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Quasi-Primary Spectrum of a Commutative Ring and a Sheaf of Rings

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Abstract

In this work, the set of quasi-primary ideals of a commutative ring with identity is equipped with a topology and is called the quasi-primary spectrum. Some topological properties of this space are examined. Further, a sheaf of rings on the quasi-primary spectrum is constructed and it is shown that this sheaf is the direct image sheaf with respect to the inclusion map from the prime spectrum of a ring to the quasi-primary spectrum of the same ring.

1. Introduction

The set of all prime ideals of a commutative ring R, called the *prime spectrum* of R, denoted by $Spec(R)$, is a well-known concept in commutative algebra. This set is equipped with the famous Zariski topology, where closed sets are defined as

$$V(I) = \{P \in Spec(R): I \subseteq P\}$$

for any ideal, I of R. Topological properties of $Spec(R)$ are widely examined throughout the years and can be found in many of the standard commutative algebra and algebraic geometry references. Besides, there is a famous sheaf construction, named *the structure sheaf*, on $Spec(R)$ which is a very useful tool to connect algebraic geometry and commutative algebra. For details of the structure sheaf, the reader may consult [1-3].

In [4], the authors generalized the Zariski topology on $Spec(R)$ to the set of primary ideals of a commutative ring R, denoted by $Prim(R)$, and they called it the *primary spectrum* of R. They defined the closed sets as

$$V_{rad}(I) = \{Q \in Prim(R): I \subseteq \sqrt{Q}\}$$

for any ideal I of R where $\sqrt{Q}$ denotes the radical of Q. They showed that these closed sets satisfy axioms of a topology on $Prim(R)$. They investigated some topological properties of this space and compared them with the well-known properties of $Spec(R)$. We note that, since any prime ideal is primary and equal to its radical, the space $Spec(R)$ is in fact a subspace of $Prim(R)$.

When [4] is examined in detail, it can be realized that the given topological construction depends only on the fact that the radical of a primary ideal is prime. So, this topology is in fact valid on a much larger set, the set of ideals whose radicals are prime. These types of ideals are first introduced by L. Fuchs in [5]. He named them *quasi-primary* ideals. We aim to investigate the set of quasi-primary ideals of a commutative ring R equipped with a topology similar to the one defined in [4] and to construct a sheaf of rings on this topological space.

In Section 2, after observing certain general topological aspects of the quasi-primary spectrum, we deal with irreducibility and irreducible components of this space. Moreover, we examine the disconnectedness of the space and finally prove that the dimension of the quasi-primary spectrum of a Noetherian local ring is finite. In

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Section 3, we construct a sheaf of rings on the quasi-primary spectrum of and prove that this sheaf is actually the direct image sheaf under the inclusion map from the prime spectrum to the quasi-primary spectrum.

2. The Quasi-Primary Spectrum of a Ring

Throughout this paper, all rings are commutative with identity. In this section, we define a topology on the set of all quasi-primary ideals of a ring and examine some properties of this topological space.

First, we present some (known) properties of quasi-primary ideals we need in the rest of the paper. Let R be a ring. Following [5], an ideal I of R is called *quasi-primary* if the radical of I, denoted by $\sqrt{I}$ is prime.

Lemma 2.1 *Let I be an ideal of a ring R and S a multiplicative subset of R. Denote the localization of R with respect to S by R_S.*

i. If I is primary, then I is quasi-primary.
ii. If I is a quasi-primary ideal, the I has only one minimal prime ideal.
iii. If IR_S is a quasi-primary ideal of R_S, then $IR_S \cap R$ is a quasi-primary ideal of R.
iv. If I is a quasi-primary ideal of R such that $\sqrt{I} \cap S = \emptyset$, then IR_S is a quasi-primary ideal of R_S.

