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# ON (QUASI-)MORPHIC PROPERTY OF SKEW POLYNOMIAL RINGS

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Abstract. The main objective of this paper is to study (quasi-)morphic property of skew polynomial rings. Let $R$ be a ring, $\sigma$ be a ring homomorphism on $R$ and $n \geq 1$. We show that $R$ inherits the quasi-morphic property from $R[x;\sigma]/(x^{n+1})$. It is also proved that the morphic property over $R[x;\sigma]/(x^{n+1})$ implies that $R$ is a regular ring. Moreover, we characterize a unit-regular ring $R$ via the morphic property of $R[x;\sigma]/(x^{n+1})$. We also investigate the relationship between strongly regular rings and centrally morphic rings. For instance, we show that for a domain $R$, $R[x;\sigma]/(x^{n+1})$ is (left) centrally morphic if and only if $R$ is a division ring and $\sigma(r) = u^{-1}ru$ for some $u \in R$. Examples which delimit and illustrate our results are provided.

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## 1. Introduction

Throughout this paper, we assume that $R$ is an associative ring (not necessarily commutative) with unity. If $X \subseteq R$, then the notation $\mathrm{r.ann}_R(X)$ ($\mathrm{l.ann}_R(X)$) denotes the right (left) annihilator of $X$ with elements from $R$ and it is defined by $\{r \in R \mid Xr = 0\}$ ($\{r \in R \mid rX = 0\}$). The notations $\mathrm{U}(R)$, $\mathrm{J}(R)$ and $\mathrm{C}(R)$ stand for the group of unit elements, the Jacobson radical and the center of $R$, respectively. If $M$ is an $R$-module, then the ring of $R$-endomorphisms of $M$ is denoted by $\mathrm{End}_R(M)$. We remind that the ring of polynomials in indeterminate $x$ over $R$ is denoted by $R[x]$.

Recall that a ring $R$ is said to be *(unit-)regular* if for every $a \in R$, there exists $u \in R$ ($u \in \mathrm{U}(R)$) such that $a = aua$. Moreover, $R$ is called *strongly regular* if for every $a \in R$, $a \in Ra^2$. It is routine to see that strongly regular rings are unit-regular and every unit-regular ring is regular (for more information, see [7]). It is a known theorem of Ehrlich [6], that a ring $R$ is unit-regular if and only if $R$ is regular and for each $a \in R$, $R/Ra \simeq \mathrm{l.ann}_R(a)$. In 2004, Nicholson and Campos [12], called a ring $R$ *left morphic* if for every $a \in R$, $R/Ra \simeq \mathrm{l.ann}_R(a)$. Equivalently, a ring $R$ is left morphic if and only if for every $a \in R$, there exists an element $b \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(b)$. In addition, the ring $R$ is called *left centrally morphic* if $b$ can be chosen an element of $\mathrm{C}(R)$ [10, Section 5]. In 2007, Camillo and Nicholson [2], generalized this concept to quasi-morphic rings. They called a ring $R$ *left quasi-morphic* provided that for every $a \in R$, there exist elements $b, c \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$. Right (quasi-)morphic rings are defined in the same way. A left and right (quasi-)morphic ring is called *(quasi-)morphic*. These notions have been also studied by several authors, see for example [1], [3], [4], [5], [8] and [9]. Clearly, every left centrally morphic ring is left morphic and left morphic rings are left quasi-morphic. While for a commutative ring $R$, these three concepts coincide. Moreover, the class of regular (resp., strongly regular, unit-regular) rings is contained in the class of quasi-morphic (resp., centrally morphic, morphic) rings. We summarize these relations in the following diagram which all of the implications are strict. For more details, see [2], [10] and [12].

$$\begin{tikzcd}
\text{strongly regular} \arrow[r] \arrow[d] & \text{unit-regular} \arrow[r] \arrow[d] & \text{regular} \arrow[d] \\
\text{centrally morphic} \arrow[r] & \text{morphic} \arrow[r] & \text{quasi-morphic}
\end{tikzcd}$$

Unit-regular rings are precisely regular and (left) morphic rings [6, Theorem 1]. Following [2] and [3], all known examples of quasi-morphic rings are either morphic rings, or regular rings, or the direct products of these rings. However Lee and Zhou in 2009, investigated the (quasi-)morphic property of $R[x]/(x^{n+1})$ $(n \geq 1)$ to construct examples of quasi-morphic rings that are neither regular, nor morphic, nor the direct products of regular rings and morphic rings [10, Example 14]. For instance, they proved that a ring $R$ is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic where $n \geq 1$ [10, Theorem 11]. In addition, they showed that if $R$ is a regular ring then for any $n \geq 1$, $R[x]/(x^{n+1})$ $(n \geq 1)$ is quasi-morphic [10, Theorem 4] and the converse has been asked as an open question [10, Question 1]. Moreover, if $R[x]/(x^{n+1})$ is left (quasi-)morphic where $n \geq 1$, then $R$ is so [10, Lemma 10]. Besides, Lee and Zhou in 2007 [11, Corollary 3], showed that $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ is left morphic provided that $R$ is a unit-regular ring and $\sigma : R \to R$ is a ring homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$.

These motivate us to study (quasi-)morphic property of $R[x;\sigma]/(x^{n+1})$ where $\sigma$ is a ring homomorphism over $R$ and $n \geq 1$. In Section 2, we show that if $R[x;\sigma]/(x^{n+1})$ is left quasi-morphic, then $R$ is also left quasi-morphic (Theorem 2.3). Moreover, it is shown that the same result is valid for the morphic property provided that $\sigma$ is an isomorphism (Theorem 2.9). In addition, we prove that if $R[x;\sigma]/(x^{n+1})$ is morphic, then $R$ is a regular ring (Theorem 2.16). Among other results, we give a new characterization of unit-regular rings by showing that if $\sigma : R \to R$ is an isomorphism such that $\sigma(e) = e$ for all idempotents $e \in R$, then $R[x;\sigma]/(x^{n+1})$ is morphic if and only if $R$ is unit-regular (Theorem 2.22).

In Section 3, we study the relation between strongly regular rings and centrally morphic rings. For instance, it will shown that a ring $R$ is strongly regular provided that $R[x;\sigma]/(x^{n+1})$ is left centrally morphic (Theorem 3.3), however the converse is not valid. A left centrally morphic ring $R[x;\sigma]/(x^{n+1})$ has been characterized provided that $R$ is a domain (Theorem 3.4). Moreover, we give an example showing that there exists a strongly regular ring $R$ and an isomorphism $\sigma$ over $R$ such that $\sigma(e) = e$ for all $e^2 = e \in R$, however, for every $n \geq 1$, $R[x;\sigma]/(x^{n+1})$ is not left centrally morphic (Example 3.8). As applications of our observations, some of the results in [10], have been generalized to the ring $R[x;\sigma]/(x^{n+1})$.

