Universal Journal of Mathematics and Applications, 5 (3) (2022) 107-116
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1105072

Numerical Stability of Runge-Kutta Methods for Differential Equations with Piecewise Constant Arguments with Matrix Coefficients

Hefan Yin[1*] and Qi Wang[1]

[1] *School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, 510006, China*
[*] *Corresponding author*

Article Info

Keywords: *Runge-Kutta methods, Analytical stability, Numerical stability.*
2010 AMS: *65L07, 65L20.*
Received: *18 April 2022*
Accepted: *15 September 2022*
Available online: *30 September 2022*

Abstract

The paper discusses the analytical stability and numerical stability of differential equations with piecewise constant arguments with matrix coefficients. Firstly, the Runge-Kutta method is applied to the equation and the recurrence relationship of the numerical solution of the equation is obtained. Secondly, it is proved that the Runge-Kutta method can preserve the convergence order. Thirdly, the stability conditions of the numerical solution under different matrix coefficients are given by Padé approximation and order star theory. Finally, the conclusions are verified by several numerical experiments.

1. Introduction

With more and more research on delay differential equations [1–5], it has been widely used in various fields, such as population research [6,7], epidemiology [8,9], electrodynamics [10,11], neural network system [12,13] and so on. As a special class of delay differential equations, differential equations with piecewise constant arguments (EPCA) are difficult to solve accurately because of their complex structure. Therefore, a series of numerical methods are introduced, for example, Euler method [14], improved Euler method [15] and Runge-Kutta methods [16]. Meanwhile, scholars also paid more attention to the study of the properties of the numerical solution of EPCA. Gao [17] considered the numerical oscillation and non-oscillation for EPCA. It is proved that oscillation of the analytic solution is preserved by the Runge-Kutta methods under some conditions. The conditions under which the non-oscillation of analytic solution is preserved by the Runge-Kutta methods were obtained. In [18], Wang introduced the condition that Runge-Kutta methods was called $BN_f - stable$ and established two classes of Runge-Kutta methods which satisfied the condition. It is shown that a class of Runge-Kutta methods can preserve their original convergence order for EPCA but not the other class of Runge-Kutta methods. At the end of the paper, the asymptotic stability results of Runge-Kutta methods are obtained. In [19], a class of linear impulsive EPCA were considered. From the paper, it is proved that the θ-methods preserved stability of the equations. Stability conditions of Runge-Kutta methods were obtained. Liang et al. [20] considered the conditions of numerical stability of EPCA with matrix coefficients through the Runge-Kutta methods. The different types of matrix coefficient L are classified and discussed in [20]. Different from [20], we will consider a more complex equation and obtain new conclusions in this work.
This paper deals with the stability of the numerical solution of the following EPCA with matrix coefficients:

$$
\begin{aligned}
&x'(t) = Lx(t) + Mx\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\
&x(0) = x_0,
\end{aligned}
\tag{1.1}
$$

where $[\cdot]$ designates the greatest-integer function, $L, M \in \mathbb{C}^{d\times d}, L$ is nonsingular, and $x_0 \in \mathbb{C}^d$ is a given initial value. The general form of

Email addresses and ORCID numbers: yinhefan@126.com, 0000-0001-7182-3272 (H. Yin), bmwzwq@126.com, 0000-0003-3578-2551 (Q. Wang),

this type equation is

$$
\begin{aligned}
&x'(t) = f(t, x(t), x(\alpha(t))), t \geq 0, \\
&x(0) = x_0,
\end{aligned}
\tag{1.2}
$$

where the argument $\alpha(t)$ has intervals of constancy. Because the argument deviation of Eq. (1.1) is positive in $\left[n, n+\frac{1}{2}\right)$ and negative in $\left[n+\frac{1}{2}, n+1\right)$, Eq. (1.1) is also said to be of alternately advanced and retarded type.

2. Analytical Stability

Definition 2.1 ([21]). *A solution of Eq. (1.1) on $[0,\infty)$ is a function $x(t)$ satisfies the conditions:*
(i) $x(t)$ is continuous on $[0,\infty)$;
(ii) The derivative $x'(t)$ exists at each point $t \in [0,\infty)$, with the possible exception of the points $t = n+\frac{1}{2}, n = 1,2,\ldots$, where one-sided derivatives exist;
(iii) Eq. (1.1) is satisfied on $\left[0,\frac{1}{2}\right)$ and each interval $\left[n-\frac{1}{2}, n+\frac{1}{2}\right)$ for $n = 1,2,\ldots$

Theorem 2.2 ([21]). *Eq. (1.1) has a unique solution on $[0,\infty)$*

$$
x(t) = M(T(t))M_0^{\left[t+\frac{1}{2}\right]}x_0,
\tag{2.1}
$$

where $T(t) = t - \left[t+\frac{1}{2}\right], M(t) = e^{Lt} + \left(e^{Lt} - I\right)L^{-1}M, M_0 = M\left(-\frac{1}{2}\right)^{-1}M\left(\frac{1}{2}\right)$.