Proof. (i) and (ii) is obvious. For (iii) and (iv) it is enough to observe that $\sqrt{I}R_S = \sqrt{IR_S}$. □

Let

$$QPrim(R) = \{I \subseteq R : I \text{ is a quasi-primary ideal}\}.$$

For any subset S of R, let us define the set

$$V_q(S) = \{Q \in QPrim(R) : S \subseteq \sqrt{Q}\}.$$

Observe that, for any subset S of R, if $I = (S)$ we have $V_q(S) = V_q(I)$. If $S = \{a\}$ for $a \in R$, we write $V_q(S) = V_q(a)$.

In [4], the authors defined a topology on the set $Prim(R)$ of primary ideals of a commutative ring using the sets $V_{rad}(I) = \{Q \in Prim(R) : I \subseteq \sqrt{Q}\}$ as the closed sets. In this construction, they only used the property that a primary ideal has a prime radical. So, we realized that the topology axioms for closed sets are in fact satisfied by the sets

$$V_q(I) = \{Q \in QPrim(R) : I \subseteq \sqrt{Q}\}$$

where I any ideal of R. Thus, $QPrim(R)$ is a topological space with closed sets $V_q(I)$ where I is an ideal of R. Since any primary ideal is quasi-primary, we have $Prim(R)$ as a subspace of $QPrim(R)$.

For the sake of completeness, we note some properties (from Proposition 2.2 to Corollary 2.6) of $V_q(I)$ without proofs. For details, see [4].

Proposition 2.2 *Let I, J be ideals of R and $\{I_\lambda\}_{\lambda \in \Lambda}$ a family of ideals of R. Then the followings hold:*

i. If $I \subseteq J$, then $V_q(J) \subseteq V_q(I)$.
ii. $V_q(0) = QPrim(R)$ and $V_q(R) = \emptyset$.
iii. $V_q(I \cap J) = V_q(IJ) = V_q(I) \cup V_q(J)$.
iv. $V_q(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V_q(I_\lambda)$.
v. $V_q(I) = V_q(\sqrt{I})$.

Proof. Similar to [4, Remark 2.1] and [4, Proposition 2.3]. □

Corollary 2.3 *The family $\{V_q(I) : I \text{ is an ideal of } R\}$ satisfies the axioms of closed sets of a topology on $QPrim(R)$.*

This topology is called *Zariski topology* on $QPrim(R)$ and the space $QPrim(R)$ is named as the *quasi-primary spectrum* of R. We note that any open set in $QPrim(R)$ is of the form $QPrim(R) \setminus V_q(S)$ for some subset S of R.

Consider the set $U_a = QPrim(R) \setminus V_q(a)$ for any $a \in R$.

Theorem 2.4 *Let R be a ring. The family $\{U_a\}_{a \in R}$ is a base for the Zariski topology on $QPrim(R)$.*

Proof. Similar to [4, Theorem 3.1]. □

Note that $U_0 = \emptyset$ and $U_r = QPrim(R)$ for every unit $r \in R$.

Theorem 2.5 *Let R be a ring and $a, b \in R$. The followings hold:*

i. $\sqrt{(a)} = \sqrt{(b)}$ if and only if $U_a = U_b$.
ii. $U_{ab} = U_a \cap U_b$.
iii. $U_a = \emptyset$ if and only if a is nilpotent.
iv. U_a is quasi-compact.

Proof. Similar to [4, Theorem 3.2]. □

Corollary 2.6 *Let R be a ring. The space $QPrim(R)$ is quasi-compact.*

Quasi-primary ideals were firstly introduced and examined thoroughly in [5]. It is generally studied on rings satisfying maximal conditions; in other words, every ascending chain of ideals is finite. It is also noted that the quasi-primary ideals in rings satisfying maximal conditions can be characterized as follows: A quasi-primary ideal is either a power of a prime ideal or an intermediate ideal between two powers of one and the same prime ideal. In view of this fact, the following theorem is given for rings satisfying maximal condition.

