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$\mathfrak{X}$-ELEMENTS IN MULTIPLICATIVE LATTICES - A GENERALIZATION OF J-IDEALS, n-IDEALS AND r-IDEALS IN RINGS

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Abstract. In this paper, we introduce the concept of an $\mathfrak{X}$-element with respect to an M-closed set $\mathfrak{X}$ in multiplicative lattices and study properties of $\mathfrak{X}$-elements. For a particular M-closed subset $\mathfrak{X}$, we define the concepts of r-elements, n-elements and J-elements. These elements generalize the notion of r-ideals, n-ideals and J-ideals of a commutative ring with identity to multiplicative lattices. In fact, we prove that an ideal I of a commutative ring R with identity is a n-ideal (J-ideal) of R if and only if it is an n-element (J-element) of $Id(R)$, the ideal lattice of R.

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1. Introduction

The ideal theory of commutative rings with identity is very rich. Many researchers have defined different ideals ranging from prime ideals, maximal ideals, primary ideals to recently introduced r-ideals, n-ideals and J-ideals. More details about r-ideals, n-ideals, and J-ideals can be found in Mohamadian [14], Tekir et al. [15] and Khashan and Bani-Ata [12] respectively.

Ward and Dilworth [16] introduced the concept of multiplicative lattices to generalize the ideal theory of commutative rings with identity. Analogously, the concepts of a prime element, maximal element, primary elements are defined.

The study of prime elements and their generalization in a multiplicative lattice is the main focus of many researchers. Different classes of elements and generalizations of a prime element in multiplicative lattices were studied; see Burton [3], Joshi and Ballal [7], Jayaram [8], Jayaram and Johnson [9,10], Jayaram et al. [11], Manjarekar and Bingi [13].

We have observed a unifying pattern in the results of J-ideals, n-ideals and r-ideals of rings. This motivates us to introduce a new class of elements, namely an

$\mathfrak{X}$-element in multiplicative lattices. Hence this study will unify many of the results proved for these ideals and generalize them to multiplicative lattice settings. In fact, by replacing an M-closed set $\mathfrak{X}$ by $Z(L)$, the set of zero-divisors of L, or $J(L)$, the set of Jacobson radical of L, we get the notion of r-element, J-element, etc. Hence this justifies the name $\mathfrak{X}$-element. These elements are the generalizations of r-ideals, n-ideals and J-ideals of a commutative ring with identity. In fact, we prove that an ideal I of a commutative ring R with identity is an n-ideal(a J-ideal) of R if and only if it is an n-element (a J-element) of $Id(R)$, the ideal lattice of R.

Now, we begin with the necessary concepts and terminology given in [1,2,4,6,16].

Definition 1.1 (See [1,2,4,6,16])**.** A nonempty subset I of a lattice L is a *semi-ideal* if $x \le a \in I$ implies $x \in I$. A semi-ideal I of L is an *ideal* if $a \vee b \in I$ whenever $a, b \in I$. An ideal (semi-ideal) I of a lattice L is a *proper* ideal (semi-ideal) of L if $I \ne L$. A proper ideal (semi-ideal) I is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$, and it is *minimal* if it does not properly contain another prime ideal (prime semi-ideal). For $a \in L$, let $(a] = \{x \in L \,|\, x \le a\}$. The set $(a]$ is the *principal ideal generated by* a. An element x of a lattice L is called *meet irreducible* if $x = y \wedge z$ implies $x = y$ or $x = z$.

A lattice L is *complete* if for any subset S of L, we have $\bigvee S, \bigwedge S \in L$. The smallest element and the greatest element of a lattice L is denoted by 0 and 1 respectively.

The concept of multiplicative lattices was introduced by Ward and Dilworth [16] to study the abstract commutative ideal theory of commutative rings.

A complete lattice L is a *multiplicative lattice* if there exists a binary operation "$\cdot$" called the *multiplication* on L satisfying the following conditions:

(1) $a \cdot b = b \cdot a$ for all $a, b \in L$.

(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in L$.

(3) $a \cdot (\bigvee_\alpha b_\alpha) = \bigvee_\alpha (a \cdot b_\alpha)$ for all $a, b_\alpha \in L$, $\alpha \in \Lambda$(an index set).

(4) $a \cdot 1 = a$ for all $a \in L$.

Note that in a multiplicative lattice L, $a \cdot b \le a \wedge b$ for $a, b \in L$. For this, let $a = a \cdot 1 = a \cdot (b \vee 1) = a \cdot b \vee a$. Thus $a \cdot b \le a$. Similarly, $a \cdot b \le b$. This proves that $a \cdot b \le a \wedge b$. Moreover, if $a \le b$ in L, then $a \cdot c \le b \cdot c$ for every $c \in L$. Also, if $a \le b$ and $c \le d$ then $a \cdot c \le b \cdot d$.

An element c of a complete lattice L is *compact* if $c \le \bigvee_\alpha a_\alpha$, $\alpha \in \Lambda$ (Λ is an index set) implies $c \le \bigvee_{i=1}^{n} a_{\alpha_i}$, where $n \in \mathbb{Z}^+$. The set of all compact elements of a lattice L is denoted by L_*.

A lattice L is *compactly generated* or *algebraic* if for every $x \in L$, there exist $x_\alpha \in L_*$ for $\alpha \in \Lambda$(an index set) such that $x = \bigvee_\alpha x_\alpha$, that is, every element is a join of compact elements. Equivalently, if L is a compactly generated lattice and if $a \nleq b$ for $a, b \in L$, then there exists a nonzero compact element $c \in L_*$ such that $c \leq a$ and $c \nleq b$.

A multiplicative lattice L is 1-*compact* if 1 is a compact element of L. A multiplicative lattice L is *compact* if every element of L is a compact element.

A multiplicative lattice L is a *c-lattice* if L is 1-compact, compactly generated multiplicative lattice in which the product of two compact elements is compact. Note that the ideal lattice of a commutative ring R with identity is always a c-lattice.