Definition 2.3. *The zero solution of Eq. (1.1) is asymptotically stable if any solution $x(t)$ of Eq. (1.1) satisfies*

$$
\lim_{t\to\infty} x(t) = 0.
$$

Lemma 2.4 ([21]). *The zero solution of Eq. (1.1) is asymptotically stable, if and only if the eigenvalues λ_j $(j = 1,\cdots,r)$ of the matrix M_0 satisfy the inequality $|\lambda_j| < 1$.*

From [22], we suppose that $\|\cdot\|$ denotes the matrix norm derived from a vector norm on $\mathbb{C}^d$ and $\mu[\cdot]$ denotes the logarithmic norm of the matrix which defined by

$$
\mu[L] = \lim_{\Delta\to 0^+} \frac{\|I_d + \Delta L\| - 1}{\Delta},
$$

where I_d is the $d \times d$ identity matrix.

Theorem 2.5. *The zero solution of Eq. (1.1) is asymptotically stable if*
(i) $\mu[L] < 0$;
(ii) $\|M\| < -\mu[L]$.

Proof. Suppose that $x\left(n-\frac{1}{2}\right) = c_n, x\left(n+\frac{1}{2}\right) = c_{n+1}$ for interval $\left[n-\frac{1}{2}, n+\frac{1}{2}\right]$. According to Eq. (1.1), we use the method of constant variation, let

$$
x(t) = a(t)e^{Lt},
\tag{2.2}
$$

so

$$
x'(t) = a'(t)e^{Lt} + a(t)Le^{Lt},
\tag{2.3}
$$

$$
a\left(n-\frac{1}{2}\right) = c_n e^{-L\left(n-\frac{1}{2}\right)},
\tag{2.4}
$$

then we substitute Eq. (2.3) into Eq. (1.1) and obtain

$$
a'(t) = e^{-Lt}Mc_n,
\tag{2.5}
$$

integrate both sides of Eq. (2.5), we have

$$
\int_{n-\frac{1}{2}}^{t} a'(t)dt = Mc_n \int_{n-\frac{1}{2}}^{t} e^{-Lt}dt,
\tag{2.6}
$$

i.e.,

$$
a(t) - a\left(n-\frac{1}{2}\right) = \left(e^{-L\left(n-\frac{1}{2}\right)} - e^{-Lt}\right)L^{-1}Mc_n,
\tag{2.7}
$$

in other words

$$
a(t) = \left(e^{-L\left(n-\frac{1}{2}\right)} - e^{-Lt}\right)L^{-1}Mc_n + c_n e^{-L\left(n-\frac{1}{2}\right)},
\tag{2.8}
$$

therefore

$$x(t)=\left(e^{L\left(t-n+\frac{1}{2}\right)}-I\right)L^{-1}Mc_n+c_ne^{L\left(t-n+\frac{1}{2}\right)}. \tag{2.9}$$

Let $t=n+\frac{1}{2}$, we have

$$c_{n+1}=\left(e^L+\left(e^L-I\right)L^{-1}M\right)c_n, \tag{2.10}$$

i.e.,

$$\lambda=\frac{c_{n+1}}{c_n}=e^L+\left(e^L-I\right)L^{-1}M. \tag{2.11}$$

Now we just need to prove $\|\lambda\|<1$. From (2.11), we have

$$\|\lambda\|=\left\|e^L+\left(e^L-I\right)L^{-1}M\right\|\le\left\|e^L\right\|+\left\|\left(e^L-I\right)L^{-1}\right\|\|M\|, \tag{2.12}$$

by the condition (i), we know that $\mu[L]\neq 0$, so

$$\left\|\left(e^L-I\right)L^{-1}\right\|=\left\|\int_0^1 e^{Ls}ds\right\|\le\int_0^1\left\|e^{Ls}\right\|ds\le\int_0^1 e^{\mu[L]s}ds=\frac{1}{\mu[L]}\left(e^{\mu[L]}-1\right).$$

Noting that $e^{\mu[L]}-1$ and $\mu[L]$ have the same sign, by the condition (ii), we have

$$\|\lambda\|\le\left\|e^L\right\|+\left\|\left(e^L-I\right)L^{-1}\right\|\|M\|\le e^{\mu[L]}+\frac{\|M\|}{\mu[L]}\left(e^{\mu[L]}-1\right)<e^{\mu[L]}-\left(e^{\mu[L]}-1\right)=1.$$

