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All Dehn Fillings of the Whitehead Link Complement are Tetrahedron Manifolds

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Abstract

In this paper we show that Dehn surgeries on the oriented components of the Whitehead link yield tetrahedron manifolds of Heegaard genus ≤ 2. As a consequence, the eight homogeneous Thurston 3–geometries are realized by tetrahedron manifolds of Heegaard genus ≤ 2. The proof is based on techniques of Combinatorial Group Theory, and geometric presentations of manifold fundamental groups.

Keywords: *Tetrahedron manifold, Whitehead link, Dehn surgery, Heegaard genus, group presentation.*

AMS Subject Classification (2020): *Primary: 57M12 57M25; Secondary: 57M50.*

1. Introduction and results

Let us consider the Whitehead link $\mathcal{W}$ in the oriented 3–sphere $\mathbb{S}^3$. See Figure 1. Let $\mathcal{W}(m/n; p/q)$ denote the closed connected orientable 3–manifold obtained by m/n and p/q Dehn surgeries on the oriented components of $\mathcal{W}$. Such manifolds were studied in [11] and [12], and their topological classification follows from the results of [10].

Let $G(m/n; p/q)$ be the fundamental group of $\mathcal{W}(m/n; p/q)$. It was shown in [6], Theorem 3.2, that $G(m/n; p/q)$ admits a finite balanced presentation with generators a and b and relators

$$\begin{aligned} a^{p+q}b^{-n}a^{-q}b^{n}a^{q}b^{n}a^{-q}b^{-n} &= 1 \\ b^{m+n}a^{-q}b^{-n}a^{q}b^{n}a^{q}b^{-n}a^{-q} &= 1 \end{aligned} \tag{1}$$

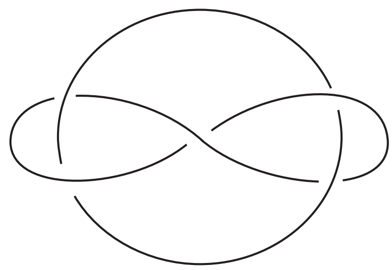

Figure 1. The Whitehead link $\mathcal{W}$ and the surgery manifold $\mathcal{W}(m/n; p/q)$.

Such a presentation is *geometric*, that is, it corresponds to a spine of the manifold. In particular, it arises from a Heegaard diagram of genus 2. If $\mathcal{W}(m/n; p/q)$ admits a hyperbolic structure, then it has Heegaard genus 2. In

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general, since (1) is a geometric presentation, we see that $\mathcal{W}(m/n; p/q)$ has Heegaard genus at most 2. However, this sentence is well known as $\mathcal{W}$ has tunnel number 1. For presentations of Groups in terms of Generators and Relations we refer to [8] and [9].

The face identification procedure is a very classical method for constructing closed 3–manifolds. It is well-known that such spaces can be combinatorially described as quotients of polyhedral 3–balls by pairwise identifications of their boundary faces. The interiors of such 3–balls become open 3–balls in the quotients. Their boundaries become embedded 2–polyhedra which are spines of the quotient manifolds. A unified method to treat the theory of discontinuous transformation groups in a space of constant curvature and the face identification procedure has been developed by Molnár in [14]. See also [15]. Then Molnár and Szirmai [17] study an interesting family of compact 3–manifolds, i.e., space forms, that are derived from famous Euclidean and non-Euclidean polyhedral tilings by the unified method of face identifications. For the classification of face-transitive periodic three-dimensional tilings see [7].

Following [3], [4], [13] and [25], a *tetrahedron manifold* is, by definition, a closed connected orientable 3–manifold obtained by a suitable subdivision of the edges of a standard tetrahedron plus an identification of pairs of the boundary faces. We now state our main result.

Theorem 1. *Any surgery 3–manifold $\mathcal{W}(m/n; p/q)$ is a tetrahedron manifold of Heegaard genus ≤ 2.*

For the proof, we use the finite presentation in (1), the classification results from [10], and the following theorem which is a consequence of the Geometrization Theorem in combination with the Mostow-Prasad Rigidity Theorem [20] [21]. See [1], Theorem 3.17.

