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Square numbers, square pyramidal numbers, and generalized Fibonacci polynomials

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Abstract — In this paper, we derive two interesting formulas for square and square pyramidal numbers. We focus on the linear recurrence relation with constant coefficients for square and square pyramidal numbers. Then we deal with the relationship between generalized Fibonacci polynomials and these numbers. Also, we give some determinant representations of these numbers.

Subject Classification (2020): 11B39, 11B37.

1. Introduction

A square number is an integer that is the square of an integer. The nth square number is denoted by n^2 or S_n. For example, $S_0 = 0$, $S_1 = 1$, $S_2 = 4$, $S_3 = 9$ (sequence A000290 in the OEIS [1]). The difference between two square number is given by the identity

$$n^2 = (n-1)^2 + 2(n-1) + 1, \ (S_n = S_{n-1} + 2(n-1) + 1) \tag{1.1}$$

The S_n is also equal to the sum of the first n odd numbers as follows:

$$S_n = \sum_{i=1}^{n} (2i-1)$$

For calculating square numbers, there are various recursive approaches. For example,

$$S_n = S_{n-1} + (n-1) + n,$$
$$S_n = S_{n-1} + 2n - 1,$$
$$S_n = 2S_{n-1} - S_{n-2} + 2$$

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Another interesting sequence obtained in the form of sums of square numbers is the square pyramidal numbers. The number of stacked spheres in a pyramid with a square base is counted by a square pyramidal number, which is a natural number (sequence A000330 in the OEIS [2]). The nth square pyramidal number is denoted by P_n. P_n can be obtained in different ways[3–5], some of which are

$$\begin{aligned} P_n &= \sum_{i=0}^{n} S_i \\ &= 1+4+\cdots+n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \\ &= \binom{n+2}{3} + \binom{n+1}{3} \\ &= \frac{1}{4} T_{2n} \\ &= \frac{1}{4}\binom{2n+2}{3} \\ &= 2(t_1 + t_2 + \cdots + t_{n-1}) + t_n \end{aligned}$$

where t_n and T_n are the nth triangular and triangular pyramidal numbers, respectively.

Square pyramid numbers are one of the numbers that are discussed in Greek mathematics [6]. It is known that Archimedes worked on formulas related to these numbers [7]. The problem of finding the formula for the sums of the progressions of the squares can be found in the work of Fibonacci [8]. Lucas and Watson focused on the cannonball problem and studied square pyramid numbers [9, 10]. In [11], the authors examined spreading and covering numbers and discussed the relationship between spreading numbers and square pyramidal numbers. More recently, Agarwal has studied several different types of figurative numbers in detail and discussed their computation through simple recurrence relations, patterns and properties, and mutual relationships [12].

The acquisition of a sequence using linear recurrence both expands the field of application and offers new opportunities for obtaining its properties. The presence of linear recurrence of triangular numbers and triangular pyramidal numbers is seen in [13]. When the literature is examined, it is noteworthy that there are few studies on the repetition relations of square numbers and square pyramidal numbers. It can be said that one of the important reasons for this is that these numbers are obtained with the help of easier formulas without the need for recurrence. The following question is likely to come to mind. "One cannot imagine a much simpler formula than n^2, so what is the virtue of expressing this in a much more complicated fashion?" Our main focus here is not to find a simpler formula, but rather to offer a different perspective by showing that these numbers can be obtained by linear recurrence.

In Section 2, we obtain a linear recurrence for each square and square pyramidal number. In Section 3, we show that square numbers and square pyramid numbers are the special cases of generalized Fibonacci polynomials defined by MacHenry in [14]. Thus, the opportunities offered by generalized Fibonacci polynomials can be used for these numbers. Some of these are combinatorial calculations, determinant and permanent representation, generating matrices, etc.

2. Recurrence Relation

In this section, we focus on the recurrence relation of square numbers and square pyramidal numbers.

Theorem 2.1. Let $n \geq 2$ be any integer and S_n be the nth square number. Then,

$$S_n = 4S_{n-1} + (-7)S_{n-2} + 8S_{n-3} - 8S_{n-4} + \cdots (-1)^n 8S_1 \tag{2.1}$$

Proof.