By Lemma 2.4, the proof is completed. □

3. Runge-Kutta Methods and Convergence

In this section, we consider the Runge-Kutta methods (A,B,C) to solve the given equation.The following is the form of the Butcher column of the Runge-Kutta methods:

$$\begin{array}{c|c} C & A \\ \hline & B^T \end{array}$$

where the matrix $A=\{a_{ij}\}$, the weight vector B^T with $B_1+B_2+\cdots+B_v=1$ and the knot vector C with $0\le C_1\le C_2\le\cdots\le C_v\le 1$. Let $h=1/2m$ be a given stepsize with integer $m\ge 1$ and the gridpoints t_n be defined by $t_n=nh(n=0,1,2\cdots)$. Applying the Runge-Kutta methods to Eq. (1.2) leads to a numerical process of the following type, generating approximations $x_1,x_2,x_3,\cdots$ to the exact solution $x(t)$ of Eq. (1.2) at the gridpoints $t_n(n=1,2,3,\cdots)$

$$\begin{aligned} x_{n+1}&=x_n+h\sum_{i=1}^{v}B_i\left(Ly_i^{(n)}+Mz_i^{(n)}\right),\\ y_i^{(n)}&=x_n+h\sum_{j=1}^{v}a_{ij}\left(Ly_j^{(n)}+Mz_j^{(n)}\right),\quad i=1,2,\cdots,v, \end{aligned} \tag{3.1}$$

where x_n is the numerical approximation to $x(t)$ at $t_n,y_i^{(n)}$ and $z_i^{(n)}$ are the numerical approximations to $x(t_n+C_ih)$ and $x\left(\left[t_n+C_ih+\frac{1}{2}\right]\right)$, respectively. If we denote $n=2km+l$, $L(k)=\{0,1,\cdots,m-1\}$ for $k=0$ and $L(k)=\{-m,-m+1,\cdots,m-2,m-1\}$ for $k\ge 1$, then $z_i^{(2km+l)}$ can be defined as x_{2km} according to Definition 2.1. Let $Y^{(n)}=\left(\left(y_1^{(n)}\right)^T,\left(y_2^{(n)}\right)^T,\cdots,\left(y_V^{(n)}\right)^T\right)^T$, then Eq. (3.1) reduces to

$$\begin{aligned} x_{2km+l+1}&=x_{2km+l}+h\left(B^T\otimes L\right)Y^{(2km+l)}+hMx_{2km},\quad l\in L(k),\\ Y^{(2km+l)}&=(e\otimes I_d)x_{2km+l}+h(A\otimes L)Y^{(2km+l)}+h(Ae\otimes M)x_{2km}, \end{aligned} \tag{3.2}$$

where $e=(1,1,\cdots,1)^T$, $A=\left(a_{ij}\right)_{v\times v}$, $B=(B_1,B_2,\cdots,B_v)^T$ and $\otimes$ denotes the Kronecker product.
Therefore, we have

$$x_{2km+l+1}=R(Z)x_{2km+l}+\varphi(Z,Y)x_{2km},\quad l\in L(k), \tag{3.3}$$

where $Z=hL$, $Y=hM$, $\varphi(Z,Y)=\left(\mathrm{B}^T\otimes Z\right)(I_{vd}-A\otimes Z)^{-1}(Ae\otimes Y)+Y$ and $R(Z)=I_d+\left(\mathrm{B}^T\otimes Z\right)(I_{vd}-A\otimes Z)^{-1}(e\otimes I_d)$ is the stability function of the Runge-Kutta methods.
Let the Runge-Kutta methods be of order q, then there is a constant K such that for sufficiently small h [23]- [24],

$$|e^Z-R(Z)|\le Kh^{q+1}, \tag{3.4}$$

we have

$$e^{-Lt_{2km+l+1}}x(t_{2km+l+1})=e^{-Lt_{2km+1}}x(t_{km+l})+\left(e^{-Lt_{2km+1}}-e^{-Lt_{2km+t+1}}\right)L^{-1}Mx(t_{2km}). \tag{3.5}$$

So

$$x(t_{2km+l+1}) = e^{Lh}x(t_{km+l}) + \left(e^{Lh} - I_d\right) L^{-1}Mx(t_{2km}),$$

i.e.,

$$x(t_{2km+l+1}) = e^{Z}x(t_{km+l}) + \left(e^{Z} - I_d\right) Z^{-1}Yx(t_{2km}). \tag{3.6}$$

Moreover

$$\begin{aligned}
\varphi(Z,Y) &= \left(\mathrm{B}^T \otimes Z\right)(I_{vd} - A\otimes Z)^{-1}(Ae\otimes Y) + Y \\
&= \left(\left(\mathrm{B}^T \otimes Z\right)(I_{vd} - A\otimes Z)^{-1}(Ae\otimes I_d) + I_d\right) Y \\
&= \left(\mathrm{B}^T \otimes Z\right)(I_{vd} - A\otimes Z)^{-1}\left((Ae\otimes I_d) + (I_{vd} - A\otimes Z)\left(e\otimes Z^{-1}\right)\right) Y \\
&= \left(\mathrm{B}^T \otimes Z\right)(I_{vd} - A\otimes Z)^{-1}\left(e\otimes Z^{-1}\right) Y \\
&= (R(Z) - I_d)Z^{-1}Y.
\end{aligned}$$