## 2. (Quasi-)morphic property of $R[x;\sigma]/(x^{n+1})$

Let $R$ be a ring. Throughout the paper, we assume that $\sigma : R \to R$ is a ring homomorphism with $\sigma(1) = 1$. The skew polynomial ring $T := R[x;\sigma]$ is defined to be the set of all left polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$ with coefficients $a_0, \cdots, a_n \in R$. Addition is defined as usual and multiplication is defined by $xr = \sigma(r)x$ where $r \in R$ and so by induction, $x^nr = \sigma^n(r)x^n$ where $n \geq 2$. It is clear that $x^nT \subseteq Tx^n$ $(n \geq 1)$. Therefore $Tx^n$ is a two sided ideal of $T$ which is denoted by $(x^n)$. While $x^nT$ is not necessarily an ideal, unless $\sigma$ is surjective.

Let $n \geq 1$ and $S := R[x;\sigma]/(x^{n+1})$. In whole of the paper, $\alpha = \sum\limits_{i=0}^{n} a_ix^i \in S$ means that $\alpha$ is modulo the ideal $(x^{n+1})$. In [10], authors studied the (quasi-)morphic property of $R[x]/(x^{n+1})$ $(n \geq 1)$. Here we investigate this property for the ring $R[x;\sigma]/(x^{n+1})$ $(n \geq 1)$. First we remind the following lemma which is proved in [12, Lemma 1] for latter uses.

**Lemma 2.1.** *The following are equivalent for an element $b$ in a ring $R$:*

(a) *$b$ is left morphic, that is $R/Rb \simeq \mathrm{l.ann}_R(b)$;*

(b) *There exists $a \in R$ such that $Rb = \mathrm{l.ann}_R(a)$ and $\mathrm{l.ann}_R(b) = Ra$;*

(c) *There exists $a \in R$ such that $Rb = \mathrm{l.ann}_R(a)$ and $\mathrm{l.ann}_R(b) \simeq Ra$.*

**Proof.** Follows from [12, Lemma 1]. □

We note that the notions of morphic and quasi-morphic for the ring $R[x;\sigma]/(x^{n+1})$ do not coincide. See the following example.

**Example 2.2.** Let $R$ be a regular ring which is not unit-regular (for example $R = \mathrm{End}_D(V)$ where $V$ is an infinite dimensional vector space over a division ring $D$). Since $R$ is regular, by [10, Theorem 4], $R[x]/(x^{n+1})$ $(n \geq 1)$ is a quasi-morphic ring. If $R[x]/(x^{n+1})$ is left morphic, then $R$ is left morphic (Corollary 2.10). This contradicts the fact that regular and left morphic rings are precisely unit-regular [6, Theorem 1].

**Theorem 2.3.** *Let $R$ be a ring, $\sigma : R \to R$ be a homomorphism and $n \geq 1$ be an integer. If $R[x;\sigma]/(x^{n+1})$ is left (right) quasi-morphic then so is $R$.*

**Proof.** Assume that $S := R[x;\sigma]/(x^{n+1})$ is left quasi-morphic and $a$ is a nonzero arbitrary element of $R$. Therefore there exists an element $\alpha = \sum\limits_{i=0}^{n} a_i x^i \in S$ such that $\mathrm{l.ann}_S(a) = S\alpha$. It is easy to see that $\mathrm{l.ann}_R(a) = Ra_0$. By our assumption on $S$, $Sax^n = \mathrm{l.ann}_S(\beta)$ where $\beta = \sum\limits_{i=0}^{n} b_i x^i \in S$. Thus $ax^n\beta = 0$ and so $\sum\limits_{i=0}^{n} a\sigma^n(b_i)x^{n+i} = 0$. Thus $a\sigma^n(b_0) = 0$ and so $Ra \subseteq \mathrm{l.ann}_R(\sigma^n(b_0))$. Let $r \in \mathrm{l.ann}_R(\sigma^n(b_0))$. Therefore $rx^n\beta = \sum\limits_{i=0}^{n} r\sigma^n(b_i)x^{n+i} = r\sigma^n(b_0)x^n = 0$. Thus $rx^n \in \mathrm{l.ann}_S(\beta) = Sax^n$ and so $rx^n = \gamma a x^n$ where $\gamma = \sum\limits_{i=0}^{n} c_i x^i \in S$. Hence $rx^n = c_0 a x^n$ and so $r = c_0 a \in Ra$. Therefore $\mathrm{l.ann}_R(\sigma^n(b_0)) \subseteq Ra$. Thus $Ra = \mathrm{l.ann}_R(\sigma^n(b_0))$. In right quasi-morphic case, the proof is similarly. □

**Proposition 2.4.** *Let $R$ be a ring and $\sigma$ be a ring homomorphism over $R$ such that $R[x;\sigma]/(x^{n+1})$ is left morphic $(n \geq 1)$. Then $R$ is left morphic provided that one of the following conditions hold:*

(a) *For every $a \in R$, $Ra \simeq R\sigma(a)$;*
(b) *For every $a \in R$, $\mathrm{l.ann}_R(a) \simeq \mathrm{l.ann}_R(\sigma(a))$.*

**Proof.** Let $S := R[x;\sigma]/(x^{n+1})$ be left morphic and $a \in R$ be a nonzero arbitrary element. Therefore there exists $\alpha = \sum_{i=0}^{n} b_i x^i \in S$ such that $\mathrm{l.ann}_S(ax^n) = S\alpha$ and $Sax^n = \mathrm{l.ann}_S(\alpha)$. It is routine to see that $\mathrm{l.ann}_R(a) = Rb_0$ and $Ra = \mathrm{l.ann}_R(\sigma^n(b_0))$. Considering the condition (a), we have $R\sigma^n(b_0) \simeq Rb_0$. Thus

$$R/Ra = R/\mathrm{l.ann}_R(\sigma^n(b_0)) \simeq R\sigma^n(b_0) \simeq Rb_0 = \mathrm{l.ann}_R(a).$$