Theorem 2.7 *[5, Theorem 4] If Q_1 and Q_2 are quasi-primary ideals having the radicals P_1 and P_2 respectively, and $P_1 \subseteq P_2$. Then Q_1Q_2 is also quasi-primary having the radical P_1.*

Theorem 2.8 *Let R be a ring satisfying maximal condition and Q_1 and Q_2 be quasi-primary ideals of R such that $Q_1 \subseteq Q_2$. If $Q_1 \in V_q(I)$, then $Q_1Q_2 \in V_q(I)$ for any ideal I of R.*

Proof. Let I be an ideal and Q_1, Q_2 two quasi-primary ideals such that $Q_1 \subseteq Q_2$ in a ring R satisfying maximal condition. Suppose $Q_1 \in V_q(I)$. Then $I \subseteq \sqrt{Q_1} \subseteq \sqrt{Q_2}$ by the assumption. So, we obtain $I \subseteq \sqrt{Q_1Q_2}$ by Theorem 2.7, which yields $Q_1Q_2 \in V_q(I)$. □

It is known that Theorem 2.7 has no analogue in primary ideal theory. Similarly, Theorem 2.8 does not valid for the primary spectrum as can be seen in the following example.

Example 1 *Consider the residue class ring $R = K[X_1, X_2, X_3]/(X_1X_3 - X_2^2)$ where K is a field. It is clear that R satisfies the maximal condition. Let x_i denote the natural image of X_i in R for each $i \in \{1,2,3\}$. Then, the ideal $P = (x_1, x_2)$ is a prime ideal of R but P^2 is not primary [6, Example 4.12]. It is trivial that P^2 is a quasi-primary ideal of R. Now take $Q_1 = Q_2 = P$. Then, we see that $P \in V_q(P) \cap V_{rad}(P)$, however, $P^2 \in V_q(P) \backslash V_{rad}(P)$.*

Now, let us determine the closure of a point $Q \in QPrim(R)$. The closure $Cl(Q)$ of Q is

$$Cl(Q) = \bigcap_{Q \in V_q(S)} V_q(S) = \bigcap_{S \subseteq \sqrt{Q}} V_q(S) = V_q(Q).$$

Definition 2.9 *A topological space X is irreducible if X is nonempty and X cannot be written as a union of two proper closed subsets, or equivalently, any two nonempty open subsets of X intersect.*

Theorem 2.10 *Let R be a ring. Then $QPrim(R)$ is an irreducible space if and only if the nilradical of R, $\mathcal{N}(R)$, is quasi-primary.*

Proof. Let $\mathcal{N}(R)$ be a quasi-primary ideal and U, V be two non-empty open subsets of $QPrim(R)$. Suppose that $Q_1 \in U \backslash V$. Then there exists a subset S of R such that $U = X \backslash V_q(S)$. This implies $Q_1 \notin V_q(S)$, that is, $S \nsubseteq \sqrt{Q_1}$. Then we get $S \nsubseteq \sqrt{\mathcal{N}(R)}$ since $\mathcal{N}(R) \subseteq \sqrt{Q_1}$. Hence, $\mathcal{N}(R) \notin V_q(S)$. Then, we obtain $\mathcal{N}(R) \in U$. In a similar way, we get $\mathcal{N}(R) \in V$ which yields $U \cap V \neq \emptyset$. For the converse part, let $QPrim(R)$ be an irreducible space and assume that $\mathcal{N}(R)$ is not quasi-primary. Then $\sqrt{\mathcal{N}(R)}$ is not prime. Then there exist $a, b \in R$ such that $a, b \notin \sqrt{\mathcal{N}(R)}$ but $ab \in \mathcal{N}(R)$. Since $a \in \sqrt{\mathcal{N}(R)}$, $V_q(a) \neq QPrim(R)$, that is, $U_a \neq \emptyset$. Similarly, we get $U_b \neq \emptyset$. In addition, $U_{ab} = \emptyset$ since $ab \in \mathcal{N}(R)$. As a result, $U_a \cap U_b = U_{ab} = \emptyset$ for two non-empty open subsets U_a and U_b which means $QPrim(R)$ is not irreducible. □

There is a one-to-one correspondence between points of $QPrim(R)$ and irreducible closed subsets of $QPrim(R)$. The next theorem gives that correspondence.