From Eq. (3.4) and Eq. (3.6), if $x(t_{2km}) = x_{2km}$ and $x(t_{2km+l}) = x_{2km+l}$, then

$$\|x(t_{2km+l+1}) - x_{2km+l+1}\| = \left\|\left(e^{Z} - R(Z)\right)\left(x(t_{2km+l}) + Z^{-1}Yx(t_{2km})\right)\right\| \le Kh^{q+1}\left(1 + \left\|Z^{-1}Y\right\|\right) \max_{k-\frac{1}{2}\le t\le k+\frac{1}{2}} |x(t)|, \tag{3.7}$$

which implies that for Eq. (1.1) the Runge-Kutta method is also convergent of order q.

4. Numerical Stability

4.1. The general asymptotic stability

In this section, we will study the conditions of numerical stability for any initial value. We introduce the set Σ consisting of all pairs (L,M), which satisfies Theorem 2.5, i.e., $\Sigma = \left\{(L,M) \in \mathbb{C}^{d\times d} \times \mathbb{C}^{d\times d} : \mu[L] < 0, \|M\| < -\mu[L]\right\}$. From Eq. (3.4), we obtain

$$\begin{pmatrix} x_{2km} \\ x_{2km+1} \\ \vdots \\ x_{2km+m-1} \\ x_{(2k+1)m} \end{pmatrix} = T_1 \begin{pmatrix} x_{2km-1} \\ x_{2km} \\ \vdots \\ x_{2km+m-2} \\ x_{2km+m-1} \end{pmatrix}, \begin{pmatrix} x_{2km-1} \\ x_{2km} \\ \vdots \\ x_{2km+m-2} \\ x_{2km+m-1} \end{pmatrix} = T_2 \begin{pmatrix} x_{2km-2} \\ x_{2km-1} \\ \vdots \\ x_{2km+m-3} \\ x_{2kmm+m-2} \end{pmatrix}, \cdots, \begin{pmatrix} x_{(2k-1)m+1} \\ x_{(2k-1)m+2} \\ \vdots \\ x_{2km} \\ x_{2km+1} \end{pmatrix} = T_m \begin{pmatrix} x_{(2k-1)m} \\ x_{(2k-1)m+1} \\ \vdots \\ x_{2km-1} \\ x_{2km} \end{pmatrix}.$$

where

$$T_1 = \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & I_d \\ 0 & \varphi(Z,Y) & 0 & \cdots & R(Z) \end{pmatrix}, T_2 = \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & I_d \\ 0 & 0 & \varphi(Z,Y) & \cdots & R(Z) \end{pmatrix},$$

$$T_m = \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & I_d \\ 0 & 0 & 0 & \cdots & \varphi(Z,Y) + R(Z) \end{pmatrix}.$$

Let $X_{2k} = \left(x_{2km}^T, x_{2km+1}^T, \cdots, x_{2km+m}^T\right)^T$ and $T = \prod_{i=1}^m T_i$, we obtain

$$X_{2k} = TX_{2k-1}, \quad k = 1, 2, \cdots, \tag{4.1}$$

where

$$T = \begin{pmatrix} 0 & \cdots & 0 & B_{1,m+1} \\ 0 & \cdots & 0 & B_{2,m+1} \\ \vdots & \ldots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{m+1,m+1} \end{pmatrix},$$

$$B_{i,m+1} = I_d + \left(R(Z)^{i-1} - I_d\right)\left(I_d + Z^{-1}Y\right).$$

Definition 4.1. *Process (4.1) for Eq. (1.1) is called asymptotically stable at (L,M) if and only if for all $h=1/2m$,*
(i) $I_{vd}-A\otimes Z$ is invertible,
(ii) for any given $x_i(1\le i\le m)$, process (4.1) defines $X_{2k}(k=1,2\cdots)$ that satisfy $X_{2k}\to 0$ for $k\to\infty$.

Definition 4.2. *The set of all pairs (L,M) at which the process (4.1) for Eq. (1.1) is asymptotically stable for all $h=1/2m$ is called the asymptotical stability region denoted by S, i.e., $S=\left\{(L,M):\rho\left(R(Z)^m+(R(Z)^m-I_d)Z^{-1}Y\right)<1\right\}$.*

Lemma 4.3 ([20]). *If the Runge-Kutta method is A-stable and $\mu[L]<0$, then for any integer m,*

$$\frac{1-\|R(Z)^m\|}{\left\|(I_d-R(Z)^m)Z^{-1}\right\|}\ge\frac{1-\|R(Z)\|}{\left\|(I_d-R(Z))Z^{-1}\right\|}. \tag{4.2}$$

Theorem 4.4. *If the Runge-Kutta method is A-stable, for all Z with $\mu[Z]<0$,*

$$\|R(Z)\|-\mu[Z]\left\|(I_d-R(Z))Z^{-1}\right\|\le 1,$$

then $\Sigma\subset S$.