Theorem 2. *Let M and M' be closed connected orientable prime 3–manifolds with isomorphic fundamental groups. If M and M' are not lens spaces, then M and M' are homeomorphic.*

As a consequence of Theorem 1 and the classification results from [10], we get

Corollary 3. *The eight homogeneous Thurston 3–geometries are realized by tetrahedron manifolds of Heegaard genus ≤ 2.*

To clarify the statement of Corollary 3 we explicitly confirm that all eight Thurston's geometries occur on some manifolds of type $\mathcal{W}(m/n; p/q)$. Following [2], let $K_{[b_1,b_2]}$ denote the 2–bridge knot in $\mathbb{S}^3$ that corresponds to a continued fraction $\beta/\alpha = [b_1, b_2] = 1/(b_1 - 1/b_2)$. Then α is odd, and we may assume that β is even with $1 < \beta < \alpha$. As remarked in [2], at least one of the coefficients b_i, $i = 1, 2$, is even, and we may set $b_1 = 2n$, for some integer n, as $K_{[b_1,b_2]}$ is equivalent to $K_{[b_2,b_1]}$. Now doing a p/q surgery on $K_{[2n,2]}$ is the same as doing $-1/n$ and p/q surgeries on the components of the Whitehead link $\mathcal{W}$. Recall that $K_{[2n,2]}$ is a twist knot, and $K_{[2,2]}$ is the trefoil knot $T(3, 2)$. Thus, for every $n \geq 1$, we have the homeomorphism $\mathcal{W}(-1/n; p/q) \cong K_{[2n,2]}(p/q)$. For $n = 1$, we get $\mathcal{W}(-1; p/q) \cong K_{[2,2]}(p/q)$, where $K_{[2,2]} = T(3, 2)$. From [19] and [24], let M be the closed 3–manifold obtained by Dehn surgery on $T(3, 2)$ with coefficient p/q, and set $\sigma = 6q + p$.

If $\sigma = 0$ (e.g., $p = -6$ and $q = 1$), then M is reducible, and it is homeomorphic to $\mathbb{R}P^3 \# L(3, 1)$.

If $|\sigma| = 1$, then M is homeomorphic to the lens space $L(|p|, 4q)$. In this case, M has Heegaard genus one. It admits spherical geometry (i.e., $\mathbb{S}^3$–geometry) as $p \neq 0$. In particular, for $|p| = 1$, we get the 3–sphere $\mathbb{S}^3$.

If $|\sigma| \geq 2$, then M is a Seifert fibered 3–manifold with three exceptional fibers, that is,

$$M \equiv (O\, 0\, o\colon\ -1\ (3, 1)\ (2, 1)\ (p + 6q, q)).$$

From [18], let us consider two parallel curves in a solid torus V with surgery instructions $(1, b)$ and (α, β). Without changing the meridian of the resulting solid torus, we can exchange the surgery instructions for $(1, b + n)$ and $(\alpha, \beta - \alpha n)$, respectively. Applying this construction to the above surgery manifold, for $|\sigma| \geq 2$, it follows that M can be completely represented by the following Seifert coefficients:

$$M \equiv (O\, 0\, o\colon 0\ (3, 1)\ (2, -1)\ (p + 6q, q)).$$

Thus the Euler number of the Seifert fibration for M is given by

$$\mathbf{e} = \frac{1}{2} - \frac{1}{3} - \frac{q}{p + 6q} = \frac{p}{6(p + 6q)}.$$

The Euler characteristic of the base orbifold is given by

$$\chi = 2 - (1 - \frac{1}{3}) - (1 - \frac{1}{2}) - (1 - \frac{1}{p + 6q}) = \frac{6 - 6q - p}{6(p + 6q)}.$$

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$\mathbf{e} = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\mathbf{e} \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{PSL}$

By [22], the relevant geometries of a Seifert fibered 3–manifold is determined by $\mathbf{e}$ and χ according to the above table.

In our case, we have:

- $\chi = 0$, $\mathbf{e} = 0$ $(p = 0,\ q = 1)$. Then the surgery manifold

$$M \equiv (O\,0\,o\colon\ -1\,(3,1)\,(2,1)\,(6,1))$$

is an Euclidean manifold (see [18], p.63), hence it admits an $\mathbb{E}^3$–geometry.

- $\chi < 0$, $\mathbf{e} = 0$ $(p = 0,\ q < 0$ or $q > 1)$. Then the resulting surgery manifold admits a $(\mathbb{H}^2 \times \mathbb{R})$–geometry.

- $\chi = 0$, $\mathbf{e} \neq 0$ $(p \neq 0,\ 6q = 6 - p)$. Then the surgery manifold admits a Nil–geometry.

- $\chi < 0$, $\mathbf{e} \neq 0$ (hence $p \neq 0$). Then the surgery manifold admits a $\widetilde{PSL}$–geometry.

The surgery manifold $\mathcal{W}(m/n; p/q)$ with $n = 0$ (hence $m = 1$) is homeomorphic to the lens space $L(p, q)$, which has $\mathbb{S}^3$–geometry for $p \neq 0$, and $(\mathbb{S}^2 \times \mathbb{S}^1)$–geometry for $p = 0$ as $L(0, 1) \cong \mathbb{S}^2 \times \mathbb{S}^1$.
From [3], the surgery manifold $\mathcal{W}(m/n; p/q)$, with $p = 0$, $q = 1$ and $m = \mathbf{n} - 2$, $n = 1$ for $\mathbf{n} > 2$, admits a Sol geometry.
From [4], the surgery manifold $\mathcal{W}(m/n; p/q)$, with $p = 4\mathbf{m} + 1$, $q = \mathbf{m}$ and $m = 2\mathbf{n} + 3$, $n = \mathbf{n} + 1$, admits a hyperbolic geometry. Hyperbolic manifolds arising by surgeries on the components of the Whitehead link have been studied in [10], [11] and [12]. Symmetries in the eight homogeneous Thurston 3-geometries have been described by Molnár and Szirmai in [16].