Therefore $R$ is a left morphic ring. If the condition (b) holds then

$$Ra = \mathrm{l.ann}_R(\sigma^n(b_0)) \simeq \mathrm{l.ann}_R(b_0)$$

and so by Lemma 2.1, we are done. □

**Remark 2.5.** We remark that there are examples of nontrivial ring homomorphisms over a ring $R$ satisfying the conditions (a) and (b) of the above proposition. For instance, let $R$ be a unit-regular ring and $\sigma : R \to R$ be any ring homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$ (for example, see [11, Example 4]). It is well known that $R$ is unit-regular if and only if every its element can be written as a unit times an idempotent if and only if every its element can be written as an idempotent times a unit. Now, let $a$ be a nonzero arbitrary element in $R$. Therefore there exist idempotents $e, f \in R$ and units $u, v \in R$ such that $a = ue$ and $a = fv$. Since $u, v \in \mathrm{U}(R)$, clearly $\sigma(u), \sigma(v) \in \mathrm{U}(R)$. Thus we have

$$Ra = Rue = Re = R\sigma(e) = R\sigma(u)\sigma(e) = R\sigma(ue) = R\sigma(a),$$
$$\mathrm{l.ann}_R(a) = \mathrm{l.ann}_R(fv) = \mathrm{l.ann}_R(f) = \mathrm{l.ann}_R(\sigma(f)\sigma(v)) = \mathrm{l.ann}_R(\sigma(a)).$$

Recall that a ring $R$ is called *directly finite* if $ab = 1$ in $R$ implies $ba = 1$. Every left (right) morphic ring is directly finite [12, Proposition 6]. However the converse does not hold necessarily.

**Lemma 2.6.** *Let $R$ be a ring, $\sigma : R \to R$ be a ring homomorphism and $n \geq 1$. If $R[x;\sigma]/(x^{n+1})$ is left (right) morphic then $R$ is directly finite.*

**Proof.** It follows from [12, Proposition 6]. □

In the following, we give an example illustrating that the "morphic" condition in the above lemma is necessary and we cannot replace it with the slightly weaker condition "quasi-morphic". Before we give our example, we include a needed result from [10].

**Theorem 2.7.** [10, Corollary 16] *Let $R$ be a regular ring and $\sigma : R \to R$ be a homomorphism such that for all idempotents $e \in R$, $\sigma(e) = e$. Then $R[x;\sigma]/(x^2)$ is left quasi-morphic.*

**Example 2.8.** Let $R = S \times T$ where $S$ is a noncommutative strongly regular ring and $T$ is a regular ring which is not directly finite (for example $T = \mathrm{End}_D(V)$ where $V$ is an infinite dimensional vector space over a division ring $D$). Therefore $R$ is a regular ring which is not unit-regular. It is well known that a strongly regular ring $S$ is commutative if and only if every unit element of $S$ is central. Therefore there exists a unit element $v \in S$ which is not central. Let $u := (v, 1_T)$. Clearly $u \in \mathrm{U}(R)$ and it is not central. We consider the ring isomorphism $\sigma : R \to R$ defined by $\sigma(r) = uru^{-1}$. Thus $\sigma \neq 1$. It is routine to see that $\sigma(e) = e$ where $e^2 = e \in R$. Therefore by Theorem 2.7, $S := R[x;\sigma]/(x^2)$ is left quasi-morphic (we note that by Theorem 2.9, it is not left morphic) however $R$ is not directly finite.

**Theorem 2.9.** *Let $R$ be a ring and $\sigma : R \to R$ be a ring isomorphism. If $R[x;\sigma]/(x^{n+1})$ ($n \geq 1$) is left (right) morphic then $R$ is left (right) morphic and* $\mathrm{J}(R) = 0$.

**Proof.** Assume that $n \geq 1$ and $S := R[x;\sigma]/(x^{n+1})$ is left morphic. Let $a$ be a nonzero arbitrary element of $R$. Therefore there exists $\alpha = \sum_{i=0}^{n} a_i x^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = Sa$ and $\mathrm{l.ann}_S(a) = S\alpha$. It is easy to check that $Ra_0 = \mathrm{l.ann}_R(a)$. Now it is enough to prove that $Ra = \mathrm{l.ann}_R(a_0)$. Since $a\alpha = 0$, $Ra \subseteq \mathrm{l.ann}_R(a_0)$. Let $r \in \mathrm{l.ann}_R(a_0)$. Therefore $x^n r\alpha = \sigma^n(ra_0)x^n = 0$ and so $x^n r \in \mathrm{l.ann}_S(\alpha) = Sa$. Thus there exists $\beta = \sum_{i=0}^{n} b_i x^i \in S$ such that $x^n r = \beta a$ and it shows that $\sigma^n(r) = b_n\sigma^n(a)$. Since $\sigma$ is an isomorphism, $\sigma^n(s) = b_n$ for some $s \in R$. Therefore $\sigma^n(r) = \sigma^n(sa)$ and so $r = sa \in Ra$. Hence $R$ is left morphic. Now we show that $\mathrm{J}(R) = 0$. To see it, let $a \in \mathrm{J}(R)$. Since $R$ is left morphic, there exists $b \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $\mathrm{l.ann}_R(b) = Ra$. Moreover, $\mathrm{l.ann}_S(bx^n) = S\gamma$ where $\gamma = \sum_{i=0}^{n} t_i x^i \in S$. Clearly $x \in \mathrm{l.ann}_S(bx^n)$. Therefore $x = \beta\gamma$ for some $\beta = \sum_{i=0}^{n} r_i x^i \in S$. Thus $r_0t_0 = 0$ and $r_0t_1 + r_1\sigma(t_0) = 1$. It is easy to see that $Rt_0 = \mathrm{l.ann}_R(b)$. Therefore $Rt_0 = Ra$ and so $t_0 \in \mathrm{J}(R)$. Since $\sigma$ is an isomorphism, $\sigma(t_0) \in \mathrm{J}(R)$. Therefore $r_0t_1 = 1 - r_1\sigma(t_0) \in \mathrm{U}(R)$ and so $r_0 \in R$ is right invertible. By Lemma 2.6, $R$ is directly finite and so $r_0 \in \mathrm{U}(R)$. Thus $t_0 = 0$ and so $a = 0$.
In case $S$ is right morphic, the proof is similar. □

As an application of Theorems 2.3 and 2.9, we can deduce the following corollary which is proved in [10, Lemma 10].

**Corollary 2.10.** *Let $R$ be a ring and $n \geq 1$ be an integer. If $R[x]/(x^{n+1})$ is left (quasi-)morphic, then so is $R$.*

**Proof.** It follows from Theorems 2.3 and 2.9 by setting $\sigma = 1$. □

Next, we consider conditions under which the ring $R[x;\sigma]/(x^{n+1})$ is never left quasi-morphic.