Theorem 2.11 *Let Y be a subset of $QPrim(R)$. Then Y is an irreducible closed subset of $QPrim(R)$ if and only if $Y = V_q(Q)$ for some $Q \in QPrim(R)$.*

Proof. Let $Y = V_q(Q)$ for any $Q \in QPrim(R)$. Since $V_q(Q) = Cl(Q)$ and $Cl(Q)$ is irreducible, Y is an irreducible closed subset of $QPrim(R)$. Conversely, let Y be an irreducible closed subset of $QPrim(R)$. Then $Y = V_q(I)$ for some ideal I of R. Now suppose that $I \notin QPrim(R)$. Then $\sqrt{I}$ is not prime. Then there are elements $a, b \in R$ such that $ab \in I$ but $a, b \notin \sqrt{I}$. Thus, $Y = V_q(I) \subseteq V_q(ab) = V_q(a) \cup V_q(b)$. Also, $V_q(a) \neq V_q(I)$ and $V_q(b) \neq V_q(I)$ due to $a, b \notin \sqrt{I}$. Therefore, we conclude that Y is reducible, which contradicts our assumption. □

Let I be an ideal of a ring satisfying maximal condition. Then, by [5, Theorem 5], the ideal I is an intersection of a finite number of quasi-primary ideals, say $Q_1, \ldots, Q_n$ with radicals $P_1, \ldots, P_n$, respectively. Hence, $\sqrt{I} = \cap_{i=1}^{n} \sqrt{Q_i} = \cap_{i=1}^{n} P_i$, that is, there is no prime ideal containing I other than P_i's where $i = 1, \ldots, n$. Then, for any ideal I in a ring satisfying maximal condition, every closed subset $V_q(I)$ can be written as the finite union of irreducible closed sets, that is,

$$V_q(I) = V_q(P_1) \cup \ldots \cup V_q(P_n)$$

by Proposition 2.2 (iii) and (v).

Let V be a closed subset of a topological space X. Recall that a dense point of V is called a *generic point*. By the above theorem, we conclude that every irreducible closed subset of $QPrim(R)$ has a generic point.

The maximal irreducible subsets of a topological space X are called *irreducible components*.

Theorem 2.12 *Irreducible components of* $QPrim(R)$ *are the closed sets* $V_q(Q)$ *where* $\sqrt{Q}$ *is a minimal prime ideal of* R.

Proof. By Theorem 2.11, any irreducible closed subset of $QPrim(R)$ can be written of the form $V_q(Q)$ for some quasi-primary ideal Q of R. Assume that $V_q(Q)$ is not maximal. Then $V_q(Q) \subset V_q(Q')$ for some quasi-primary ideal Q' of R. Since $V_q(Q') = V_q(\sqrt{Q'})$ we have $\sqrt{Q'} \subset \sqrt{Q}$. Hence, $\sqrt{Q}$ is not minimal. Conversely, assume that $\sqrt{Q}$ is not a minimal prime ideal. Then there is a prime ideal P of R such that $P \subset \sqrt{Q}$. Then we get $V_q(Q) \subset V_q(P)$. Therefore, $V_q(Q)$ is not a maximal irreducible set. □

Theorem 2.13 *Let* R *be a ring. The following are equivalent:*

i. $QPrim(R)$ is disconnected.
ii. $R \cong R_2 \times R_2$ where R_1 and R_2 are nonzero rings.
iii. R contains an idempotent.