An element p of a multiplicative lattice L with $p \neq 1$ is *prime* if $a \cdot b \leq p$ implies $a \leq p$ or $b \leq p$. It is not difficult to prove that an element p (with $p \neq 1$) of a c-lattice L is *prime* if $a \cdot b \leq p$ for $a, b \in L_*$ implies $a \leq p$ or $b \leq p$. An element p is said to be a *minimal prime element* if there is no prime element q such that $q < p$. An ideal P of a commutative ring R with identity is prime if and only if it is a prime element of $Id(R)$, the ideal lattice of R.

Let L be a c-lattice and $a \in L$. Then the *radical* of a is denoted by $\sqrt{a}$ and given by $\sqrt{a} = \bigvee\{x \in L_* \mid x^n \leq a \text{ for some } n \in \mathbb{N}\}$. Note that if any compact element $c \leq \sqrt{a}$, then $c^m \leq a$ for some $m \in \mathbb{N}$. An element a of a c-lattice is a *radical element*, if $a = \sqrt{a}$. A c-lattice is called a *domain* if 0 is a prime element of L.

A proper element i of a c-lattice L is called *primary element* if whenever $a \cdot b \leq i$ for some $a, b \in L$ then either $a \leq i$ or $b \leq \sqrt{i}$.

A proper element m of a multiplicative lattice is said to be *maximal*, if $m \leq n < 1$, then $m = n$. The set of all maximal elements of L is denoted by $\mathrm{Max}(L)$. The *Jacobson radical* of L is the set $J(L) = \bigwedge\{m \mid m \in \mathrm{Max}(L)\}$. It is easy to observe that a maximal element of a c-lattice L is prime. A c-lattice L is said to be *local*, if L has the unique maximal element m. In this case, we write $(L; m)$.

A non-empty subset $\mathfrak{X}$ of L_*(set of all compact elements) in a multiplicative lattice L is *multiplicatively closed* if $s_1 \cdot s_2 \in \mathfrak{X}$, whenever $s_1, s_2 \in \mathfrak{X}$. Thus a multiplicatively closed subset A of L is a subset of L_*.

A non-empty subset $\mathfrak{X}$ of a multiplicative lattice L is called *M-closed* if $a, b \in \mathfrak{X}$, then $a \cdot b \in \mathfrak{X}$. Obviously, every ideal, in particular the principal ideal $(i]$ generated by i, is an M-closed subset of L.

From the definitions, it is clear that every multiplicatively closed subset A of a multiplicative lattice L is a M-closed subset of L. The converse is not true. For example, one can see that $[0, 1]$ of reals is a multiplicative lattice with respect to the usual multiplication. However, $\{1\}$ is a M-closed subset but not a multiplicatively closed set, as 1 is not a compact element. This can be seen as $1 = \bigvee F$ where $F = \{x \mid x \in (0,1)\}$. But there is no finite subset F_1 of F such that $1 = \bigvee F_1$. Further, if L is a compact lattice or finite, then $L = L_*$ and hence both definitions coincide with each other. Clearly, if p is a prime element of a c-lattice L, then $L \setminus (p]$ is an M-closed subset of L.

In a multiplicative lattice L, an element $a \in L$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$, and L is *reduced* if the only nilpotent element is 0. The set of all nilpotent elements of a multiplicative lattice L is denoted by $\mathrm{Nil}(L)$. We denote the set $Z(L)$ of zero-divisors in L by the set $Z(L) = \{x \in L \mid x \cdot y = 0 \text{ for some } y \in L \setminus \{0\}\}$. Clearly, $\mathrm{Nil}(L) \subseteq Z(L)$. Let L be a multiplicative lattice and $a, b \in L$. Then $(a : b) = \bigvee\{x \mid x \cdot b \leq a\}$. Note that $x \cdot b \leq a \Leftrightarrow x \leq (a : b)$. Clearly, $a \leq (a : b)$ and $(a : b) \cdot b \leq a$ for $a, b \in L$. If $a \in L$, then $ann_L(a) = \bigvee\{x \in L \mid a \cdot x = 0\}$.

For undefined concepts in lattices, see Grätzer [6].

2. $\mathfrak{X}$-elements in multiplicative lattices

We introduce the concept of an $\mathfrak{X}$-element in multiplicative lattices.

Definition 2.1. Let L be a multiplicative lattice and $\mathfrak{X}$ be an M-closed subset of L. A proper element i of L is called an $\mathfrak{X}$-*element*, if $a \cdot b \leq i$ with $a \notin \mathfrak{X}$ implies $b \leq i$ for all $a, b \in L$.

Example 2.2. Consider a lattice K whose Hasse diagram is shown in Figure 1. Clearly, in K, $a \vee c = b \vee c = d$ and $a \wedge c = b \wedge c = 0$. On K, define the trivial multiplication $x \cdot y = 0 = y \cdot x$ for every $x, y \notin \{1\}$ and $x \cdot 1 = x = 1 \cdot x$ for every $x \in K$. It is easy to see that K is a multiplicative lattice. Moreover, K is non-reduced, as $a^2 = b^2 = c^2 = d^2 = 0$. If $\mathfrak{X} = \{0, a, b, c, d\}$, then every proper element of K is an $\mathfrak{X}$-element of K.

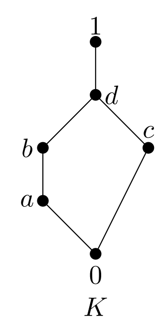

FIGURE 1. A multiplicative lattice in which every proper element is an $\mathfrak{X}$-element

Remark 2.3. Note that a proper element of a multiplicative lattice L is an $\mathfrak{X}$-element or not, depends on an M-closed subset $\mathfrak{X}$ under consideration. If x is an

$\mathfrak{X}_1$-element with respect to an M-closed subset $\mathfrak{X}_1$, then x may or may not be an $\mathfrak{X}_2$-element with respect to an M-closed subset $\mathfrak{X}_2$ different from $\mathfrak{X}_1$.

Also, note that if L is a multiplicative lattice and $\mathfrak{X} = \{1\}$ is an M-closed subset of L, then L does not contain an $\mathfrak{X}$-element.