2. A class of tetrahedron manifolds

Let us consider the simplicial complex $P(m/n; p/q)$ which triangulates the boundary of the standard tetrahedron as indicated in Figure 2. Let A_i be a vertex of the tetrahedron, for $i = 0, 1, 2, 3$. The oriented edges A_0A_2 and A_3A_0 are subdivided into four sequences of oriented edges, labeled by the oriented sequences b^n, a^q, b^n and a^q, where the exponent is taken with sign $+1$ (resp. -1) if the walking sense of A_0A_2 and A_3A_0 is coherent (resp. opposite) to the arrow marked in Figure 2. The oriented edge A_2A_3 is labeled by b^m, that is, it is subdivided by m oriented edges of the same length, labeled by b. The oriented edges A_1A_2 and A_3A_1 are subdivided into four sequences of oriented edges, labeled by the oriented sequences a^q, b^n, a^q and b^n. As before, the exponent is taken with sign $+1$ (resp. -1) if the walking sense of A_1A_2 and A_3A_1 is coherent (resp. opposite) to the arrow drawn in Figure 2. We identify in pairs the boundary faces of the standard tetrahedron, i.e., $F_1 \equiv \overline{F}_1$ and $F_2 \equiv \overline{F}_2$. The faces are to be paired so that the index stars in Figure 2 match up. For $(m, n) = (p, q) = 1$, the resulting space, denoted by $M(m/n; p/q)$, has one vertex, two 1–cells, also labeled by a and b, two 2–cells, and one 3–cell. Since the Euler characteristic vanishes, the quotient space $M(m/n; p/q)$ is a closed connected orientable 3–manifold. The face pairing forces an identification of the cells of $P(m/n; p/q)$ into a 2–polyhedron, which is a spine of the manifold $M(m/n; p/q)$. We can immediately obtain a finite presentation for the fundamental group of $M(m/n; p/q)$ by considering the two spine relations:

Theorem 4. *The simplicial complex $P(m/n; p/q)$ with the identifications described above defines the tetrahedron manifold $M(m/n; p/q)$ of Heegaard genus ≤ 2. The two cycle relations provide the finite geometric presentation in (1) for the fundamental group of $M(m/n; p/q)$. Such a presentation corresponds to a spine of the constructed manifold.*

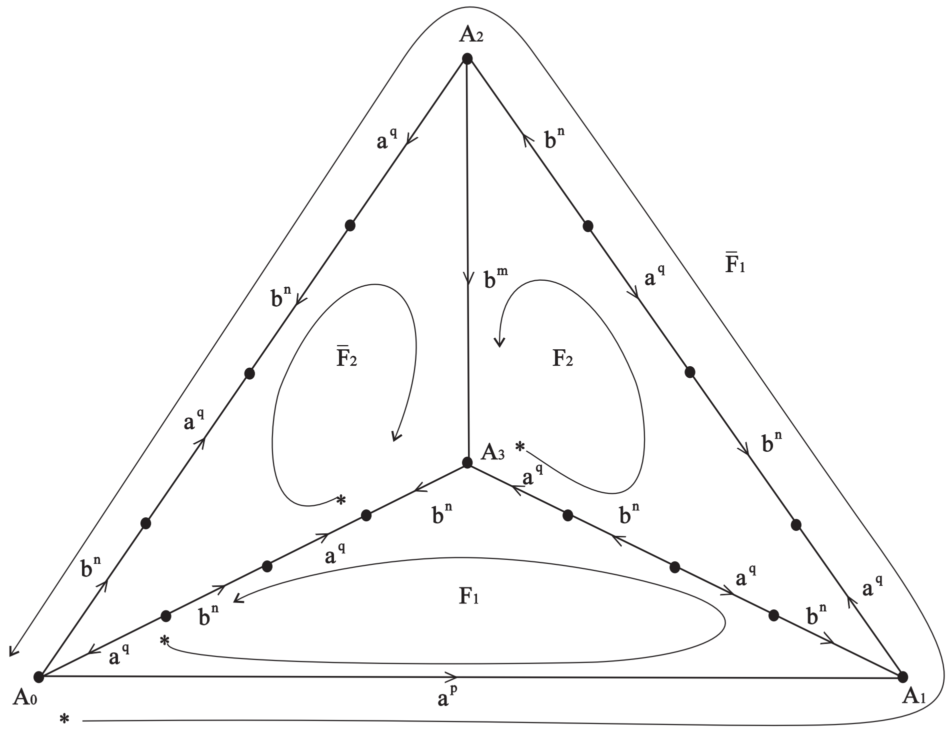

Figure 2. Polyhedral schemata of the tetrahedron manifolds $M(m/n;p/q)$.