Proof. (i)⟹(ii) Assume that $QPrim(R)$ is disconnected. Then $QPrim(R) = V_q(I) \cap V_q(J)$ for some ideals I and J of R where $V_q(I) \cap V_q(J) = \emptyset$. Then we have $I + J = R$ and $I \cap J = IJ$. So, we get $R = R/I \times R/J$.
(ii) ⟹ (iii) Assume that $R \cong R_2 \times R_2$ where R_1 and R_2 are nonzero rings via an isomorphism ϕ. Then $\phi^{-1}(1,0)$ is a nontrivial idempotent of R.
(iii) ⟹ (i) Assume that $e \in R$ is an idempotent. Then $QPrim(R) = X_1 \cup X_2$ where $X_1 = \{Q \in QPrim(R): e \in \sqrt{Q}\}$ and $X_1 = \{Q \in QPrim(R): 1 - e \in \sqrt{Q}\}$. Observe that $X_1 \cap X_2 = \emptyset$. Thus, $QPrim(R)$ is disconnected. □

The *dimension* of a topological space X is the number n such that X has a chain of irreducible closed sets

$$V_1 \subset V_2 \subset \cdots \subset V_n$$

and no such chain more that n terms.

Theorem 2.14 *Let* R *be a Noetherian local ring. Then the dimension of* $QPrim(R)$ *is finite.*

Proof. Let

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

be a chain of irreducible subsets. This chain can be written as

$$V_q(Q_1) \subset V_q(Q_2) \subset \cdots \subset V_q(Q_n) \subset \cdots$$

where $Q_i \in QPrim(R)$. Let $P_i = \sqrt{Q_i}$ for each i. Then we have

$$\cdots \subset P_n \subset \cdots \subset P_2 \subset P_1.$$

By [7, Corollary 11.11], the dimension of R is finite. So, the above chain of prime ideals must terminate. Therefore, the dimension of $QPrim(R)$ is finite, and in fact equal to the dimension of R. □

3. A Sheaf of Rings on the Quasi-Primary Spectrum

In this section we define a sheaf of rings on the quasi-primary spectrum. Let $\phi: R \to R'$ be a ring homomorphism. For any $Q \in QPrim(R')$, it is easy to show that $f^{-1}(Q) \in QPrim(R)$. So, f induces the map

$$\phi^a: QPrim(R') \to QPrim(R)$$

which is called the *associated map* of ϕ.

For any $A \subseteq R$ we have $(\phi^a)^{-1}(V(A)) = V(\phi(A))$. So, the map ϕ^a is continuous.

Let $S \subseteq R$ be a multiplicative subset of R. Let $\phi: R \to R_S$ be the canonical homomorphism. Since $\sqrt{IR_S} = \sqrt{I}R_S$ for any ideal I of R, the map ϕ^a is an inclusion. The set $U_S = \phi^a(QPrim(R_S))$ is equal to the set of quasi-primary ideals of R whose radicals are disjoint from S. There is a one-to-one correspondence between quasi-primary ideals of R_S and quasi-primary ideals of R whose radicals are disjoint from S. So, the space $QPrim(R_S)$ is homeomorphic to the subspace U_S of $QPrim(R)$.

In particular, if $S = \{f^i: i \in \mathbb{N}\}$ then $U_S = QPrim(R) \backslash V_q(f) = U_f$. So, basis sets U_f are homeomorphic to $QPrim(R_f)$ where R_f is the localization of R with respect to the multiplicative subset $S = \{f^i: i \in \mathbb{N}\}$.

Lemma 3.1 $U_a \subseteq U_b$ *if and only if* $a \in \sqrt{(b)}$ *for any* $a, b \in R$.