Lemma 2.4. *Let L be a multiplicative lattice and $\mathfrak{X}$ be an M-closed subset of L. If i is an $\mathfrak{X}$-element of L, then $(i] \subseteq \mathfrak{X}$. In particular, if $(i] = \mathfrak{X}$, then i is an $\mathfrak{X}$-element of L if and only if i is a prime element of L.*

Proof. Suppose i is an $\mathfrak{X}$-element of a multiplicative lattice L and let $x \in (i]$. Suppose on the contrary that $x \notin \mathfrak{X}$. Clearly, $x \cdot 1 \leq i$ with $x \notin \mathfrak{X}$. Since i is an $\mathfrak{X}$-element, we get $1 \leq i$, a contradiction to the fact that i is a proper element of L. Therefore $x \in \mathfrak{X}$ and hence $(i] \subseteq \mathfrak{X}$.

Now, we prove the "in particular" part. Suppose that $(i] = \mathfrak{X}$ and i is an $\mathfrak{X}$-element of L. Let $a, b \in L$ such that $a \cdot b \leq i$ and $a \nleq i$, i.e., $a \notin \mathfrak{X}$. As i is an $\mathfrak{X}$-element, $b \leq i$. So i is a prime element of L.

Conversely, suppose that $(i] = \mathfrak{X}$ and i is a prime element of L. Let $a, b \in L$ such that $a \cdot b \leq i$ with $a \notin \mathfrak{X} = (i]$. By primeness of i, $b \leq i$. Thus i is an $\mathfrak{X}$-element of L. □

Remark 2.5. The converse of the Lemma 2.4 need not be true in general, i.e., if i is a proper element of a multiplicative lattice L such that $i \in \mathfrak{X}$, then i need not be an $\mathfrak{X}$-element of L. Consider the ideal lattice L of the ring $\mathbb{Z}_{12}$. Clearly, L is a non-reduced lattice. Put $\mathfrak{X} = \{(0), (6)\}$. Then $(0) \in \mathfrak{X}$, however (0) is not an $\mathfrak{X}$-element of L, because $(3) \cdot (4) \leq (0)$ and $(3) \notin \mathfrak{X}$ does not imply that $(4) \leq (0)$.

Also, $(6) \in \mathfrak{X}$, but (6) is not an $\mathfrak{X}$-element of L, because $(2) \cdot (3) \leq (6)$ and $(3) \notin \mathfrak{X}$ does not imply that $(2) \leq (6)$.

Lemma 2.6. *Let L be a multiplicative lattice and $\mathfrak{X}$ and $\mathfrak{X}'$ be M-closed subsets of L such that $\mathfrak{X} \subseteq \mathfrak{X}'$. If i is an $\mathfrak{X}$-element of L, then i is an $\mathfrak{X}'$-element of L.*

Proof. It follows from the definition of an $\mathfrak{X}$-element. □

Lemma 2.7. *Let $(L; m)$ be a local lattice. Then every proper element of L is an $\mathfrak{X}$-element for $\mathfrak{X} = (m]$.*

Proof. Let $a \cdot b \leq i$ and $a \notin \mathfrak{X} = (m]$. Since L is local, $a = 1$. Hence in this case $b \leq i$. Thus i is an $\mathfrak{X}$-element of L. □

Lemma 2.8. *Assume that every proper element of a c-lattice L is an $\mathfrak{X}$-element, where $\mathfrak{X} = (m]$ and $m \in L \setminus \{1\}$. Then m is the unique maximal element of L.*

Proof. Let i be a proper element of L which is also an $\mathfrak{X}$-element. Then by Lemma 2.4, $i \leq m$. This is true for all proper elements i of L. In particular, it is true for all maximal elements m' too. This proves that L has the unique maximal element m. Clearly, m is a meet-irreducible element. For this, suppose that $m = y \wedge z$ for some $y,\ z \in L$. Hence $m \leq y$ and $m \leq z$. As m is the unique maximal element of L and $m = y \wedge z$, we get either $y \leq m$ or $z \leq m$. Thus $m = y$ or $m = z$. Therefore m is a meet-irreducible element of L. □

Lemma 2.9. *Let L be a multiplicative lattice and $\mathfrak{X}$ be an M-closed subset of L. If $\{i_j\}$, where $j \in \Lambda$ (an index set), is a non-empty set of $\mathfrak{X}$-elements of L, then $\bigwedge_j i_j$ is also an $\mathfrak{X}$-element.*

Proof. Obvious. □

Remark 2.10. The join of two $\mathfrak{X}$-elements is not necessarily an $\mathfrak{X}$-element. Consider the ideal lattice L of $\mathbb{Z}_{15}$ with $\mathfrak{X} = \{(0), (3), (5)\}$. Then $(3), (5)$ are $\mathfrak{X}$-elements of L, but $(3) \vee (5) = (1)$ is not an $\mathfrak{X}$-element, as (1) is not a proper element of L.

Lemma 2.11. *Let i be a proper element of a c-lattice L and $\mathfrak{X}$ be an M-closed subset of L. Then i is an $\mathfrak{X}$-element of L if and only if $i = (i : a)$ for all $a \notin \mathfrak{X}$. In particular, if i is an $\mathfrak{X}$-element of L, then $(i : a)$ is an $\mathfrak{X}$-element of L for all $a \notin \mathfrak{X}$.*

Proof. Suppose i is an $\mathfrak{X}$-element of L and let $a \notin \mathfrak{X}$. We always have $i \leq (i : a)$. Let x be any compact element such that $x \leq (i : a)$. Therefore $x \cdot a \leq i$. Since i is an $\mathfrak{X}$-element and $a \notin \mathfrak{X}$, we get $x \leq i$. Hence $(i : a) \leq i$, as L is a c-lattice. Therefore $i = (i : a)$ for all $a \notin \mathfrak{X}$.