**Proposition 2.11.** *Let $T$ be a directly finite ring, $R = T \times T$ and $\sigma : R \to R$ be a ring homomorphism. Under taking one of the following conditions, the ring $R[x;\sigma]/(x^{n+1})$ is not left quasi-morphic for every $n \geq 1$.*

(a) *For every $a = (t, 0) \in R$, $\sigma(a)a = 0$;*
(b) *For every $a = (0, t) \in R$, $\sigma(a)a = 0$.*

**Proof.** Let $n \geq 1$ and $S = R[x;\sigma]/(x^{n+1})$. Assume that the condition (a) holds. We show that $S$ is not left quasi-morphic. To see this, let $b = (0, 1) \in R$. On the contrary, suppose that $S$ is left quasi-morphic. Therefore there exists an element $\alpha := \sum_{i=0}^{n} a_i x^i \in S$ such that $\mathrm{l.ann}_S(bx^n) = S\alpha$. Thus $a_0 b = 0$ and so $a_0 = (t, 0)$ for some $t \in T$. Since $(1, 0) \in \mathrm{l.ann}_S(bx^n)$, $a_0 \neq 0$ and also $(1, 0) = \gamma\alpha$ where $\gamma = \sum_{i=0}^{n} c_i x^i \in S$. Therefore $c_0 a_0 = (1, 0)$ and so $t$ is left invertible in $T$. Since $T$ is directly finite, $t \in \mathrm{U}(T)$. On the other hand, $x \in \mathrm{l.ann}_S(bx^n)$ and so $x = (\sum_{i=0}^{n} d_i x^i)\alpha$ where each $d_i \in R$. Therefore $d_0 a_0 = 0$ and $d_0 a_1 + d_1 \sigma(a_0) = 1$. Since $t \in T$ is a unit element, $d_0 = (0, l)$ where $l \in T$. We note that by our assumption, $\sigma(a_0)a_0 = 0$. Therefore $a_0 = d_0 a_1 a_0 + d_1 \sigma(a_0) a_0 = (0, l) a_1 (t, 0) = 0$, a contradiction. Under the condition (b), just take $b = (1, 0)$ and the proof is analogous. □

**Corollary 2.12.** *Let $T$ be a ring, $R = T \times T$ and $\sigma : R \to R$ be a ring homomorphism. For every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is not left morphic provided that one of the following conditions hold:*

(a) *For every $a = (t, 0) \in R$, $\sigma(a)a = 0$;*
(b) *For every $a = (0, t) \in R$, $\sigma(a)a = 0$.*

**Proof.** We note that if $R[x; \sigma]/(x^{n+1})$ is left morphic for some $n \geq 1$, then by Lemma 2.6, $R$ is directly finite and so $T$ is directly finite. This contradicts Proposition 2.11. □

**Proposition 2.13.** *Let $T$ be a directly finite ring, $R = T \times T$ and $\sigma : R \to R$ defined by $\sigma(t_1, t_2) = (t_2, t_1)$. Then for every $n \geq 1$, $R[x; \sigma]/(x^{n+1})$ is neither left nor right quasi-morphic.*

**Proof.** In view of Proposition 2.11, $S := R[x; \sigma]/(x^{n+1})$ $(n \geq 1)$ is not left quasi-morphic. We show that it is also not right quasi-morphic. To see this, suppose that $S$ is right quasi-morphic. Therefore there exists $\alpha = \sum_{i=0}^{n} b_i x^i \in S$ such that $\mathrm{r.ann}_S(bx^n) = \alpha S$ where $b = (1, 0) \in R$. Let $n$ be an odd number. In this case, $\sigma^n(t_1, t_2) = (t_2, t_1)$ where $(t_1, t_2) \in R$. Therefore $b\sigma^n(b_0) = 0$. Let $b_0 = (t_0, t_1) \in R$. Thus $b(t_1, t_0) = 0$ and so $t_1 = 0$. It is also clear that $(1, 0) \in \mathrm{r.ann}_S(bx^n)$. This implies that there exists $c_0 \in T$ such that $t_0 c_0 = 1$. Since $T$ is directly finite, $t_0$ is unit in $T$. On the other hand, $x \in \mathrm{r.ann}_S(bx^n)$. Therefore there exists $s_0, s_1 \in R$ such that $b_0 s_0 = 0$ and $b_0 s_1 + b_1 \sigma(s_0) = 1$. Let $s_0 = (d_0, d_1) \in R$. Therefore $(t_0, 0)(d_0, d_1) = 0$ and so $t_0 d_0 = 0$. Since $t_0 \in \mathrm{U}(T)$, $d_0 = 0$. Hence $s_0 = b_0 s_1 s_0 + b_1 \sigma(s_0) s_0 = 0$. Therefore $b_0 s_1 = 1$, a contradiction.

In case $n$ is even, regarding that $\sigma^n$ is identity, the proof is similar to the case $n$ is odd. □

The next example illustrates that the converse of Theorems 2.3 and 2.9, does not hold necessarily even with the stronger condition "$\sigma$ is an isomorphism over $R$".

**Example 2.14.** Assume that $F$ is a field, $R = F \times F$ and $\sigma : R \to R$ is an isomorphism defined by $\sigma(a,b) = (b,a)$. We note that $R$ is a unit-regular ring and so it is morphic. While by Proposition 2.13, for every $n \geq 1$, $R[x;\sigma]/(x^{n+1})$ is not left (right) quasi-morphic.

In continue, we investigate the relation between "regularity of a ring $R$" and "morphic property of $R[x;\sigma]/(x^{n+1})$". Before we prove our results, we recall the following theorem from [11].

**Theorem 2.15.** [11, Corollary 3] *Let $R$ be a unit-regular ring with a homomorphism $\sigma : R \to R$ such that $\sigma(e) = e$ for all idempotents $e \in R$. Then $R[x;\sigma]/(x^{n+1})$ is left morphic for each $n \geq 1$.*

**Theorem 2.16.** *Let $R$ be a ring, $\sigma : R \to R$ be a homomorphism and $n \geq 1$. If $R[x;\sigma]/(x^{n+1})$ is morphic then $R$ is regular.*