Proof. Assume that $U_a \subseteq U_b$ for some $a, b \in R$. Then, for any $Q \in QPrim(R)$, we have $a \notin \sqrt{Q}$ implies $b \notin \sqrt{Q}$. That means $b \in \sqrt{Q}$ implies $a \in \sqrt{Q}$. Since $QPrim(R)$ contains prime ideals, this observation yields that $a \in \sqrt{(b)}$. Conversely, assume that $a \in \sqrt{(b)}$ for some $a, b \in R$. Let $q \in U_a$. Then $a \notin \sqrt{Q}$. Since a is contained in the intersection of all prime ideals that contain b, we obtain that $b \notin \sqrt{Q}$. Therefore, we have $Q \in U_b$. □

Our aim is to construct a sheaf of rings on $QPrim(R)$. We assign to each open set U_a the ring $\mathcal{F}(U_a) := R_a$, ring of quotients with respect to the multiplicative subset $\{1, a, a^2, \dots\}$, and define the restriction maps

$$res_{U_b,U_a}: R_b \to R_a, \quad r/b^m \mapsto t^m r/a^{nm}.$$

Since, by Lemma 3.1, we have $U_a \subseteq U_b$ if and only if $a^n = tb$ for some positive integer n and $t \in R$, the map res_{U_b,U_a} is well-defined.

For an arbitrary open set U of $QPrim(R)$ let

$$\mathcal{F}(U) = \varprojlim \mathcal{F}(U_a)$$

where the projective limit is taken over all $U_a \subseteq U$ relative to the system of homomorphisms res_{U_b,U_a} for $U_a \subseteq U_b$.

For $U \subseteq V$, each family $\{v_i\} \in \mathcal{F}(V)$ consisting of $v_i \in R_{a_i}$ with $U_{a_i} \subseteq V$ defines a subfamily $\{a_j\}$ consisting of the a_j for those indexes j with $U_{a_j} \subseteq U$. Then $\{v_j\} \in \mathcal{F}(U)$. Define

$$res_{V,U}: \mathcal{F}(V) \to \mathcal{F}(U), \quad \{v_i\} \mapsto \{v_j\} .$$

With this construction, $\mathcal{F}$ turns to be a sheaf of rings on $QPrim(R)$. In fact, this sheaf is the direct image sheaf under the inclusion map from $Spec(R)$ into $QPrim(R)$:

Theorem 3.2 *The sheaf $\mathcal{F}$ on $QPrim(R)$ is equal to the direct image sheaf ι_* under the inclusion map $\iota: Spec(R) \to QPrim(R)$.*

Proof. The inclusion map ι is continuous. For any open set U of $QPrim(R)$, direct image sheaf ι_* is defined as follows:

$$\iota_*(U) = \mathcal{O}(\iota^{-1}(U))$$

where $\mathcal{O}$ denotes the structure sheaf on $Spec(R)$. For $a \in R$, we have

$$\begin{aligned} \iota^{-1}(U_a) &= \{P \in SpecR: \iota(P) \in U_a\} \\ &= \{P \in SpecR: P \in U_a\} \\ &= \{P \in SpecR: a \notin \sqrt{P} = P\} \end{aligned}$$

The final set is a principal open set for $Spec(R)$ and the corresponding ring for this set is R_a. So, we get $\iota_*(U_a) = R_a = \mathcal{F}(U_a)$.

For $U_a \subseteq U_b$, we have $res_{U_b,U_a} = \rho^{X_b}_{X_a}$ where $\rho^{X_b}_{X_a}$ is the restriction map from principal open set X_b to X_a of $Spec(R)$ with respect to the structure sheaf $\mathcal{O}$. Thus, the sheafs $\mathcal{F}$ and $\iota_*\mathcal{F}$ are the same. □

Similar to the structure sheaf $\mathcal{O}$ on $Spec(R)$, the stalk $\mathcal{F}_Q$ of $\mathcal{F}$ at a point $Q \in QPrim(R)$ is $R_{\sqrt{Q}}$. Therefore, we conclude that $(QPrim(R), \mathcal{F})$ is a locally ringed space.

Declaration of Ethical Standards

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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