Conversely, suppose that $i = (i : a)$ for all $a \notin \mathfrak{X}$. Let $c, d \in L$ such that $c \cdot d \ \leq i$ with $c \ \notin \mathfrak{X}$. We claim that $d \leq i$. Since $c \cdot d \ \leq i$, we have $d \leq (i : c)$. As $c \ \notin \mathfrak{X}$, by the assumption $(i : c) = i$, we have $d \leq i$. Therefore, i is an $\mathfrak{X}$-element of L. Further, the "in particular" part is easy to observe. □

Lemma 2.12. *Let i be a proper element of a multiplicative lattice L and $\mathfrak{X}$ be an M-closed subset of L. Then the following statements are equivalent.*

(1) *i is an $\mathfrak{X}$-element of L.*
(2) *$(i : a)$ is an $\mathfrak{X}$-element of L for every $a \nleq i$.*
(3) *$\big((i : a)\big] \subseteq \mathfrak{X}$ for all $a \nleq i$.*

Proof. (1) $\Rightarrow$ (2): Suppose that i is an $\mathfrak{X}$-element of L with $j \nleq i$. Clearly, $(i : j) \neq 1$. Let $a, b \in L$ such that $a \cdot b \leq (i : j)$ with $a \notin \mathfrak{X}$. So $a \cdot b \cdot j \leq i$. As i is an $\mathfrak{X}$-element and $a \notin \mathfrak{X}$, we get $b \cdot j \leq i$, i.e., $b \leq (i : j)$. Therefore $(i : j)$ is an $\mathfrak{X}$-element of L.

(2) $\Rightarrow$ (3): It follows from Lemma 2.4.

(3) $\Rightarrow$ (1): Suppose that $((i:a)] \subseteq \mathfrak{X}$ for all $a \nleq i$. Let $c, d \in L$ such that $c \cdot d \leq i$ with $c \notin \mathfrak{X}$. We claim that $d \leq i$. Suppose $d \nleq i$. So by the assumption $\Big((i:d)\Big] \subseteq \mathfrak{X}$ and $c \leq (i:d)$, we have $c \in \mathfrak{X}$, a contradiction. Therefore $d \leq i$. Hence i is an $\mathfrak{X}$-element of L. □

Lemma 2.13. *Let L be a multiplicative lattice and $\mathfrak{X}$ be an M-closed subset of L. If i is a maximal $\mathfrak{X}$-element of L, that is, maximal among all $\mathfrak{X}$-elements of L, then i is a prime element of L.*

Proof. Suppose i is a maximal $\mathfrak{X}$-element of L. Let $a, b \in L$ such that $a \cdot b \leq i$ and $a \nleq i$. Since i is an $\mathfrak{X}$-element and $a \nleq i$, by Lemma 2.12, $(i:a)$ is an $\mathfrak{X}$-element of L. As i is a maximal $\mathfrak{X}$-element of L and $i \leq (i:a)$, we get $(i:a) = i$. Therefore $b \leq i$. □

As mentioned earlier, $\mathfrak{X} = (j]$ is always an M-closed subset of a multiplicative lattice L.

Lemma 2.14. *Let j be a proper element of a c-lattice L and $\mathfrak{X} = (j]$. Then a proper element i is an $\mathfrak{X}$-element if and only if the condition $(+)$:*

$(+)$: for all $a,\ b\ \in L_*$ (set of all compact elements), $a \cdot b\ \leq i$ with $a\ \notin \mathfrak{X}$ implies $b\ \leq i$.

Proof. Assume that the condition $(+)$ holds. Let $a, b \in L$ such that $a \cdot b\ \leq i$ with $a\ \notin \mathfrak{X}$. As L is a c-lattice and $a\ \nleq j$, there exists $(0 \neq) x \in L_*$ such that $x \leq a$ and $x\ \nleq j$. Now, let y be a compact element such that $y \leq b$. As $x \cdot y \leq a \cdot b \leq i$ with $x\ \notin \mathfrak{X} = (j]$, by the condition $(+)$, $y \leq i$. Thus every compact element $\leq b$ is $\leq i$ and L is a c-lattice, we get $b \leq i$. Hence i is an $\mathfrak{X}$-element. The converse is obvious. □

Lemma 2.15. *Let L be a c-lattice and $\mathfrak{X} = (j]$. Then for a prime element i of L with $j \leq i$, i is an $\mathfrak{X}$-element if and only if $i = j$.*

Proof. Assume that i is a prime element which is also an $\mathfrak{X}$-element of L. By Lemma 2.4, we have $i \leq j$. This together with $j \leq i$, we have $i = j$. Conversely, assume that $i = j$ and i is prime. To prove i is an $\mathfrak{X}$-element, assume that $a \cdot b \leq i$ and $a \notin \mathfrak{X}$. Then by primeness of i and $i = j$, we have $b \leq i$. □

Corollary 2.16. *Let L be a c-lattice and $\mathfrak{X} = (j]$. Then for a maximal element i of L, i is an $\mathfrak{X}$-element if and only if $i = j$.*

Proof. Assume that i is a maximal element which is also an $\mathfrak{X}$-element of L. By Lemma 2.4, we have $i \leq j$. Thus by the maximality of i, we have $i = j$. Conversely, assume that $i = j$. Since i is maximal, it is prime. Thus the result follows from Lemma 2.15. □

Theorem 2.17. *Let L be a c-lattice and $\mathfrak{X} = (j]$, where*

$$j = \bigwedge\{i_k \mid i_k \text{ is a prime element of } L\}.$$

Then the following statements are equivalent.

(1) *There exists an $\mathfrak{X}$-element in L.*
(2) *j is a prime element of L.*

Moreover, if the set $Min(L)$ of all minimal prime elements in L is finite, then all the above conditions are equivalent to $(++)$: $|Min(L)| = 1$.