**Proof.** Let $S := R[x;\sigma]/(x^{n+1})$ be morphic and $a \in R$ be any nonzero element. By Theorem 2.3, $R$ is quasi-morphic. Therefore there exists an element $b \in R$ such that $Ra = \mathrm{l.ann}_R(b)$. Let $\alpha := bx^n$. Since $S$ is left morphic, there exists $\beta = \sum_{i=0}^{n} b_i x^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = S\beta$ and $S\alpha = \mathrm{l.ann}_S(\beta)$. We note that $S$ is also right morphic. Thus there exists $\gamma \in S$ such that $\beta S = \mathrm{r.ann}_S(\gamma)$. Besides, $\mathrm{r.ann}_S(\alpha) = \mathrm{r.ann}_S(S\alpha) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\beta)) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\beta S) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\mathrm{r.ann}_S(\gamma))) = \mathrm{r.ann}_S(\gamma) = \beta S$. Clearly $x \in \mathrm{l.ann}_S(\alpha) = S\beta$ and $x \in \mathrm{r.ann}_S(\alpha) = \beta S$. Therefore there exist $\sum_{i=0}^{n} r_i x^i$ and $\sum_{i=0}^{n} s_i x^i$ in $S$ such that $x = (\sum_{i=0}^{n} r_i x^i)(\sum_{i=0}^{n} b_i x^i)$ and $x = (\sum_{i=0}^{n} b_i x^i)(\sum_{i=0}^{n} s_i x^i)$. Therefore $r_0 b_0 = 0$, $r_0 b_1 + r_1\sigma(b_0) = 1$, $b_0 s_0 = 0$ and $b_0 s_1 + b_1\sigma(s_0) = 1$. Now we have the following:

$$r_0 = r_0(b_0 s_1 + b_1\sigma(s_0)) = r_0 b_1 \sigma(s_0),$$
$$\sigma(s_0) = (r_0 b_1 + r_1\sigma(b_0))\sigma(s_0) = r_0 b_1 \sigma(s_0).$$

Thus $r_0 = \sigma(s_0)$ and so $b_0 = (b_0 s_1 + b_1\sigma(s_0))b_0 = b_0 s_1 b_0 + b_1 r_0 b_0 = b_0 s_1 b_0$. Therefore $b_0$ is regular. Since $\mathrm{l.ann}_S(\alpha) = S\beta$, it is routine to see that $Rb_0 = \mathrm{l.ann}_R(b) = Ra$. We show that $a$ is regular. To see it, let $e := s_1 b_0$. It is easy to see that $e^2 = e$ and $Rb_0 = Re$. Therefore $Ra = Re$ and so $a = ae = as_1 b_0$. Since $b_0 \in Ra$, $b_0 = ta$ where $t \in R$. Therefore $a = as_1 ta$ and so $a$ is regular, as desired. $\square$

We note that by Example 2.14, the converse of the above theorem does not hold true in general.

We conclude with the following direct applications of Theorem 2.16.

**Corollary 2.17.** *Let $R$ be a ring, $n \geq 1$ and $\sigma : R \to R$ be a homomorphism. If $R[x;\sigma]/(x^{n+1})$ is morphic, then the following statements hold:*

(a) *If $\sigma$ is an isomorphism then $R$ is unit-regular;*

(b) *If $R$ is commutative then it is unit-regular.*

**Proof.** Since $R[x;\sigma]/(x^{n+1})$ is morphic, by Theorem 2.16, $R$ is a regular ring.
(a) follows from Theorem 2.9 and the fact that every regular and left morphic ring is unit-regular.
(b) is clear. □

**Corollary 2.18.** *Let $R$ be a ring, $\sigma : R \to R$ be a homomorphism and $R[x;\sigma]/(x^{n+1})$ be a morphic ring where $n \geq 1$. The following statements are equivalent:*

(a) *$R$ is unit-regular;*

(b) *$R$ is morphic;*

(c) *$R$ is left morphic.*

**Proof.** (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are clear.
(c) $\Rightarrow$ (a) This follows from Theorem 2.16. □

**Corollary 2.19.** *Let $R$ be a domain, $n \geq 1$ and $\sigma$ be a ring homomorphism on $R$. The following statements are equivalent:*

(a) *$R[x;\sigma]/(x^{n+1})$ is left morphic;*

(b) *$R[x;\sigma]/(x^{n+1})$ is left quasi-morphic;*

(c) *$R$ is a division ring.*

**Proof.** (a) $\Rightarrow$ (b) This is clear.
(b) $\Rightarrow$ (c) This follows from Theorem 2.3 and the fact that left quasi-morphic domains are precisely division rings [2, Lemma 1].
(c) $\Rightarrow$ (a) This follows from Theorem 2.15. □

In the following, we characterize unit-regular rings via the morphic property of $R[x;\sigma]/(x^{n+1})$. Before we proceed, we include the following lemmas from [11] and [12].

Recall from [11] that elements $a$ and $b$ in a ring $R$ are said to be equivalent to each other whenever $b = uav$ for some units $u$ and $v$ in $R$.

**Lemma 2.20.** *If $c$ is a left (right) morphic element in a ring $R$, then for every $u \in \mathrm{U}(R)$, the same is true for $cu$ and $uc$.*

**Proof.** It follows from [12, Lemma 3]. □

**Lemma 2.21.** *Let $R$ be a ring, $\sigma : R \to R$ be a homomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$ and let $S = R[x;\sigma]/(x^{n+1})$ where $n \geq 0$. Then the following are equivalent:*

(a) *$R$ is a unit-regular ring;*

(b) *Each $\alpha \in S$ is equivalent to $e_0 + e_1x + \cdots + e_nx^n$, where the $e_i$s ($0 \leq i \leq n$) are orthogonal idempotents of $R$ and for each $1 \leq i \leq n$, $e_i \in (1-e_{i-1})\cdots(1-e_0)R(1-e_0)\cdots(1-e_{i-1})$.*

**Proof.** The result follows by [11, Thoerem 2]. □

Now we prove the next theorem as a generalization of Theorem 11 in [10].

**Theorem 2.22.** *Let $R$ be a ring, $\sigma : R \to R$ be an isomorphism such that $\sigma(e) = e$ for all idempotents $e \in R$ and $n \geq 1$. Then the ring $R[x;\sigma]/(x^{n+1})$ is morphic if and only if $R$ is unit-regular.*

**Proof.** ($\Rightarrow$) It follows from Corollary 2.17.
($\Leftarrow$) Let $R$ be unit-regular and $S := R[x;\sigma]/(x^{n+1})$ where $n \geq 1$. By Theorem 2.15, $S$ is left morphic. We prove that $S$ is also right morphic. Let $\alpha \in S$ be an arbitrary element. By Lemma 2.21, there exist orthogonal idempotents $e_0, e_1, \cdots, e_n$ in $R$ such that $\alpha$ is equivalent to $\gamma := e_0 + e_1x + \cdots + e_nx^n$ and for each $1 \leq i \leq n$, $e_i \in (1-e_{i-1})\cdots(1-e_0)R(1-e_0)\cdots(1-e_{i-1})$. Now by Lemma 2.20, it is enough to show that $\gamma$ is right morphic. To see this, let $b_i = (1-e_0)(1-e_1)\cdots(1-e_{n-i})$ for each $0 \leq i \leq n$. Since $e_ie_j = e_je_i = 0$, $(1-e_i)(1-e_j) = (1-e_j)(1-e_i)$ where $0 \leq i \neq j \leq n$. Therefore each $b_i$ is idempotent and $e_ib_j = b_je_i = 0$ where $0 \leq i \leq n-j$ and $0 \leq j \leq n$. Let $\beta := b_0 + b_1x + \cdots + b_nx^n$. We first prove that $\mathrm{r.ann}_S(\beta) = \gamma S$. It is easy to see that $\beta\gamma = 0$ and so $\gamma S \leq \mathrm{r.ann}_S(\beta)$. Now assume that $\delta = r_0 + r_1x + \cdots + r_nx^n \in \mathrm{r.ann}_S(\beta)$. Therefore $\beta\delta = 0$ and so $b_0r_t + b_1\sigma(r_{t-1}) + b_2\sigma^2(r_{t-2}) + \cdots + b_t\sigma^t(r_0) = 0$ where $0 \leq t \leq n$. Therefore $b_0r_0 = 0$ and so $r_0 = e_0r_0 + e_1r_0 + \cdots + e_nr_0$. We claim that $e_1r_0 = e_2r_0 = \cdots = e_nr_0 = 0$. To see this, note that for all $t = 1, \cdots, n$, we have:

$$e_{n-(t-1)}b_0r_t + e_{n-(t-1)}b_1\sigma(r_{t-1}) + e_{n-(t-1)}b_2\sigma^2(r_{t-2}) + \cdots + e_{n-(t-1)}b_t\sigma^t(r_0) = 0$$

Since $e_{n-(t-1)}b_i = 0$ ($0 \leq i \leq t-1$) and $e_{n-(t-1)}b_t = e_{n-(t-1)}$, it follows that $\sigma^t(e_{n-(t-1)}r_0) = \sigma^t(e_{n-(t-1)})\sigma^t(r_0) = e_{n-(t-1)}\sigma^t(r_0) = e_{n-(t-1)}b_t\sigma^t(r_0) = 0$. Since $\sigma$ is a monomorphism, $e_{n-(t-1)}r_0 = 0$ where $1 \leq t \leq n$. Thus $r_0 = e_0r_0$. The same reasoning applies that for each $1 \leq i \leq n$, $r_i = (e_0 + e_1 + \cdots + e_i)r_i$. Now we note that each $r_ix^i = (e_0 + e_1 + \cdots + e_i)r_ix^i = (e_0 + e_1 + \cdots + e_i)x^is_i$ where each

$s_i \in R$ and $\sigma^i(s_i) = r_i$. Therefore, these show that $\delta \in e_0R + (e_0 + e_1)xR + \cdots + (e_0 + e_1 + \cdots + e_n)x^nR$. Moreover, it is routine to see that for every $0 \leq i \leq n$,

$$(e_0 + e_1 + \cdots + e_i)x^i = e_0x^i + e_1x^i + \cdots + e_ix^i = \gamma e_0x^i + \gamma e_1x^{i-1} + \cdots + \gamma e_i = \gamma \sum_{j=0}^{i} e_jx^{i-j} \in \gamma S.$$

Thus we deduce that $e_0R + (e_0 + e_1)xR + \cdots + (e_0 + e_1 + \cdots + e_n)x^nR \leq \gamma S$ and so $\delta \in \gamma S$. Therefore $\mathrm{r.ann}_S(\beta) \leq \gamma S$. Similar argument applies that $\mathrm{r.ann}_S(\gamma) = b_0R + b_1xR + \cdots + b_nx^nR = \beta S$. The proof is now completed. □

The next two examples exhibit that in the above theorem, both conditions "$\sigma$ is an isomorphism on $R$" and "$\sigma(e) = e$ for all $e^2 = e \in R$" are not superfluous, respectively.

**Example 2.23.** The condition "$\sigma$ is an isomorphism over $R$" in Theorem 2.22, is needed. Let $F$ be a field and $\sigma : F \to F$ be a homomorphism such that $\sigma(F) \neq F$. Clearly the only idempotents in $F$ are 0 and 1. Therefore the condition $\sigma(e) = e$ does hold where $e^2 = e \in F$. Then by Theorem 2.15, $F[x;\sigma]/(x^2)$ is left morphic. However by [12, Example 8], it is not right morphic.

**Example 2.24.** By Proposition 2.13, there exists a unit-regular ring $R$ and an isomorphism $\sigma : R \to R$ such that $\sigma(e) \neq e$ for some $e^2 = e \in R$ while for each $n \geq 1$, $R[x;\sigma]/(x^{n+1})$ is not even left (right) quasi-morphic (for instance, $R$ and $\sigma$ are what mentioned in Example 2.14).

Now as an application of Theorem 2.22, we can deduce the following result which is also proved by Lee and Zhou in [10, Theorem 11].

**Corollary 2.25.** *Let $n \geq 1$ be an integer. Then a ring $R$ is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic.*

**Proof.** It follows from Theorem 2.22 by taking $\sigma = 1$. □

## 3. Centrally morphic property of the ring $R[x;\sigma]/(x^{n+1})$

Recall from [10, Section 5] that a ring $R$ is called *left centrally morphic* if for each $a \in R$, there exists $b \in \mathrm{C}(R)$ such that $Ra = \mathrm{l.ann}_R(b)$ and $Rb = \mathrm{l.ann}_R(a)$. Right centrally morphic rings are defined analogously. A left and right centrally morphic ring is called *centrally morphic*. Clearly every left centrally morphic ring is left morphic. Strongly regular rings and commutative morphic rings are centrally morphic however there exists a centrally morphic ring $R$ that is neither strongly regular nor commutative (for more details, see [10, Lemma 21 and Example 22]). We begin with the following proposition.

**Proposition 3.1.** *Let $R$ be a ring. The following statements are equivalent:*

(a) *$R$ is left centrally morphic;*

(b) *For every $a \in R$, there exist elements $b, c \in \mathrm{C}(R)$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$.*

**Proof.** (a)$\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (a) Let $a$ be any arbitrary element of $R$. By our assumption, $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$ where $b, c \in \mathrm{C}(R)$. It is enough to show that $\mathrm{l.ann}_R(b) = Ra$. Since $b \in \mathrm{C}(R)$ and $ba = 0$, $Ra \subseteq \mathrm{l.ann}_R(b)$. Besides we note that $\mathrm{l.ann}_R(b) = \mathrm{r.ann}_R(Rb) = \mathrm{r.ann}_R(\mathrm{l.ann}_R(a)) = \mathrm{r.ann}_R(\mathrm{l.ann}_R(aR))$. Since $Ra$ is a two sided ideal of $R$, $aR \subseteq Ra$ and so $\mathrm{r.ann}_R(\mathrm{l.ann}_R(aR)) \subseteq \mathrm{r.ann}_R(\mathrm{l.ann}_R(Ra))$. Therefore $\mathrm{l.ann}_R(b) \subseteq \mathrm{r.ann}_R(\mathrm{l.ann}_R(Ra))$. Moreover,

$$\mathrm{r.ann}_R(\mathrm{l.ann}_R(Ra)) = \mathrm{r.ann}_R(\mathrm{l.ann}_R(\mathrm{r.ann}_R(c))) = Ra.$$

Thus $\mathrm{l.ann}_R(b) \subseteq Ra$, as desired. $\square$

Next, we investigate the relation between "strongly regular property of a ring $R$" and "centrally morphic property of $R[x;\sigma]/(x^{n+1})$". This is motivated by Theroem 2.22 and also the following result from [10, Theorem 20].