Proof. Let $(L,M)\in\Sigma$. By Lemma 4.3, we obtain

$$\begin{aligned}
\left\|R(Z)^m+(R(Z)^m-I_d)Z^{-1}Y\right\| &\le \|R(Z)^m\|+\left\|(R(Z)^m-I_d)Z^{-1}\right\|\|Y\|\\
&\le \frac{\|R(Z)^m\|(1-\|R(Z)\|)+\left\|(R(Z)^m-I_d)Z^{-1}\right\|\|Y\|(1-\|R(Z)\|)}{1-\|R(Z)\|}\\
&= \frac{\|R(Z)^m\|(1-\|R(Z)\|)+\left\|(R(Z)^m-I_d)Z^{-1}\right\|\|Y\|(1-\|R(Z)\|)}{1-\|R(Z)\|}-1+1\\
&= \frac{(\|R(Z)^m\|-1)(1-\|R(Z)\|)+(1-\|R(Z)^m\|)\left\|(I_d-R(Z))Z^{-1}\right\|\|Y\|}{1-\|R(Z)\|}+1\\
&< \frac{(1-\|R(Z)^m\|)\left(\|R(Z)\|-1-\mu[Z]\left\|(I_d-R(Z))Z^{-1}\right\|\right)}{1-\|R(Z)\|}+1\\
&\le 1,
\end{aligned}$$

therefore, $\rho\left(R(Z)^m+(R(Z)^m-I_d)Z^{-1}Y\right)<1$. By Definition 4.2, the proof is completed. □

4.2. In the case of 2-norm and L being a normal matrix

In this section, we suppose that L is a normal matrix, i.e., $LL^*=L^*L$, with eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_d$ and $\|\cdot\|$ denotes the spectral norm, i.e.,

$$\|L\|=\max\left\{\sqrt{\lambda}:\lambda \text{ is an eigenvalue of } L^*L\right\}.$$

Lemma 4.5 ([20]). *If the Runge-Kutta method is A-stable and (L,M) satisfies Theorem 2.5, then $x_n\to 0$ as $n\to\infty$ if and only if*

$$|R(\aleph_i)|-\mathrm{Re}(\aleph_i)\left|\frac{1-R(\aleph_i)}{\aleph_i}\right|\le 1,\quad i=1,2,\cdots,d,$$

where $\aleph_i=h\lambda_i$ and λ_i is an eigenvalue of L.

Let $\Sigma_1=\{(L,M):(L,M)\in\Sigma$ and L is a normal matrix$\}$.

Theorem 4.6. *If the Runge-Kutta method is A-stable, then $\Sigma_1\subset S$ if and only if for all $\aleph$ with $\mathrm{Re}(\aleph)<0$,*

$$|R(\aleph)|-\mathrm{Re}(\aleph)\left|\frac{1-R(\aleph)}{\aleph}\right|\le 1.$$

Proof. By Lemma 4.5, for A-stable Runge-Kutta method, $\Sigma_1\subset S$ if and only if for $\aleph_i=h\lambda_i$ and λ_i being an eigenvalue of L,

$$|R(\aleph_i)|-\mathrm{Re}(\aleph_i)\left|\frac{1-R(\aleph_i)}{\aleph_i}\right|\le 1,\quad i=1,2,\cdots,d,$$

then the proof is completed by the arbitrariness of L. □

4.3. In the case of 2-norm and L being a real symmetric matrix

In this section, we suppose that L is a real symmetric matrix and $\|\cdot\|$ denotes the spectral norm. Let

$$\Sigma_2 = \{(L,M) : (L,M) \in \Sigma \text{ and } L \text{ is a real symmetric matrix}\}.$$

Theorem 4.7. *If the Runge-Kutta method is A-stable, then $\Sigma_2 \subset S$ if and only if $0 \le R(\aleph) \le 1$ for all $\aleph < 0$.*

Proof. It is obvious that L is a normal matrix, in the view of Theorem 4.6, $\Sigma_2 \subset S$ if and only if for all $\aleph < 0$,

$$|R(\aleph)| - \mathrm{Re}(\aleph)\left|\frac{R(\aleph)-1}{\aleph}\right| \le 1,$$

the eigenvalues of real symmetric matrices must be real numbers, so we have

$$|R(\aleph)| - \mathrm{Re}(\aleph)\left|\frac{R(\aleph)-1}{\aleph}\right| = |R(\aleph)| + |R(\aleph)-1| \le 1,$$

which is equivalent to $0 \le R(\aleph) \le 1$. □

Corollary 4.8. *Suppose that the Runge-Kutta method is A-stable and $R(\aleph)$ is the (r,s)-Padé approximation to $e^{\aleph}$, then r is odd if and only if $0 < R(\aleph) \le e^{\aleph}$ for all $\aleph < 0$.*