Proof. (1) ⇒ (2): Suppose there exists an $\mathfrak{X}$-element i in L. Let $\beta = \{x \mid x$ is an $\mathfrak{X}$-element in $L\}$. As $i \in \beta$, β is a poset under induced partial order of L. Let $j_1 \leq j_2 \leq \cdots \leq j_n \leq \cdots$ be a chain $\mathcal{C}$ in β. Using Lemma 2.14, we prove that $t = \bigvee_{\alpha=1}^{\infty} j_\alpha$ is in β, i.e., $t = \bigvee_{\alpha=1}^{\infty} j_\alpha$ is an $\mathfrak{X}$-element of L. Let $a, b \in L_*$ (set of all compact element of L) such that $a \cdot b \leq t = \bigvee_{\alpha=1}^{\infty} j_\alpha$ and $a \nleq j$. As L is a c-lattice and $a, b \in L_*$, we get $a \cdot b \in L_*$. Therefore $a \cdot b \leq j_1 \vee j_2 \vee \cdots \vee j_n$ for some $j_1, j_2, \cdots, j_n \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, we must have $j_1 \vee j_2 \vee \cdots \vee j_n = j_\gamma$ for some γ, where $1 \leq \gamma \leq n$. Thus $a \cdot b \leq j_\gamma$ with $a \nleq j$. Since j_γ is an $\mathfrak{X}$-element of L, we have $b \leq j_\gamma$. Thus $b \leq \bigvee_{\alpha=1}^{\infty} j_\alpha = t$. Therefore t is an $\mathfrak{X}$-element of L. By Zorn's Lemma, β has a maximal element, say w, that is, w is a maximal $\mathfrak{X}$-element. By Lemma 2.13, w is a prime element of L. Hence $j \leq w$. Also, w is an $\mathfrak{X}$-element, we have by Lemma 2.4, $w \leq j$. Thus $j = w$. Hence j is prime.

(2) ⇒ (1): Let j be a prime element. By Lemma 2.15, j is an $\mathfrak{X}$-element.

We now prove (2) ⇔ $(++)$.

Assume that j is a prime element. Since $j = \wedge\{i_k \mid i_k$ is a prime element of $L\}$ and the set $Min(L)$ is finite, without loss of generality, we assume that

$$j = \bigwedge_{k=1}^{n}\{i_k \mid i_k \in Min(L)\}.$$

By primeness of j and $j \leq i_k$ for all k with $i_k \in Min(L)$, we have $j = i_k$ for all k. Thus $|Min(L)| = 1$.

Conversely, assume that $|Min(L)| = 1$. Let p be the only minimal prime element in L. Then $p \leq i_k$ for every k. Hence $j = p$. This proves that j is prime. □

Lemma 2.18. [5, Lemma 2.5] *Let L be a c-lattice and $a \in L$. Then radical of a is given by $\sqrt{a} = \wedge\{p \in L$: p is a minimal prime element over $a\}$.*

Lemma 2.19. *Let L be a c-lattice and $\mathfrak{X} = (j]$, where*

$$j = \bigwedge\{i_k \mid i_k \text{ is a prime element of } L\}.$$

Then a proper element i is an $\mathfrak{X}$-element of L if and only if i is a primary element of L and $\sqrt{i} = j$.

Proof. Suppose i is an $\mathfrak{X}$-element of L. By Lemma 2.4, $(i] \subseteq \mathfrak{X}$. Hence $i \leq \sqrt{i} \leq \sqrt{j} = j$. Clearly, by Lemma 2.18, $j \leq \sqrt{i}$. Hence $j = \sqrt{i}$. Let $a, b \in L$ such that $a \cdot b \leq i$. By Theorem 2.17, j is a prime element of L. Hence either $a \leq j$ or $b \leq j$. So either $a \leq \sqrt{i}$ or $b \leq \sqrt{i}$. Hence i is a primary element of L. Conversely, suppose that i is a primary element of L and $\sqrt{i} = j$. Let $a, b \in L$ such that $a \cdot b \leq i$ and $a \nleq j$. Since i is a primary element and $a \nleq j = \sqrt{i}$, we get $b \leq i$. Thus i is an $\mathfrak{X}$-element of L. □

Lemma 2.20. *Let L be a c-lattice and $\mathfrak{X} = (j]$, where $j = \wedge\{i_k \mid i_k$ is a maximal element of $L\}$. Then a proper element i is an $\mathfrak{X}$-element of L if and only if i satisfies the following condition*
(A): If $a \cdot b \leq i$, then $a \leq i$ or $b \leq m$, where $m = \wedge\{i_k \mid i_k$ is a maximal element $\geq i\}$ and $m = j$.

Proof. It follows on similar lines as that of Lemma 2.19. □

Lemma 2.21. *Let L be a c-lattice and $\mathfrak{X}$ be an M-closed subset of L. Let k be an element of L such that $k \notin \mathfrak{X}$. If i_1 and i_2 are $\mathfrak{X}$-elements with $i_1 k = i_2 k$, then $i_1 = i_2$. Further, if i is an element such that ik is an $\mathfrak{X}$-element, then $ik = i$.*

Proof. Clearly, $i_1 k \leq i_2$ with $k \notin \mathfrak{X}$. Since i_2 is an $\mathfrak{X}$-element, we have $i_1 \leq i_2$. On similar lines we can prove that $i_2 \leq i_1$. Thus $i_1 = i_2$. Now, we prove the "further" part. Since ik is an $\mathfrak{X}$-element and $ik \leq ik$ with $k \notin \mathfrak{X}$, we have $i \leq ik$. The reverse inequality is always true. Hence $i = ik$. □

It is well-known that a proper ideal P of a commutative ring R with identity is prime if and only if $R \smallsetminus P$ is a multiplicatively closed subset of R. Analogously, a proper element p of a c-lattice L is prime if and only if $L \smallsetminus (p]$ is an M-closed subset of L. To characterize $\mathfrak{X}$-elements, we define $\mathfrak{X}$-multiplicatively closed subset of L as follows.

Definition 2.22. Let $\mathfrak{X}$ be an M-closed subset of a c-lattice L. A non-empty subset A of L_* with $(L_* \smallsetminus \mathfrak{X}) \subseteq A$ is called a $\mathfrak{X}$-*multiplicatively closed subset* of L, if $a_1 \in (L_* \smallsetminus \mathfrak{X})$ and $a_2 \in A$, then $a_1 \cdot a_2 \in A$.

Remark 2.23. If A is a $\mathfrak{X}$-multiplicatively closed subset of L, then A need not be a multiplicatively closed subset of L. Consider a multiplicative lattice K given in Example 2.2. If we take $\mathfrak{X} = \{0,\ a,\ b,\ c,\ d\}$, then $A = \{1,\ c,\ d\}$ is an $\mathfrak{X}$-multiplicatively closed subset of K but $A = \{1,\ c,\ d\}$ is not a multiplicatively closed subset of K, as $c, d \in A$ and $c \cdot d = 0 \notin A$.