**Theorem 3.2.** [10, Theorem 20] *Let $n \geq 1$ be an integer. Then $R$ is strongly regular if and only if $R[x]/(x^{n+1})$ is a (left) centrally morphic ring.*

**Theorem 3.3.** *Let $R$ be a ring, $n \geq 1$ and $\sigma$ be a ring homomorphism on $R$. If $R[x;\sigma]/(x^{n+1})$ is left centrally morphic then $R$ is strongly regular.*

**Proof.** Let $S := R[x;\sigma]/(x^{n+1})$ be left centrally morphic and $a$ be a nonzero arbitrary element of $R$. By Theorem 2.3, $R$ is left quasi-morphic. Therefore $Ra = \mathrm{l.ann}_R(b)$ for some $b \in R$. By our assumption on $S$, there exists $\beta = \sum_{i=0}^{n} b_i x^i \in \mathrm{C}(S)$ such that $\mathrm{l.ann}_S(bx^n) = S\beta$. Since $x \in \mathrm{l.ann}_S(bx^n) = S\beta$, there exists $\gamma = \sum_{i=0}^{n} r_i x^i \in S$ such that $x = \gamma\beta$. Therefore

$$r_0 b_0 = 0 \text{ and } r_0 b_1 + r_1 \sigma(b_0) = 1.$$

We note that $\beta \in \mathrm{C}(S)$ implies that $b_i \sigma^i(r) = r b_i$ where $0 \leq i \leq n$ and $r \in R$. In particular, $b_0 \in \mathrm{C}(R)$. Moreover, $\beta x = x\beta$ and so $\sigma(b_0) = b_0$. Therefore we have

$$b_0 = b_0(r_0 b_1 + r_1 \sigma(b_0)) = r_0 b_0 b_1 + b_0 r_1 b_0 = b_0^2 r_1.$$

Therefore $b_0 \in Rb_0^2$ and so $Rb_0 = Rb_0^2$. Since $\mathrm{l.ann}_S(bx^n) = S\beta$, $Rb_0 = \mathrm{l.ann}_R(b)$ and so $Ra = Rb_0$. Therefore $Ra^2 = Rb_0^2 = Rb_0 = Ra$ and so $a$ is strongly regular. $\square$

The converse of Theorem 3.3, does not necessarily hold even in case $\sigma$ is an isomorphism on $R$. In view of Example 2.14, we showed that there exists a strongly regular ring $R$ and an isomorphism $\sigma : R \to R$ such that for each $n \geq 1$, $R[x;\sigma]/(x^{n+1})$ is not left (right) quasi-morphic. We note that in this example, there exists an idempotent $e \in R$ such that $\sigma(e) \neq e$. This motivates us to ask *"if $R$ is a strongly regular ring and $\sigma : R \to R$ is an isomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$, whether $R[x;\sigma]/(x^{n+1})$ is left centrally morphic"*. Example 3.8 is a negative answer to this question. Before we proceed, the following stated results are needed.

In the next theorem, we present necessary and sufficient conditions over a domain $R$ and a homomorphism $\sigma : R \to R$ under which $R[x;\sigma]/(x^{n+1})$ is centrally morphic.

**Theorem 3.4.** *Let $R$ be a domain, $\sigma : R \to R$ be a ring homomorphism. Then $R[x;\sigma]/(x^{n+1})$ ($n \geq 1$) is (left) centrally morphic if and only if $R$ is a division ring and there exists a nonzero element $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$.*

**Proof.** Let $R$ be a domain and $S := R[x;\sigma]/(x^{n+1})$ ($n \geq 1$). We first prove the following two claims:

Claim (1). *Let $\alpha = \sum_{i=0}^{n} a_i x^i \in S$. Then $\alpha \in \mathrm{U}(S)$ if and only if $a_0 \in \mathrm{U}(R)$.*

Proof of Claim (1): Let $a_0 \in \mathrm{U}(R)$. Thus for each $i$, $\sigma^i(a_0)$ is also unit in $R$. Assume that $b_0 := a_0^{-1}$, $b_1 := -b_0 a_1 \sigma(a_0^{-1})$ and for every $2 \leq t \leq n$, $b_t := -(a_0^{-1}a_t + b_1\sigma(a_{t-1}) + b_2\sigma^2(a_{t-2}) + \cdots + b_{t-1}\sigma^{t-1}(a_1))\sigma^t(a_0^{-1})$. It is routine to see that $\beta\alpha = \alpha\beta = 1$ where $\beta = \sum_{i=0}^{n} b_i x^i \in S$ and so $\alpha \in \mathrm{U}(S)$. The converse is clear.

Claim (2). *Let $\sigma$ be a monomorphism on $R$. If $\alpha = a_t x^t + a_{t+1}x^{t+1} + \cdots + a_n x^n \in S$ where $1 \leq t \leq n$ and $a_t \neq 0$, then $\mathrm{l.ann}_S(\alpha) = Rx^{n-t+1} + Rx^{n-t+2} + \cdots + Rx^n$.*

Proof of Claim (2): It is clear that $Rx^{n-t+1} + Rx^{n-t+2} + \cdots + Rx^n \subseteq \mathrm{l.ann}_S(\alpha)$. Let $\theta = \sum_{i=0}^{n} s_i x^i \in \mathrm{l.ann}_S(\alpha)$. Therefore $\theta\alpha = 0$ implies that

$$s_0 a_t = 0;$$

$$s_0 a_{t+1} + s_1\sigma(a_t) = 0;$$

$$\vdots$$

$$s_0 a_n + s_1\sigma(a_{n-1}) + \cdots + s_{n-t}\sigma^{n-t}(a_t) = 0.$$

Since $R$ is a domain and $a_t \neq 0$, $s_0 = 0$. Therefore by the above equations and monomorphism condition on $\sigma$, we conclude that $s_1 = s_2 = \ldots = s_{n-t} = 0$. Thus $\theta = s_{n-t+1}x^{n-t+1} + \ldots + s_n x^n \in Rx^{n-t+1} + Rx^{n-t+2} + \cdots + Rx^n$, as desired.