3. Proof of Theorem 1.

Set $(m,n)=(p,q)=1$. Assume that $|m/n| \leq |p/q|$ as $\mathcal{W}(m/n;p/q)$ is homeomorphic to $\mathcal{W}(p/q;m/n)$. This follows from the symmetries of the Whitehead link $\mathcal{W}$. In particular, there is a self–homeomorphism of the exterior of $\mathcal{W}$ in $\mathbb{S}^3$ which interchanges two boundary tori. See, for example, [11]. Here we consider only nontrivial surgery manifolds, that is, we exclude the case that either $n=0$ or $q=0$. Such surgeries on the components of $\mathcal{W}$ produce the lens spaces (including $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$), which are tetrahedron manifolds by obvious modification of the polyhedral schemata in Figure 2. For example, if $q=0$ (hence $p=1$) the sequences of edges labeled by a^q disappear from the polyhedral scheme in Figure 2. In this case $G(m/n;\infty)$ admits a geometric presentation with generators a and b and relators $a=1$ and $b^m=1$. By Singer moves [23] on the Heegaard diagram associated to $G(m/n;\infty)$, we see that the resulting manifold is the lens space $L(m,n)$.

If $\mathcal{W}(m/n;p/q)$ and $M(m/n;p/q)$ are prime and not lens spaces, then they are homeomorphic by Theorems 2 and 4. More precisely, Theorem 4 states that these manifolds have isomorphic fundamental groups. This enables the application of Theorem 2. We now investigate the case of reducible manifolds and the case of manifolds homeomorphic to lens spaces.

Reducible manifolds. From the classification results of [10], the surgery manifold $\mathcal{W}(m/n;p/q)$ is reducible if and only if

$$(m/n,p/q) \in \{(-1,-6),(-2,-4),(-3,-3)\}.$$

In these cases, we have $\mathcal{W}(-1;-6) \cong L(3,1)\#\mathbb{R}P^3$, $\mathcal{W}(-2;-4) \cong L(4,1)\#\mathbb{R}P^3$, and $\mathcal{W}(-3;-3) \cong L(3,1)\#L(3,1)$. For such pairs of integers, we show that the manifolds $\mathcal{W}(m/n;p/q)$ and $M(m/n;p/q)$ are homeomorphic.

Case $(m/n,p/q)=(-1,-6)$. For $m=1$, $n=-1$, $p=-6$ and $q=1$, the finite presentation of $G(m/n;p/q)$ in (1) has generators a and b and relations

(2) $$a^{-5}ba^{-1}b^{-1}ab^{-1}a^{-1}b=1$$

and

(3) $$a^{-2}bab^{-1}ab = 1.$$

From (3) we get

(4) $$a^{-1}ba^{-1} = ba^{-2}b$$

which becomes $a^{-2}ba^{-2}ba^{-2} = a^{-3}ba^{-3}$, or, equivalently,

(5) $$(a^{-2}b)^3 = (a^{-3}b)^2.$$

Substituting relation (4) into (2) and doing simplifications yield the following sequence of equivalent relations:

$$a^{-4}(a^{-1}ba^{-1})b^{-1}ab^{-1}a^{-1}b = 1$$
$$a^{-4}(ba^{-2}b)b^{-1}ab^{-1}a^{-1}b = 1$$
$$a^{-4}ba^{-1}b^{-1}a^{-1}b = 1$$
$$a^{-3}(a^{-1}ba^{-1})b^{-1}a^{-1}b = 1$$
$$a^{-3}(ba^{-2}b)b^{-1}a^{-1}b = 1$$
$$a^{-3}ba^{-3}b = 1$$

hence

(6) $$(a^{-3}b)^2 = 1.$$

By (5) and (6), the group $G(-1;-6)$ admits a finite presentation with generators a and b and relations $(a^{-3}b)^2 = 1$ and $(a^{-2}b)^3 = 1$. Set $x = a^{-2}b$ and $y = a^{-3}b$ with inverse relations $a = xy^{-1}$ and $b = (xy^{-1})^2x$. Then $G(-1;-6)$ is presented by generators x and y and relations $x^3 = 1$ and $y^2 = 1$, that is, $\pi_1(M(-1;-6)) \cong \mathbb{Z}_3 * \mathbb{Z}_2$. By Theorem 3.11 of [1] there exist closed orientable 3–manifolds M_1 and M_2 with $\pi_1(M_1) \cong \mathbb{Z}_3$, $\pi_1(M_2) \cong \mathbb{Z}_2$, and $M(-1;-6) \cong M_1\#M_2$. Since $M(-1;-6)$ has Heegaard genus two, the manifold M_i, $i = 1,2$, has Heegaard genus one. It follows that $M_1 \cong L(3,1)$ and $M_2 \cong \mathbb{R}P^3$, as requested.
For this case, we can also propose an alternative argument as follows. Since $\mathcal{W}(-1;-6) \cong L(3,1)\#\mathbb{R}P^3$ and both $\pi_1(M(-1;-6))$ and $\pi_1(\mathcal{W}(-1;-6))$ have the presentation $G(-1;-6)$, we immediately know that