Proof. According to Lemma 4.3 in [24], $0 < R(\aleph) \le e^{\aleph}$ for all $\aleph < 0$ if and only if the negative real axis is contained in a white sector in the left-half plane, which is the same as to r is odd. The proof is completed. □

Theorem 4.9. *If Runge-Kutta method is A-stable and $R(\aleph)$ is the (r,s)-Padé approximation to $e^{\aleph}$, then $\Sigma_2 \subset S$ if and only if r is odd.*

Proof. By Corollary 4.8, $0 < R(\aleph) \le e^{\aleph} < 1$ for all $\aleph < 0$ if and only if r is odd, then for A-stable Runge-Kutta method, $0 < R(\aleph) < 1$ for all $\aleph < 0$ if and only if r is odd. According to Theorem 4.7, the proof is completed. □

For the higher order Runge-Kutta methods, their stability conclusions are shown in Table 1.

	Gauss-Legendre	*Radau IA, IIA*	*Lobatto IIIA, IIIB*	*Lobatto IIIC*
(r,s)	(v,v)	$(v-1,v)$	$(v-1,v-1)$	$(v-2,v)$
$\Sigma_2 \subset S$	v is odd	v is even	v is even	v is odd

Table 1: The higher order Runge-Kutta methods

5. Numerical Experiments

In this section, we give some examples to verify the conclusions in the paper. Four Runge-Kutta methods are used: 2-Gauss-Legendre, 2-Radau IA, 3-Lobatto IIIB and 3-Lobatto IIIC. Their Butcher columns are listed as follows:
2-Gauss-Legendre:

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

2-Radau IA:

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

3-Lobatto IIIB:

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 1 & \frac{1}{6} & \frac{5}{6} & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

3-Lobatto IIIC:

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

For Theorem 4.4, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -15 & -6 & -9 \\ 1 & -3 & 0 \\ 1 & -1 & -10 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 0 & -1 \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.1}$$

where L is a matrix with $\mu[L] \approx -2.4862 < 0$ and $\|M\| \approx 2.4495 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For all Z with $\mu[Z] \approx -0.0249 < 0$, we obtain Theorem 4.4 is satisfied. We plot the numerical solution of 2-Gauss-Legendre for Eq. (5.1) with $m = 50$ in Figure 5.1 and we obtain that the numerical solution is asymptotically stable.

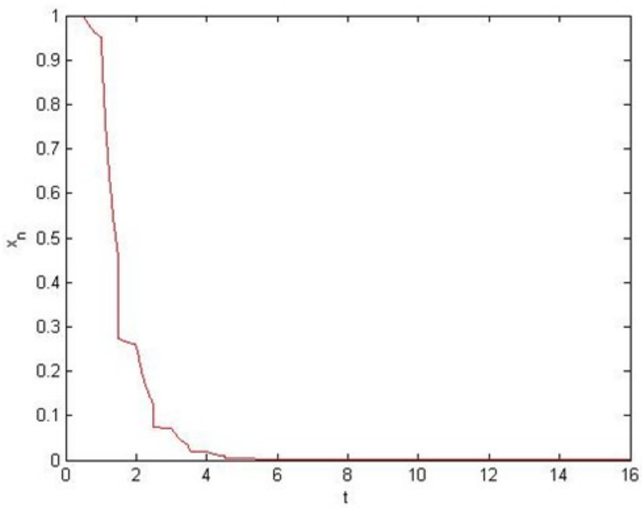

Figure 5.1: 2-Gauss-Legendre solution for Eq. (5.1) with $m = 50$.

For Theorem 4.6, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.2}$$

where L is a normal matrix with $\mu[L] = -1 < 0$ and $\|M\| \approx 0.7071 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.02 \\ -0.01 \end{pmatrix}$ with $\mathrm{Re}(\aleph) = \begin{pmatrix} -0.02 \\ -0.01 \end{pmatrix} < 0$, we obtain Theorem 4.6 is satisfied. We plot the numerical solution of 2-Radau IA for Eq. (5.2) with $m = 50$ in Figure 5.2 and we obtain that the numerical solution is asymptotically stable.