Also, if A is a multiplicatively closed subset of L, then A need not be a $\mathfrak{X}$-multiplicatively closed subset of L for some M-closed subset $\mathfrak{X}$. Consider the ideal lattice L of the ring $\mathbb{Z}_{12}$. Since L is finite, $L = L_*$. Further, $\{(1)\}$ is a multiplicatively closed subset of L, but $\{(1)\}$ is not a $\mathfrak{X}$-multiplicatively closed subset of L for $\mathfrak{X} = \{(0),\ (6)\}$, as $(L \smallsetminus \mathfrak{X}) \nsubseteq \{(1)\}$.

Lemma 2.24. *Let i be a proper element of a c-lattice L and $\mathfrak{X}$ be an M-closed subset of L. If i is an $\mathfrak{X}$-element of L, then $L_* \smallsetminus (i]$ is an $\mathfrak{X}$-multiplicatively closed subset of L. The converse is true if either $\mathfrak{X} = (j]$ or L is a compact lattice.*

Proof. Suppose that i is an $\mathfrak{X}$-element of L. By Lemma 2.4, $(L_* \smallsetminus \mathfrak{X}) \subseteq (L_* \smallsetminus (i])$. Let $a \in (L_* \smallsetminus \mathfrak{X})$ and $b \in (L_* \smallsetminus (i])$. We claim that $a \cdot b \in (L_* \smallsetminus (i])$. Suppose on the contrary that $a \cdot b \notin (L_* \smallsetminus (i])$. So $a \cdot b \leq i$. Since i is an $\mathfrak{X}$-element of L and $a \notin \mathfrak{X}$, we get $b \leq i$, a contradiction to $b \in (L_* \smallsetminus (i])$. Thus $a \cdot b \in (L_* \smallsetminus (i])$. Consequently, $L_* \smallsetminus (i]$ is an $\mathfrak{X}$-multiplicatively closed subset of L.

Conversely, suppose that $\mathfrak{X} = (j]$, being an ideal is an M-closed subset of a c-lattice L, and $L_* \smallsetminus (i]$ is an $\mathfrak{X}$-multiplicatively closed subset of L. Therefore $\big(L_* \smallsetminus (j]\big) \subseteq \big(L_* \smallsetminus (i]\big)$. In view of Lemma 2.14, to show that i is an $\mathfrak{X}$-element of L, it is enough to show that for $a, b \in L_*$ such that $a \cdot b \leq i$ with $a \notin \mathfrak{X} = (j]$, we have $b \leq i$. If $b \in (L_* \smallsetminus (i])$, then as $(L_* \smallsetminus (i])$ is an $\mathfrak{X}$-multiplicatively closed subset of L, we get $a \cdot b \in (L_* \smallsetminus (i])$, a contradiction to $a \cdot b \leq i$. Therefore i is an $\mathfrak{X}$-element of L.

If L is a compact lattice and $\mathfrak{X}$ is an M-closed subset of L, then the converse follows similarly. □

Theorem 2.25. *Let j be a proper element of a c-lattice L and $\mathfrak{X} = (j]$. Suppose $a \in L$ and $t \nleq a$ for all $t \in A$, where A is an $\mathfrak{X}$-multiplicatively closed subset of L. Then there is an $\mathfrak{X}$-element i of L such that $a \leq i$ and i is maximal with respect to $t \nleq i$ for all $t \in A$.*

Proof. Let $R = \{c \in L \mid a \leq c \text{ and } t \nleq c \text{ for all } t \in A\}$. Clearly, $a \in R$ and hence R is a poset under the induced partial order of L. Let $\mathcal{C}$ be a chain in R and $w = \bigvee\{d \mid d \in \mathcal{C}\}$. Clearly, $a \leq w$. We claim that $w \in R$. Suppose on the contrary that $w \notin R$, that is, $t \leq w$ for some $t \in A$. Since $t \in A \subseteq L_*$ is compact and $t \leq w = \bigvee\{d \mid d \in \mathcal{C}\}$, we have $t \leq d_1 \vee d_2 \vee \cdots \vee d_n$ for some $d_1, d_2, \cdots, d_n \in \mathcal{C}$. As $\mathcal{C}$ is a chain, we must have $d_1 \vee d_2 \vee \cdots \vee d_n = d_r$ for some r, where $1 \le r \le n$. Thus $t \le d_r$ for some r, where $1 \le r \le n$. Since d_r is from chain $\mathcal{C}$ and $\mathcal{C}$ is a chain of elements from R, we get d_r element of R such that $t \le d_r$. This is a contradiction to $d_r \in R$. Thus $w \in R$. Hence by Zorn's Lemma, there is a maximal element i of R. Hence $a \le i$ and $t \nleq i$ for all $t \in A$.

In view of Lemma 2.14, to prove i is an $\mathfrak{X}$-element of L, assume that $x, y \in L_*$ such that $x \cdot y \le i$ with $x \notin \mathfrak{X} = (j]$. Suppose that $y \nleq j$. Clearly, $y \le (i : x)$ and, if $i = (i : x)$, then $y \le i$ and we are done. Hence assume that $i < (i : x)$. Since i is a maximal element of R and $a \le i < (i : x)$, we have $(i : x) \notin R$. Hence, there exists a compact element $t_1 \in A$ such that $t_1 \le (i : x)$, that is, $x \cdot t_1 \le i$. Since A is an $\mathfrak{X}$-multiplicatively closed subset, we have $(L_* \setminus (j]) = (L_* \setminus \mathfrak{X}) \subseteq A$. Further, $x \in (L_* \setminus \mathfrak{X})$ and $t_1 \in A$, we get $x \cdot t_1 \in A$. Thus there exists an element $t_2 = x \cdot t_1 \in A$ such that $t_2 \le i$, a contradiction to $i \in R$. Hence i is an $\mathfrak{X}$-element of L. □

Theorem 2.26. *Let L be a compact lattice and $\mathfrak{X}$ be an M-closed subset of L. Suppose $a \in L$ and $t \nleq a$ for all $t \in A$, where A is an $\mathfrak{X}$-multiplicatively closed subset. Then there is an $\mathfrak{X}$-element i of L such that $a \le i$ and i is maximal with respect to $t \nleq i$ for all $t \in A$.*

Proof. Proof follows on similar lines as that of Theorem 2.25. □

3. Applications of $\mathfrak{X}$-elements

As noted earlier, there is a unifying pattern in the results of J-ideals, n-ideals and r-ideals of a commutative ring with identity. In this section, we prove these results of J-ideals, n-ideals and r-ideals by suitably replacing the set $\mathfrak{X}$ in multiplicative lattices. Hence most of the results of the papers [12], [14] and [15] become corollaries of our results.

