e=8	N=10	N_e=10
0.0	1.000000	1.000000	0.100E-8	1.000000	0.300E-8	1.000000	0.140E-8
0.1	0.995004	0.994978	0.252E-4	0.995002	0.201E-5	0.995004	0.101E-6
0.2	0.980065	0.979978	0.881E-4	0.980059	0.672E-5	0.980066	0.459E-6
0.3	0.955336	0.955168	0.167E-3	0.955324	0.121E-4	0.955336	0.272E-5
0.4	0.921060	0.920816	0.244E-3	0.921044	0.164E-4	0.921060	0.734E-5
0.5	0.877582	0.877282	0.299E-3	0.877564	0.182E-4	0.877582	0.145E-4
0.6	0.825335	0.825014	0.321E-3	0.825319	0.164E-4	0.825335	0.240E-4
0.7	0.764842	0.764542	0.300E-3	0.764831	0.104E-4	0.764842	0.354E-4
0.8	0.696706	0.696473	0.233E-3	0.696706	0.484E-6	0.696706	0.476E-4
0.9	0.621609	0.621148	0.123E-3	0.621622	0.129E-4	0.621609	0.598E-4
1.0	0.540302	0.540323	0.208E-4	0.540331	0.288E-4	0.540302	0.708E-4

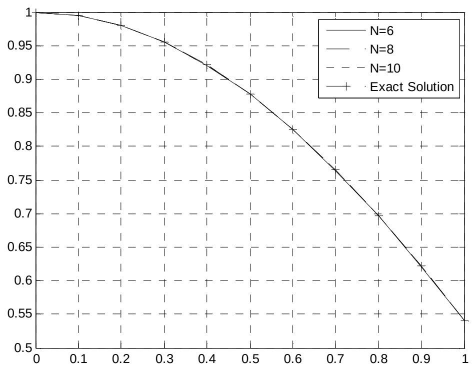

Fig.5.Numerical and exact solution of the Example4 for N=6,8,10

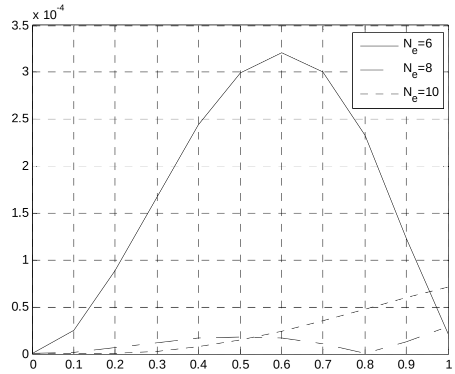

Fig.6.Error function of Example4 for various N.

Example5. Let us find the Laguerre series solution of the following second order linear differential-difference equation

$$10y''(t)+10y(t)+100y(t+2)=100\cos(t+2)$$

with $y(0)=1,\ y'(0)=0$. The exact solution of this problem is $y(t)=\cos(t)$.

Table 5
Error analysis of Example 5 for the t value

t	Exact Solution	Present Method					
		N=6	N_e=6	N=8	N_e=8	N=10	N_e=10
0.0	1.000000	1.000000	0.100E-8	1.000000	0.300E-8	1.000000	0.140E-8
0.1	0.995004	0.994978	0.252E-4	0.995002	0.201E-5	0.995004	0.101E-6
0.2	0.980065	0.979978	0.881E-4	0.980059	0.672E-5	0.980066	0.459E-6
0.3	0.955336	0.955168	0.167E-3	0.955324	0.121E-4	0.955336	0.272E-5
0.4	0.921060	0.920816	0.244E-3	0.921044	0.164E-4	0.921060	0.734E-5
0.5	0.877582	0.877282	0.299E-3	0.877564	0.182E-4	0.877582	0.145E-4
0.6	0.825335	0.825014	0.321E-3	0.825319	0.164E-4	0.825335	0.240E-4
0.7	0.764842	0.764542	0.300E-3	0.764831	0.104E-4	0.764842	0.354E-4
0.8	0.696706	0.696473	0.233E-3	0.696706	0.484E-6	0.696706	0.476E-4
0.9	0.621609	0.621148	0.123E-3	0.621622	0.129E-4	0.621609	0.598E-4
1.0	0.540302	0.540323	0.208E-4	0.540331	0.288E-4	0.540302	0.708E-4