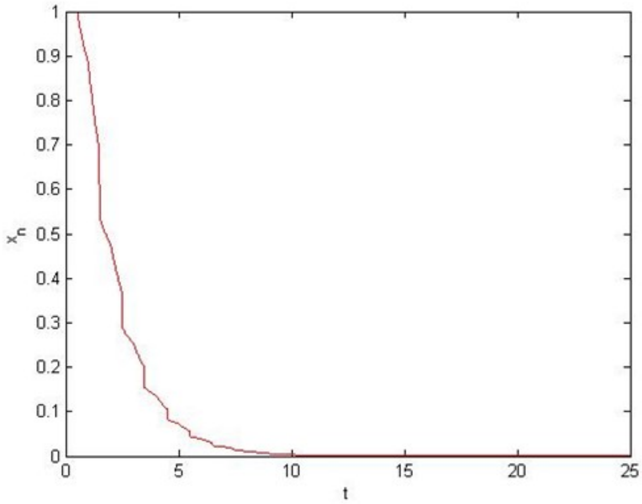

Figure 5.2: 2-Radau IA solution for Eq. (5.2) with $m = 50$.

For Theorem 4.7, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -8 & 1 & 1 \\ 1 & -6 & 1 \\ 1 & 1 & -8 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix} x\left(\left[t + \frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.3}$$

where L is a real symmetric matrix with $\mu[L] = -5 < 0$ and $\|M\| \approx 3.4338 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.09 \\ -0.08 \\ -0.05 \end{pmatrix} < 0$ with $R(\aleph) \approx \begin{pmatrix} 0.9139 \\ 0.9231 \\ 0.9512 \end{pmatrix}$, which satisfies Theorem 4.7. We plot the numerical solution of 3-Lobatto IIIB for Eq. (5.3) with $m = 50$ in Figure 5.3 and we obtain that the numerical solution is asymptotically stable.

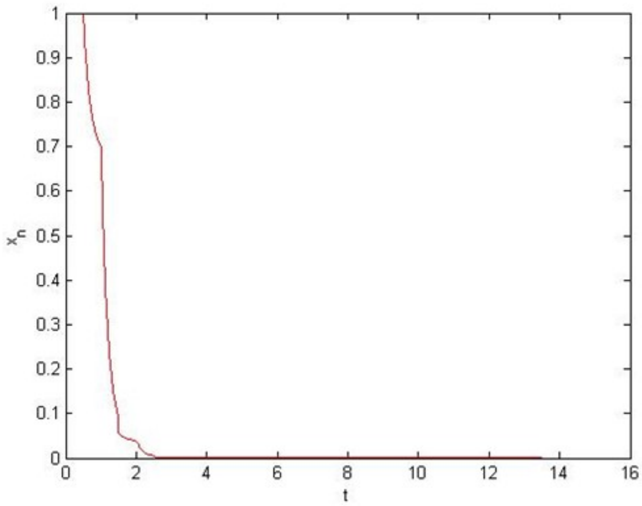

Figure 5.3: 3-Lobatto IIIB solution for Eq. (5.3) with $m = 50$.

For Theorem 4.9, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -8 & -1 & -2 \\ -1 & -3 & -1 \\ -2 & -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.4}$$

where L is a real symmetric matrix with $\mu[L] \approx -1.1270 < 0$ and $\|M\| \approx 0.8660 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.0887 \\ -0.0300 \\ -0.0127 \end{pmatrix} < 0$, $R(\aleph) \approx \begin{pmatrix} 0.9151 \\ 0.9704 \\ 0.9888 \end{pmatrix}$, $e^{\aleph} \approx \begin{pmatrix} 0.9151 \\ 0.9704 \\ 0.9888 \end{pmatrix}$, which satisfies Corollary 4.8. Therefore, Theorem 4.9 holds. We plot the numerical solution of 3-Lobatto IIIC for Eq. (5.4) with $m = 50$ in Figure 5.4 and we obtain that the numerical solution is asymptotically stable.

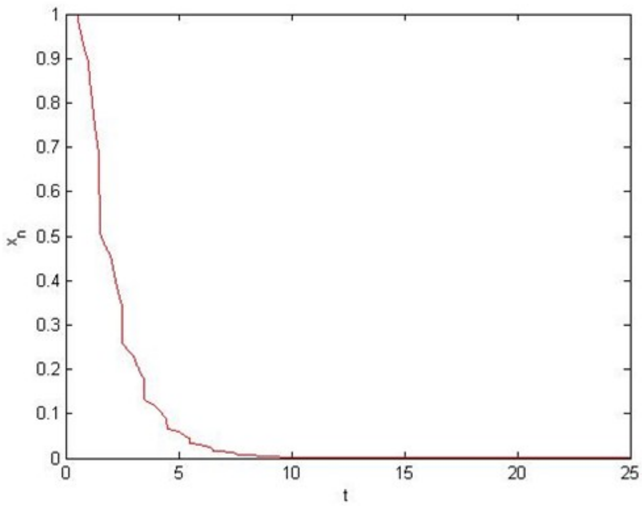

Figure 5.4: 3-Lobatto IIIC solution for Eq. (5.4) with $m = 50$.