First, we quote the definitions of J-ideals, n-ideals and r-ideals using the sets $Z(R)$, $N(R)$ and $J(R)$ ($Z(L)$, $N(L)$ and $J(L)$), the set of zero-divisors, the nil-radical and the Jacobson radical of a commutative ring R (multiplicative lattice L) respectively.

Definition 3.1. A proper ideal I of a commutative ring R with identity is called:

(1) an r-ideal, if $ab \in I$ with $ann_R(a) = (0)$ implies $b \in I$ for all $a, b \in R$ (see [14]).
(2) an n-ideal, if $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in I$ for all $a, b \in R$ (see [15]).
(3) a J-ideal, if $ab \in I$ with $a \notin J(R)$ implies $b \in I$ for all $a, b \in R$ (see [12]).

Analogously, we define the concepts of r-element, n-element and J-element in multiplicative lattices.

Definition 3.2. A proper element i of a multiplicative lattice L is called:

(1) an r-element, if $a \cdot b \leq i$ with $a \notin Z(L)$ implies $b \leq i$ for all $a,\ b \in L_*$.

(2) an n-element, if $a \cdot b \leq i$ with $a \notin \left(\sqrt{0}\,\right]$ implies $b \leq i$ for all $a,\ b \in L_*$.

(3) a J-element, if $a \cdot b \leq i$ with $a \notin \left(J(L)\right]$ implies $b \leq i$ for all $a,\ b \in L_*$.

We quote the following three results to prove that a proper ideal I of a commutative ring R with identity is an r-ideal, n-ideal and J-ideal if and only if it is an r-element, n-element and J-element of the multiplicative lattice $Id(R)$, the set of all ideals of R, respectively.

Theorem 3.3. [14, Lemma 2.5] *Let R be a commutative ring with identity and I be a proper ideal of R. Then I is an r-ideal if and only if whenever J and K are ideals of R with $J \nsubseteq Z(R)$ and $JK \subseteq I$, then $K \subseteq I$.*

Theorem 3.4. [15, Theorem 2.7] *Let R be a commutative ring with identity and I a proper ideal of R. Then the following are equivalent:*

(1) *I is an n-ideal of R.*

(2) *$I = (I : a)$ for every $a \notin \sqrt{0}$.*

(3) *For ideals J and K of R, $JK \subseteq I$ with $J \cap (R - \sqrt{0}) \neq \varnothing$ implies $K \subseteq I$.*

Theorem 3.5. [12, Proposition 2.10] *Let R be a commutative ring with identity and I a proper ideal of R. Then the following are equivalent:*

(1) *I is a J-ideal of R.*

(2) *$I = (I : a)$ for every $a \notin J(R)$.*

(3) *For ideals A and B of R, $AB \subseteq I$ with $A \nsubseteq J(R)$ implies $B \subseteq I$.*

Theorem 3.6. *Let R be a Noetherian ring with identity. Then I is an r-ideal of R if and only if I is an r-element of the multiplicative lattice $L = Id(R)$, where $Id(R)$ is the ideal lattice of R.*

Proof. Suppose that I is an r-ideal of R. Let J, K be any ideals of R such that $J \cdot K \leq I$ in L with $J \notin Z(L)$, that is, $ann_L(J) = 0_L$, where (0_R) is the least element of L, denoted by 0_L. We claim that $ann_R(J) = (0_R)$. Suppose on the contrary that $(0_R \neq) x \in ann_R(J)$. Hence $(x)J = (0_R)$, a contradiction to the $ann_L(J) = 0_L$. Hence $ann_R(J) = (0_R)$.

Now, we prove that $J \nsubseteq Z(R)$. Suppose on the contrary that $J \subseteq Z(R)$. Since R is Noetherian, $J \subseteq Z(R) = \bigcup_{i=1}^{n} P_i$, where P_i's are associate primes. By Prime

Avoidance Theorem $J \subseteq P_k$ for some k. Since P_k is an associated prime, we have $P_k = 0 : x$ for some $x \in R$. But this contradicts the fact that $ann_R(J) = (0)$. Hence $J \notin Z(R)$. By Theorem 3.3, $K \subseteq I$, i.e., $K \leq I$. Thus I is an r-element of L.

Conversely, suppose that I is an r-element of L. Let $a, b \in R$ such that $a \cdot b \in I$ with $ann_R(a) = (0_R)$. We claim that $b \in I$. Since $(a \cdot b) = (a) \cdot (b) \subseteq I$, we have $a' \cdot b' \leq I$ in L, where $a' = (a)$, $b' = (b)$. Clearly, $a' \notin Z(L)$. Hence $b' \leq I$, i.e., $b \in I$. Thus I is an r-ideal of R. □

Remark 3.7. From the proof of Theorem 3.6, it is clear that every r-element of $Id(R)$ is an r-ideal of R. However, for the converse, we need the assumption that a ring is Noetherian. It should be noted that the result is still true if we replace the "Noetherian ring" with the "ring satisfies strongly annihilator condition". By strongly annihilator condition, we mean, for a given ideal I of R, there exists $a \in I$ such that $ann_R(I) = ann_R(a)$.

Further, we are unable to find an example to show that the condition that the ring is Noetherian or satisfies strongly annihilator condition is necessary to prove the above Theorem 3.6. Hence we have:

Question 3.8. Let I be an r-ideal of a commutative ring R with identity. Is I an r-element of $Id(R)$?