$$\pi_1(M(-1;-6)) \cong \pi_1(L(3,1)\#\mathbb{R}P^3) \cong \mathbb{Z}_3 * \mathbb{Z}_2.$$

Then there exist closed orientable 3–manifolds M_1 and M_2 as above. In this case, $M_1\#M_2$ is homeomorphic to $M_1\#(-M_2)$, where $-M_2$ denotes M_2 with the opposite orientation. In fact, $M_2 = \mathbb{R}P^3$ admits an orientation reversing self-homeomorphism. More generally, it is well-known that a lens space $L(p,q)$ admits an orientation reversing self-homeomorphism if and only if $q^2 \equiv -1 \pmod p$.

Case $(m/n, p/q) = (-2,-4)$. For $m = 2$, $n = -1$, $p = 4$ and $q = -1$, the finite presentation of $G(m/n;p/q)$ in (1) has generators a and b and relations

(7) $$a^3bab^{-1}a^{-1}b^{-1}ab = 1$$

and

(8) $$baba^{-1}b^{-1}a^{-1}ba = 1.$$

From (8) we get

(9) $$babab = aba$$

which becomes $abababa = a^2ba^2$, or, equivalently,

(10) $$(ab)^4 = (a^2b)^2.$$

Substituting relation (9) into (7) and doing simplifications yield the following sequence of equivalent relations:

$$a^2(aba)b^{-1}a^{-1}b^{-1}ab = 1$$
$$a^2(babab)b^{-1}a^{-1}b^{-1}ab = 1$$
$$a^2ba^2b = 1$$

hence

$$(a^2b)^2 = 1. \tag{11}$$

By (10) and (11), the group $G(-2;-4)$ admits a finite presentation with generators a and b and relations $(a^2b)^2 = 1$ and $(ab)^4 = 1$. Set $x = ab$ and $y = a^2b$ with inverse relations $a = yx^{-1}$ and $b = xy^{-1}x$. Then $G(-2;-4)$ is presented by generators x and y and relations $x^4 = 1$ and $y^2 = 1$, that is, $\pi_1(M(-2;-4)) \cong \mathbb{Z}_4 * \mathbb{Z}_2$. Reasoning as above, we get $M(-2;-4) \cong M_1 \# M_2$ with $M_1 \cong L(4,1)$ and $M_2 \cong \mathbb{R}P^3$ because M_1 and M_2 have Heegaard genus one. Also in this case, one can apply the arguments described at the end of the previous case concerning the homeomorphism $M_1 \# M_2 \cong M_1 \#(-M_2)$.

Case $(m/n, p/q) = (-3,-3)$. For $m = p = 3$ and $n = q = -1$, the finite presentation of $G(m/n; p/q)$ in (1) has generators a and b and relations

$$a^2bab^{-1}a^{-1}b^{-1}ab = 1 \tag{12}$$

and

$$b^2aba^{-1}b^{-1}a^{-1}ba = 1. \tag{13}$$

From (13) we get

$$bab^2ab = aba \tag{14}$$

which becomes $b^2ab^2ab^2 = babab$, or, equivalently,

$$(ab)^3 = (ab^2)^3. \tag{15}$$

Substituting relation (14) into (12) and doing simplifications yield the following sequence of equivalent relations:

$$a(aba)b^{-1}a^{-1}b^{-1}ab = 1$$
$$a(bab^2ab)b^{-1}a^{-1}b^{-1}ab = 1$$
$$ababab = 1$$