6. Conclusion

In this paper, we consider the numerical stability of EPCA with matrix coefficients. For different types of matrix coefficients L, the corresponding stability conditions are obtained. In the future work, we will consider the nonlinear problems.

Funding

This study is supported by the Natural Science Foundation of Guangdong Province with the project number 2017A030313031.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] L. E. Shaikhet, *Behavior of solution of stochastic delay differential equation with additive fading perturbations*, Appl. Math. Lett., (2021), Article ID: 106640, 9 pages.

[2] N. Senu, K. C. Lee, A. Ahmadian, S. N. I. Ibrahim, *Numerical solution of delay differential equation using two-derivative Runge-Kutta type method with Newton interpolation*, Alex. Eng. J., **61** (2022), 5819-5835.
[3] K. E. M. Church, G. W. Duchesne, *Rigorous continuation of periodic solutions for impulsive delay differential equations*, Appl. Math. Comput., **415** (2022), Article ID: 126733, 27 pages.
[4] L. Berezansky, E. Braverman, *Solution estimates for linear differential equations with delay*, Appl. Math. Comput., **372** (2020), Article ID: 124962, 15 pages.
[5] C. Jamilla, R. Mendoza, I. Mez, *Solutions of neutral delay differential equations using a generalized Lambert W function*, Appl. Math. Comput., **382** (2020), Article ID: 125334, 17 pages.
[6] G. Philipp, W. Marcus, *A differential equation with state-dependent delay from cell population biology*, J. Differ. Equ., **260**(7) (2016), 6176-6200.
[7] F. Karakoc, *Asymptotic behavior of a population model with piecewise constant argument*, Appl. Math. Lett., **70** (2017), 7-13.
[8] K. F. Owusu, E. F. D. Goufo, S. Mugisha, *Modelling intracellular delay and therapy interruptions within Ghanaian HIV population*, Adv. Differ. Equ., **2020** (2020), 19 pages.
[9] C. Monica, M. Pitchaimani, *Geometric stability switch criteria in HIV-1 infection delay model*, J. Nonlinear Sci., **29** (2019), 163-181.
[10] O. Matte, *Continuity properties of the semi-group and its integral kernel in non-relativistic QED*, Rev. Math. Phys., **28**(5) (2016), 1-76.
[11] M. Dehghan, F. Shakeri, *The use of the decomposition procedure of Adomain for solving a delay differential equation arising in electrodynamics*, Phys. Scr., **78**(6) (2008), Article ID: 065004, 11 pages.
[12] Y. H. Chen, H. Y. Yu, X. Y. Meng, X. L. Xie, M. Z. Hou, J. Chevallier, *Numerical solving of the generalized Black-Scholes differential equation using Laguerre neural network*, Digit. Signal Process., **112**(5) (2021), Article ID: 103003, 11 pages.
[13] M. Akhmet, D. A. Cincin, M. Tleubergenova, Z. Nugayeva, *Unpredictable oscillations for Hopfield-type neutral networks with delayed and advanced arguments*, Mathematics, **9**(5) (2021), 19 pages.
[14] L. Zhang, M. H. Song, *Convergence of the Euler method of stochastic differential equations with piecewise continuous arguments*, Abstr. Appl. Anal., **2012** (2012), Article ID: 643783, 16 pages.
[15] S. R. Raj, M. Saradha, *Solving hybrid fuzzy fractional differential equations by improved Euler method*, Math. Theory Model., **5**(5) (2015), 106-117.
[16] Q. Wang, *The numerical asymptotically stability of a linear differential equation with piecewise constant arguments of mixed type*, Acta Appl. Math., **146** (2016), 145-161.
[17] J. F. Gao, *Numerical oscillation and non-oscillation for differential equation with piecewise continuous arguments of mixed type*, Appl. Math. Comput., **299** (2017), 16-27.
[18] W. S. Wang, *Stability of solutions of nonlinear neutral differential equations with piecewise constant delay and their discretizations*, Appl. Math. Comput., **219** (2013), 4590-4600.
[19] G. L. Zhang, *Stability of Runge-Kutta methods for linear impulsive delay differential equations with piecewise constant arguments*, J. Comput. Appl. Math., **297** (2016), 41-50.
[20] H. Liang, M. Z. Liu, Z. W. Yang, *Stability analysis of Runge-Kutta methods for systems* $u'(t) = Lu(t) + Mu([t])$, Appl. Math. Comput., **228** (2014), 463-476.
[21] J. Wiener, *General Solutions of Differential Equations*, World Scientific, Singapore, 1993.
[22] K. Dekker, J. G. Verwer, *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*, North-Holland, Amsterdam, 1984.
[23] J. C. Butcher, *The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods*, Wiley, New York, 1987.
[24] E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II, Stiff and Differential Algebraic Problems*, Springer, New York, 1996.