Theorem 3.9. *Let R be a commutative ring with identity. Then I is an n-ideal of R if and only if I is an n-element of the multiplicative lattice $L = Id(R)$, where $Id(R)$ is the ideal lattice of R.*

Proof. Suppose that I is an n-ideal of R. Let J, K be any finitely generated ideals of R such that $J \cdot K \leq I$ with $J \nleq \sqrt{0_L}$ in L. It is known that finitely generated ideals of R are compact elements of $Id(R)$. Since $J \nleq \sqrt{0_L}$, we get $J^n \neq 0_L = (0_R)$ for every $n \in \mathbb{N}$. Hence $J \cap (R \setminus \sqrt{0_R}) \neq \varnothing$. By Theorem 3.4, $K \subseteq I$, i.e., $K \leq I$. Therefore I is an n-element of L.

Conversely, suppose that I is an n-element of L. Let $a, b \in R$ such that $a \cdot b \in I$ with $a \notin \sqrt{0_R}$. We claim that $b \in I$. Since $(a \cdot b) = (a) \cdot (b) \subseteq I$, we have $a' \cdot b' \leq I$ in L, where $a' = (a)$, $b' = (b) \in L_*$. Clearly, $a' \nleq \sqrt{0_L}$. Hence $b' \leq I$, i.e., $b \in I$. Thus I is an n-ideal of R. □

Theorem 3.10. *Let R be a commutative ring with identity. Then I is a J-ideal of R if and only if I is a J-element of the multiplicative lattice $L = Id(R)$, where $Id(R)$ is the ideal lattice of R.*

Proof. Suppose that I is a J-ideal of R. Let A, B be finitely generated ideals of R (which are compact elements of $Id(R)$) such that $A \cdot B \leq I$ with $A \nleq J(L)$ in

$L = Id(R)$. Since $A \nleq J(L)$, we get $A \nsubseteq J(R)$. By Theorem 3.5, $B \subseteq I$, i.e., $B \leq I$ in L. Hence I is a J-element of L.

Conversely, suppose that I is a J-element of L. Let $a, b \in R$ such that $a \cdot b \in I$ with $a \notin J(R)$. We claim that $b \in I$. Since $(a \cdot b) = (a) \cdot (b) \subseteq I$, we have $a' \cdot b' \leq I$ in L, where $a' = (a)$, $b' = (b) \in L_*$. Clearly, $a' \nleq J(L)$. Hence $b' \leq I$, i.e., $b \in I$. Hence I is a J-ideal of R. □

Let L be a multiplicative lattice. Then one can see that each of the sets $Z(L)$, $\left(\sqrt{0}\,\right]$ and $\left(J(L)\right]$ are M-closed subsets of L. So if we replace $\mathfrak{X}$ by these sets, then we get the results of r-element, n-element and J-element respectively.

We quote some of these results for ready reference.

One can see that in a c-lattice L, $\left(L_* \smallsetminus Z(L)\right) \subseteq \left(L_* \smallsetminus \left(\sqrt{0}\,\right]\right)$ and $\left(\sqrt{0}\,\right] \subseteq \left(J(L)\right]$. For this, let $x \in \left(L_* \smallsetminus Z(L)\right)$ and $x \in \left(\sqrt{0}\,\right]$. Then $x^n = 0$ for some $n \in \mathbb{N}$. Thus $x \in Z(L)$, a contradiction. This proves the inclusion $\left(L_* \smallsetminus Z(L)\right) \subseteq \left(L_* \smallsetminus \left(\sqrt{0}\,\right]\right)$. Now, for the second inclusion, let y be any compact element such that $y \in \left(\sqrt{0}\,\right]$. Then $y^k = 0$ for some $k \in \mathbb{N}$. Let m be a maximal element of L. Then it is prime. This together with $y^k = 0 \leq m$ implies that $y \leq m$. This further yields that $y \in J(L) = \bigwedge_{k \in \Lambda} m_k$. Since L is a c-lattice and every compact element below $\sqrt{0}$ is below $J(L)$, we have $\sqrt{0} \leq J(L)$.

Hence by Lemma 2.6, we have the following result.

Proposition 3.11. *Let L be a c-lattice. Then every n-element of L is a r-element as well as it is a J-element of L.*

By Proposition 3.11, Theorems 3.6, 3.9 and 3.10, we have:

Proposition 3.12 ([12], [15])**.** *Let R be a commutative ring with identity. Then every n-ideal of R is an r-ideal as well as it is a J-ideal.*

From Lemma 2.4, we get Proposition 2.2 of [12] and Proposition 2.3 of [15]. Also, Proposition 2.4 of [15] follows from Lemma 2.9. It is easy to observe that Proposition 2.10 of [12] and Theorem 2.7 of [15] follows from Lemma 2.11. We observe that Proposition 2.13 of [12] follows from Lemmas 2.4 and 2.13. Note that Theorem 2.17 strengthens Theorem 2.12 of [15]. Lemmas 2.19 and 2.20 generalizes the equivalence of (i) and (ii) in Corollary 2.13 of [15] and Proposition 2.20 of [12] respectively. One can see that Lemma 2.21 extends Proposition 2.16 of [15] and Proposition 2.21 of [12]. Lastly, Theorem 2.23 of [15] and Proposition 2.29 of [12] follows from Theorem 2.25. For the following result, we need a little more explanation.

Proposition 3.13 ([12, Proposition 2.3]). *Let R be a commutative ring with identity. Then the following are equivalent.*

(1) *R is a local ring.*
(2) *Every proper ideal of R is a J-ideal.*

Proof. (1) $\Rightarrow$ (2): It is clear that the ideal lattice $Id(R)$ of R is a local lattice. Further, $J(L) = m$, where m is the unique maximal element of $Id(R)$. Hence by Lemma 2.7, every proper element of L is an $\mathfrak{X}$-element, where $\mathfrak{X} = (m] = \big(J(L)\big]$. That is, every proper element of L is a J-element. By Theorem 3.10, every proper ideal of R is a J-ideal.

(2) $\Rightarrow$ (1): It follows from Lemma 2.8 $\square$

Finally, the results of n-multiplicatively closed subset and J-multiplicatively closed subset can be obtained by using Lemma 2.24. Further, the results of r-ideals can be deduced from our results for Noetherian rings, since Theorem 3.6 is available for Noetherian lattice settings. If Question 3.8 has an affirmative answer, then our results will extend most of the results of r-ideals of commutative rings.

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