hence

$$(ab)^3 = 1. \tag{16}$$

By (15) and (16), the group $G(-3;-3)$ admits a finite presentation with generators a and b and relations $(ab)^3 = 1$ and $(ab^2)^3 = 1$. Set $x = ab$ and $y = ab^2$ with inverse relations $a = xy^{-1}x$ and $b = x^{-1}y$. Then $G(-3;-3)$ is presented by generators x and y and relations $x^3 = 1$ and $y^3 = 1$, that is, $\pi_1(M(-3;-3)) \cong \mathbb{Z}_3 * \mathbb{Z}_3$. Finally, we obtain $M(-3;-3) \cong L(3,1)\#L(3,1)$. In this case, the manifolds $L(3,1)\#L(3,1)$ and $L(3,1)\# - L(3,1)$ are not homeomorphic. The claimed result follows from the trick of viewing the Tietze transformations as Singer moves. See, for example, [23]. More precisely, from (15) and (16) the group $G(-3;-3)$ admits the finite presentation with generators x and y and relations $x^3(y^{-1})^3 = 1$ and $x^3 = 1$. The above Tietze transformations correspond to Singer moves on the Heegaard diagrams representing $M(-3;-3)$. So the last presentation is geometric, and we can apply an obvious extension of Theorem 2.2 of [5] with $\mathbf{p} = \mathbf{n} = 3$ and $\mathbf{k} = -1$ (according to notations in the statement of Theorem 2.2). Then $M(-3;-3)$ is homeomorphic to the manifold with Seifert invariants given by $(O\ 0\ o\colon\ -1\ \ (3,m)\ (3,-q)\ (0,1))$, which is $L(3,m)\#L(3,-q)$ with $(3,m) = (3,-q) = 1$. Then we get $M(-3;-3) \cong L(3,1)\#L(3,1)$.

Lens spaces. From the classification results of [10], the surgery manifold $\mathcal{W}(m/n; p/q)$ is homeomorphic to a lens space (including $\mathbb{S}^2 \times \mathbb{S}^1$) if and only if $(m/n, p/q)$ equals one of the following pairs: $(-1, p/q)$ with $p + 6q = \pm 1$; $(-3,-4)$; $(-3+1/n,-3)$ and $(-4+1/n,-2)$ for $n \geq 1$. In these cases we have $\mathcal{W}(-1; p/q) \cong L(|p|, 4q)$ with $p + 6q = \pm 1$, $\mathcal{W}(-3;-4) \cong L(12,5)$, $\mathcal{W}(-3+1/n;-3) \cong L(9n-3, 3n-2)$, and $\mathcal{W}(-4+1/n;-2) \cong L(8n-2, 4n+1)$. For such pairs we show that $\mathcal{W}(m/n; p/q)$ and $M(m/n; p/q)$ are homeomorphic.

Case $(m/n, p/q) = (-1, p/q)$ *with* $p + 6q = \pm 1$. For $m = 1$, $n = -1$ and $p = \pm 1 - 6q$, the finite presentation of $G(m/n; p/q)$ in (1) has generators a and b and relations

$$a^{\pm 1 - 5p}ba^{-q}b^{-1}a^{q}b^{-1}a^{-q}b = 1 \tag{17}$$

and

$$a^{-2q}ba^{q}b^{-1}a^{q}b = 1. \tag{18}$$

From (18) we get

$$a^{-q}ba^{-q} = ba^{-2q}b \tag{19}$$

which becomes $a^{-3q}ba^{-3q} = a^{-2q}ba^{-2q}ba^{-2q}$, or, equivalently,

$$(a^{-2q}b)^3 = (a^{-3q}b)^2. \tag{20}$$

Substituting relation (19) into (17) and doing simplifications yield the following sequence of equivalent relations:

$$\begin{aligned} a^{\pm 1-4q}(a^{-q}ba^{-q})b^{-1}a^{q}b^{-1}a^{-q}b &= 1 \\ a^{\pm 1-4q}(ba^{-2q}b)b^{-1}a^{q}b^{-1}a^{-q}b &= 1 \\ a^{\pm 1-4q}ba^{-q}b^{-1}a^{-q}b &= 1 \\ a^{\pm 1-3q}(a^{-q}ba^{-q})b^{-1}a^{-q}b &= 1 \\ a^{\pm 1-3q}(ba^{-2q}b)b^{-1}a^{-q}b &= 1 \\ a^{\pm 1-3q}ba^{-3q}b &= 1 \end{aligned}$$

hence

$$a^{\pm 1}(a^{-3q}b)^2 = 1. \tag{21}$$

Set $x = a$ and $y = a^{-3q}b$ with inverse relations $a = x$ and $b = x^{3q}y$. By (20) and (21) the group $G(-1; p/q)$, with $p + 6q = \pm 1$, admits the finite presentation with generators x and y and relations $x^{\pm 1}y^2 = 1$ and $(x^q y)^2 x^q y^{-1} = 1$. The above Tietze transformations correspond to Singer moves on the Heegaard diagrams representing $M(-1; p/q)$ with $p + 6q = \pm 1$. So the last presentation is geometric, and we can apply Theorem 2.2 of [5] with $\mathbf{p} = \pm 1$, $\mathbf{n} = 2$, $\mathbf{m} = q$, $\mathbf{k} = 2$, and $\mathbf{q} = -1$ (according to notations in the statement of Theorem 2.2). Then $M(-1; p/q)$ with $p + 6q = \pm 1$ is the Seifert manifold defined by the invariants $(O\ 0\ o\ :\ -1\ (3,1)\ (\pm 1, q)\ (2,1))$. By [18] this space is homeomorphic to the Seifert manifold $(O\ 0\ o\ :\ b\ (\alpha_1, \beta_1)\ (\alpha_2, \beta_2))$, where $b = -q$, $(\alpha_1, \beta_1) = (2, 1)$, $\alpha_2 = 3$, and $\beta_2 = -1$ (resp. -2) if $p = -6q + 1$ (resp. $p = -6q - 1$). But the last space is homeomorphic to the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = |p|$ and $\eta = 3q - \beta_2$. Since $\eta(4q) \equiv \pm 1 \pmod{p}$, it follows that $M(-1; p/q) \cong L(|p|, 4q)$ with $p + 6q = \pm 1$, as required.