We shall now prove the theorem. Assume that $n \geq 1$ and $S := R[x;\sigma]/(x^{n+1})$ is left centrally morphic. By Theorem 3.3, $R$ is strongly regular. Therefore $R$ is a division ring. By our assumption, there exists $\xi \in \mathrm{C}(S)$ such that $\mathrm{l.ann}_S(x^n) = S\xi$. We note that by Claim (2), $\mathrm{l.ann}_S(x^n) = Rx + Rx^2 + \cdots + Rx^n$. Therefore there exist elements $u_1, u_2, \cdots, u_n$ in $R$ such that $\xi = u_1 x + u_2 x^2 + \cdots + u_n x^n$. Since $x \in \mathrm{l.ann}_S(x^n)$, $u_1 \neq 0$. Besides, for every $r \in R$, $\xi r = r\xi$. Therefore, for all $r \in R$, $u_1\sigma(r) = ru_1$ and so $\sigma(r) = u_1^{-1} r u_1$.

Conversely, suppose that $R$ is a division ring and there exists an element $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$. We prove that $S$ is left centrally morphic. Let $\alpha = \sum_{i=0}^{n} a_i x^i \in S$ be a nonzero arbitrary element. In case $a_0 \neq 0$, by Claim (1), $\alpha$ is unit and then we are done. So suppose that $a_0 = 0$. We may also assume that $\alpha = a_t x^t + a_{t+1}x^{t+1} + \cdots + a_n x^n$ such that $1 \leq t \leq n$ and $a_t \neq 0$. Let $\beta = b_{n-t+1}x^{n-t+1} + \ldots + b_n x^n$ where each $b_i = u^i$. By Claim (2), $\mathrm{l.ann}_S(\alpha) = Rx^{n-t+1} + \ldots + Rx^n$ and $\mathrm{l.ann}_S(\beta) = Rx^t + Rx^{t+1} + \ldots + Rx^n$. Moreover, $\alpha = \alpha_1 x^t$ and $\beta = \beta_1 x^{n-t+1}$ where $\alpha_1, \beta_1 \in \mathrm{U}(S)$. Therefore $S\alpha = Sx^t$ and $S\beta = Sx^{n-t+1}$. Now it is easy to see that $S\alpha = \mathrm{l.ann}_S(\beta)$ and $S\beta = \mathrm{l.ann}_S(\alpha)$. Therefore it is enough to prove that $\beta \in \mathrm{C}(S)$. Since $\sigma(b_i) = b_i$ for each $n-t+1 \leq i \leq n$, $\beta x = x\beta$. Besides, for each $i$ and $r \in R$, $b_i x^i r = b_i \sigma^i(r) x^i = u^i u^{-i} r u^i x^i = r u^i x^i = r b_i x^i$. This implies that $\beta \in \mathrm{C}(S)$. A similar argument can be adopted to show that $S$ is also right centrally morphic. □

**Corollary 3.5.** *Let $R$ be a commutative domain and $\sigma : R \to R$ be a homomorphism. The following statements are equivalent:*

(a) *$R[x;\sigma]/(x^{n+1})$ is (left) centrally morphic where $n \geq 1$;*

(b) *$R$ is a filed and $\sigma$ is identity.*

**Proof.** (a) $\Rightarrow$ (b) It follows from Theorem 3.4.
(b) $\Rightarrow$ (a) Applying Theorem 3.2. □

**Corollary 3.6.** *Let $D$ be a division ring and $\sigma : D \to D$ be a ring homomorphism. Then $D[x;\sigma]/(x^{n+1})$ is (left) centrally morphic if and only if there exists a nonzero element $b \in D$ such that $\sigma(d) = b^{-1}db$ for all $d \in D$.*

**Proof.** It is an application of Theorem 3.4. □

**Corollary 3.7.** *Let $F$ be a field, $\sigma : F \to F$ be any nonzero ring homomorphism and $n \geq 1$. The following statements hold:*

(a) $F[x;\sigma]/(x^{n+1})$ *is left morphic.*

(b) $F[x;\sigma]/(x^{n+1})$ *is (left) centrally morphic if and only if* $\sigma$ *is identity.*

**Proof.** (a) and (b) follows from Corollaries 2.19 and 3.6, respectively. □

We end the paper with the following example to answer the question stated in the argument after Theorem 3.3.

**Example 3.8.** Let $\mathbb{C}$ be the field of complex numbers and $\sigma : \mathbb{C} \to \mathbb{C}$ be the isomorphism given by complex conjugation. Therefore in the ring $\mathbb{C}[x;\sigma]$, we have $xz = \bar{z}x$ for all $z \in \mathbb{C}$. We also note that the only nonzero idempotent elements in $\mathbb{C}$ is 1 and $\sigma(1) = 1$. Since $\sigma \neq 1$, by Corollary 3.5, the ring $\mathbb{C}[x;\sigma]/(x^{n+1})$ $(n \geq 1)$ is not left centrally morphic.

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## References

[1] L. V. An, T. G. Nam and N. S. Tung, *On quasi-morphic rings and related problems*, Southeast Asian Bull. Math., 40(1) (2016), 23-34.

[2] V. Camillo and W. K. Nicholson, *Quasi-morphic rings*, J. Algebra Appl., 6(5) (2007), 789-799.

[3] V. Camillo, W. K. Nicholson and Z. Wang, *Left quasi-morphic rings*, J. Algebra Appl., 7(6) (2008), 725-733.

[4] H. Chen, *Regular elements in quasi-morphic rings*, Comm. Algebra, 39 (2011), 1356-1364.

[5] A. J. Diesl, T. J. Dorsey and W. W. McGovern, *A characterization of certain morphic trivial extensions*, J. Algebra Appl., 10(4) (2011), 623-642.

[6] G. Ehrlich, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc., 216 (1976), 81-90.

[7] K. R. Goodearl, Von-Neumann Regular Rings, Pitman, Boston, 1979.

[8] Q. Huang and J. Chen, *$\pi$-morphic rings*, Kyungpook Math. J., 47(3) (2007), 363-372.

[9] T. K. Lee and Y. Zhou, *Morphic rings and unit regular rings*, J. Pure Appl. Algebra, 210 (2007), 501-510.

[10] T. K. Lee and Y. Zhou, *Regularity and morphic property of rings*, J. Algebra, 322 (2009), 1072-1085.

[11] T. K. Lee and Y. Zhou, *A theorem on unit regular rings*, Canad. Math. Bull., 53(2) (2010), 321-326.

[12] W. K. Nicholson and E. S. Campos, *Rings with the dual of the isomorphism theorem*, J. Algebra, 271 (2004), 391-406.

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