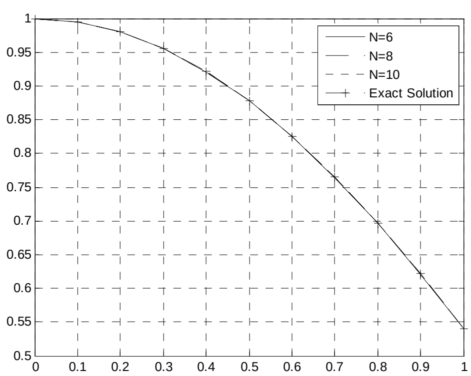

Fig.7.Numerical and exact solution of the Example5 for N=6,8,10

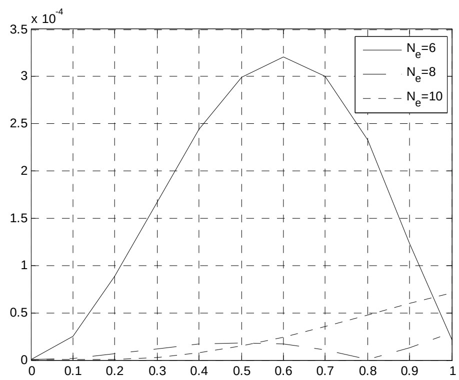

Fig.8.Error function of Example5 for various N.

We display the resulting graph of solution of Example5 for N = 6,8,10 and it is compared with exact solution in Fig.7. We plot error function for Example5 in Fig.8. This plot clearly indicates that when we increase the truncation limit N, we have less error.

Example6. Let us find the Laguerre series solution of the following second order linear differential-difference equation

$$y''(t) - y'(t) + y(t) - y(t+1) + y(t+2) = -\cos(t) - \sin(t+1) + \sin(t+2)$$

with $y(0) = 1, y'(0) = 1$. The exact solution of this problem is $y(t) = \cos t$.

Table6
Error analysis of Example 6 for the t value

t	Exact Solution	Present Method N=8	N_e=8	N=9	N_e=9	N=10	N_e=10
0.0	1.000000	0.999999	0.500E-8	0.999999	0.130E-8	1.000000	0.200E-8
0.1	0.995004	0.996889	0.188E-2	0.996889	0.188E-2	0.996889	0.188E-2
0.2	0.980067	0.987099	0.703E-2	0.987099	0.703E-2	0.987010	0.703E-2
0.3	0.955336	0.969976	0.146E-1	0.969975	0.146E-1	0.969973	0.146E-1
0.4	0.921061	0.944915	0.239E-1	0.944913	0.239E-1	0.944907	0.238E-1
0.5	0.877583	0.911365	0.338E-1	0.911363	0.338E-1	0.911353	0.338E-1
0.6	0.825336	0.868440	0.435E-1	0.868841	0.435E-1	0.868826	0.435E-1
0.7	0.764842	0.816941	0.521E-1	0.816937	0.521E-1	0.816915	0.521E-1
0.8	0.696707	0.755324	0.586E-1	0.755320	0.586E-1	0.755290	0.586E-1
0.9	0.621610	0.683375	0.621E-1	0.683746	0.621E-1	0.683707	0.621E-1
1.0	0.540302	0.602067	0.618E-1	0.602063	0.618E-1	0.602014	0.617E-1

Using the procedure in Section 3 and taking N=8,9 and 10 the matrices in Eq.(22) are computed. Hence linear algebraic system is gained. This system is approximately solved using the Maple9. The solution of the linear differential difference equation is obtained for N=8,9,10. For numerical results, see Table 6.

5. Conclusion

In recent years, the studies of high order linear differential-difference equation have attracted the attention of many mathematicians and physicists. The Laguerre collocation methods are used to solve the high order linear diferential-difference equation numerically. A considerable advantage of the method is that the Laguerre polynomial coefficients of the solution are found very easily by using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. Illustrative examples are included to demonstrate the validity and applicability of the technique and performed on the computer using a program written in Maple9. To get the best approximating solution of the equation, we take more forms from the Laguerre expansion of functions, that is , the truncation limit N must be chosen large enough. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial functions. Illustrative examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contributed to the good agreement between approximate and exact values in the numerical example.

As a result, the power of the employed method is confirmed. We assured the correctness of the obtained solutions by putting them back into the original equation with the aid of Maple, it provides an extra measure of confidence in the results. We predict that the Laguerre expansion method will be a promising method for investigating exact analytic solutions to linear differential-difference equations.The method can also be extended to the system of linear differential-difference equations with variable coefficients, but some modifications are required.

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