Case $(m/n, p/q) = (-3, -4)$. For $m = 3$, $n = -1$, $p = 4$ and $q = -1$, the finite presentation of $G(m/n; p/q)$ in (1) has generators a and b and relations

$$a^3bab^{-1}a^{-1}b^{-1}ab = 1 \tag{22}$$

and

$$b^2aba^{-1}b^{-1}a^{-1}ba = 1. \tag{23}$$

From (23) we get

$$aba = bab^2ab \tag{24}$$

which becomes $babab = b^2ab^2ab^2$, or, equivalently,

$$(ab)^3 = (ab^2)^3. \tag{25}$$

Substituting relation (24) into (22) and doing simplifications yield the following sequence of equivalent relations:

$$\begin{aligned} a^2(aba)b^{-1}a^{-1}b^{-1}ab &= 1 \\ a^2(bab^2ab)b^{-1}a^{-1}b^{-1}ab &= 1 \\ a^2babab &= 1 \end{aligned}$$

hence

$$a(ab)^3 = 1. \tag{26}$$

Set $x = a$ and $y = ab$ with inverse relations $a = x$ and $b = x^{-1}y$. By (25) and (26) the group $G(-3; -4)$ admits a finite presentation with generators x and y and relations $xy^3 = 1$ and $(x^{-1}y^2)^2x^{-1}y^{-1} = 1$. The above Tietze transformations correspond to Singer moves on the Heegaard diagrams representing $M(-3; -4)$. So the last presentation is geometric, and we can apply Theorem 2.2 of [5] with $\mathbf{p} = 1$, $\mathbf{n} = 3$, $\mathbf{m} = -1$, $\mathbf{k} = 2$, and $\mathbf{q} = -1$ (according to notations in the statement of Theorem 2.2). Then $M(-3; -4)$ is the Seifert manifold defined by the invariants $(O\ 0\ o\ :\ -1\ (1, -1)\ (3, 1)\ (3, 1))$. By [18] this space is homeomorphic to the Seifert manifold $(O\ 0\ o\ :\ b\ (\alpha_1, \beta_1)\ (\alpha_2, \beta_2))$, where $b = -1$, $(\alpha_1, \beta_1) = (3, -2)$ and $(\alpha_2, \beta_2) = (3, 1)$. But the last space is homeomorphic to the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = 12$ and $\eta = u\alpha_2 + v\beta_2$, with $u\alpha_1 - v(b\alpha_1 + \beta_1) = 1$. The latter equation becomes $3u + 5v = 1$, which gives $u = -3$ and $v = 2$. Substituting these values into the expression of η implies $\eta = -3\alpha_2 + 2\beta_2 = -7 \equiv 5$ mod 12. Thus we have $M(-3; -4) \cong L(12, 5)$, as required.

Case $(m/n, p/q) = (-3 + 1/n, -3)$. For $m = 3n - 1$, $p = 3$ and $q = -1$, the finite presentation of $G(m/n; p/q)$ in (1) has generators a and b and relations

$$a^2b^nab^{-n}a^{-1}b^{-n}ab^n = 1 \tag{27}$$

and

$$b^{2n-1}ab^na^{-1}b^{-n}a^{-1}b^na = 1. \tag{28}$$

From (27) we get

$$b^nab^n = ab^na^2b^na \tag{29}$$

which becomes $ab^nab^na = a^2b^na^2b^na^2$, or, equivalently,

$$(ab^n)^3 = (a^2b^n)^3. \tag{30}$$

Substituting relation (29) into (28) and doing simplifications yield the following sequence of equivalent relations:

$$b^{n-1}(b^nab^n)a^{-1}b^{-n}a^{-1}b^na = 1$$
$$b^{n-1}(ab^na^2b^na)a^{-1}b^{-n}a^{-1}b^na = 1$$
$$b^{n-1}ab^nab^na = 1$$

hence

$$b^{-1}(ab^n)^3 = 1. \tag{31}$$

Set $x = b^{-1}$ and $y = ab^n$ with inverse relations $a = yx^n$ and $b = x^{-1}$. By (30) and (31) the group $G(-3 + 1/n; -3)$ admits a finite presentation with generators x and y and relations $xy^3 = 1$ and $(x^ny^2)^2x^ny^{-1} = 1$. The above Tietze transformations correspond to Singer moves on the Heegaard diagrams representing $M(-3 + 1/n; -3)$. So the last presentation is geometric, and we can apply Theorem 2.2 of [5] with $\mathbf{p} = 1$, $\mathbf{n} = 3$, $\mathbf{m} = n$, $\mathbf{k} = 2$, and $\mathbf{q} = -1$ (according to notations in the statement of Theorem 2.2). Then $M(-3 + 1/n; -3)$ is the Seifert manifold defined by the invariants $(O\ 0\ o\ :\ -1\ (1, n)\ (3, 1)\ (3, 1))$. By [18] this space is homeomorphic to the Seifert manifold $(O\ 0\ o\ :\ b\ (\alpha_1, \beta_1)\ (\alpha_2, \beta_2))$, where $b = -n$, $(\alpha_1, \beta_1) = (3, -1)$ and $(\alpha_2, \beta_2) = (3, 2)$. But the last space is homeomorphic to the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = 9n - 3$ and $\eta = u\alpha_2 + v\beta_2$, with $u\alpha_1 - v(b\alpha_1 + \beta_1) = 1$. The latter equation becomes $3u + (3n + 1)v = 1$, which gives $u = 2n + 1$ and $v = -2$. Substituting these values into the expression of η implies $\eta = (2n + 1)\alpha_2 - 2\beta_2 = 6n - 1 \equiv -(3n - 2)$ mod $(9n - 3)$. Thus we have $M(-3 + 1/n; -3) \cong L(9n - 3, 3n - 2)$, as required.

Case $(m/n, p/q) = (-4 + 1/n, -2)$. For $m = 4n - 1$, $p = 2$ and $q = -1$, the finite presentation of $G(m/n; p/q)$ in (1) has generators a and b and relations

$$ab^nab^{-n}a^{-1}b^{-n}ab^n = 1 \tag{32}$$

and

$$b^{3n-1}ab^na^{-1}b^{-n}a^{-1}b^na = 1. \tag{33}$$

From (32) we get

$$b^n ab^n = ab^n ab^n a \tag{34}$$

which becomes $b^{2n}ab^{2n} = b^n ab^n ab^n ab^n$, or, equivalently,

$$(ab^{2n})^2 = (ab^n)^4. \tag{35}$$

Substituting relation (34) into (33) and doing simplifications yield the following sequence of equivalent relations:

$$b^{2n-1}(b^n ab^n)a^{-1}b^{-n}a^{-1}b^n a = 1$$
$$b^{2n-1}(ab^n ab^n a)a^{-1}b^{-n}a^{-1}b^n a = 1$$
$$b^{2n-1}ab^{2n}a = 1$$

hence

$$b^{-1}(ab^{2n})^2 = 1. \tag{36}$$

Set $x = b^{-1}$ and $y = ab^{2n}$ with inverse relations $a = yx^{2n}$ and $b = x^{-1}$. By (35) and (36) the group $G(-4 + 1/n; -2)$ admits a finite presentation with generators x and y and relations $xy^2 = 1$ and $(x^n y)^3 x^n y^{-1} = 1$. The above Tietze transformations correspond to Singer moves on the Heegaard diagrams representing the manifold $M(-4 + 1/n; -2)$. So the last presentation is geometric, and we can apply Theorem 2.2 of [5] with $\mathbf{p} = 1$, $\mathbf{n} = 2$, $\mathbf{m} = n$, $\mathbf{k} = 3$, and $\mathbf{q} = -1$ (according to notations in the statement of Theorem 2.2). Then $M(-4 + 1/n; -2)$ is the Seifert manifold defined by the invariants $(O\ 0\ o\ :\ -1\ (1, n)\ (2, 1)\ (4, 1))$. By [18] this space is homeomorphic to the Seifert manifold $(O\ 0\ o\ :\ b\ (\alpha_1, \beta_1)\ (\alpha_2, \beta_2))$, where $b = -n$, $(\alpha_1, \beta_1) = (2, 1)$ and $(\alpha_2, \beta_2) = (4, -1)$. But the last space is homeomorphic to the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = 8n - 2$ and $\eta = u\alpha_2 + v\beta_2$, with $u\alpha_1 - v(b\alpha_1 + \beta_1) = 1$. The latter equation becomes $2u + (2n - 1)v = 1$, which gives $u = n$ and $v = -1$. Substituting these values into the expression of η implies $\eta = n\alpha_2 - \beta_2 = 4n + 1$. Thus we have $M(-4 + 1/n; -2) \cong L(8n - 2, 4n + 1)$, as required.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

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