We proceed by induction on n. The result clearly holds for $n = 2$. Now suppose that the result is true for all positive integers less than or equal to n where $n > 1$. We prove it for $(n+1)$. In fact, by using definition of S_n, we obtain that

$$\begin{aligned}
S_{n+1} &= S_n + 2n + 1 \\
&= 4S_{n-1} + (-7)S_{n-2} + 8S_{n-3} - 8S_{n-4} + \cdots (-1)^n 8S_1 + 2n + 1 \\
&= 4(S_n - 2(n-1) - 1) - 7(S_{n-1} - 2(n-2) - 1) \\
&+ 8(S_{n-2} - 2(n-3) - 1) + \cdots + (-1)^n 8(S_{n-(n-2)} - 2(n-(n-1)) - 1) + 2n + 1 \\
&= 4S_n + (-7)S_{n-1} + 8S_{n-2} - 8S_{n-3} + \cdots (-1)^n 8S_2 \\
&+ (-16)(n-1) - 8 + 8(n-1) + 4 + 16(n-2) + 8 - 2(n-2) - 1 \\
&- 16(n-3) - 8 + 16(n-4) + 8 + \cdots (-1)^{n+1} 16 + (-1)^{n+1} 8 + 2n + 1
\end{aligned}$$

Let n be any odd integer. Then,

$$\begin{aligned}
S_{n+1} &= 4S_n + (-7)S_{n-1} + 8S_{n-2} - 8S_{n-3} + \cdots (-1)^n 8S_2 \\
&\quad -16(n-1-n+2+n-3-\cdots-n+(n-1)) \\
&\quad +8(n-1) + 4 - 2(n-2) - 1 + 2n + 1 \\
&= 4S_n + (-7)S_{n-1} + 8S_{n-2} - 8S_{n-3} + \cdots (-1)^n 8S_2 \\
&\quad -16\frac{(n-1)}{2} + 8(n-1) + 4 - 2(n-2) - 1 + 2n + 1 \\
&= 4S_n + (-7)S_{n-1} + 8S_{n-2} - 8S_{n-3} + \cdots (-1)^n 8S_2 + 8
\end{aligned}$$

The proof is similar when n is an even integer. Therefore, equation (2.1) holds for all positive integers $n \geq 2$.

Theorem 2.2. Let $n \geq 2$ be any integer and P_n be the nth square pyramidal number. Then,

$$P_n = 5P_{n-1} + (-11)P_{n-2} + 15P_{n-3} - 16P_{n-4} + 16P_{n-5} - \cdots (-1)^n 16P_1 \tag{2.2}$$

Proof.

We proceed by induction on n. The result clearly holds for $n = 2$. Now suppose that the result is true for all positive integers less than or equal to n where $n > 1$. We prove it for $(n+1)$. In fact, by using definition of

P_n, we obtain that

$$\begin{aligned}
P_{n+1} &= P_n + (n+1)^2 \\
&= 5P_{n-1} - 11P_{n-2} + 15P_{n-3} + 16(-P_{n-4} + P_{n-5} - \cdots (-1)^n P_1) + (n+1)^2 \\
&= 5(P_n - n^2) - 11(P_{n-1} - (n-1)^2) + 15(P_{n-2} - (n-2)^2) \\
&\quad -16(P_{n-3} - (n-3)^2) + \cdots + (-1)^{n-1} 16(P_3 - 3^2) \\
&\quad +(-1)^n 16(P_2 - 2^2) + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + 16P_{n-4} - \cdots (-1)^n 16P_2 \\
&\quad -5n^2 + 11(n-1)^2 - 15(n-2)^2 + 16(n-3)^2 - \cdots \\
&\quad +(-1)^n 16(n-(n-2))^2 + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 - 16n^2 \\
&\quad +11n^2 + 16(n-1)^2 - 5(n-1)^2 - 16(n-2)^2 + (n-2)^2 \\
&\quad +16(n-3)^2 - \cdots (-1)^n 16(n-(n-2))^2 + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 \\
&\quad +11n^2 - 5(n-1)^2 + (n-2)^2 \\
&\quad -16[n^2 - (n-1)^2 + (n-2)^2 - \cdots + (-1)(n-(n-2))^2] \\
&\quad +(n+1)^2
\end{aligned}$$

Let n be any odd integer. Then,

$$\begin{aligned}
P_{n+1} &= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 \\
&\quad + 11n^2 - 5(n-1)^2 + (n-2)^2 \\
&\quad - 16[(2n-1) + (2(n-2)-1) + (2(n-4)-1) + \cdots \\
&\quad + (2(n-(n-3))-1)] + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 \\
&\quad + 11n^2 - 5(n-1)^2 + (n-2)^2 \\
&\quad - 16[2.(\frac{n+1}{2})^2 - 2 - \frac{n-1}{2}] + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 \\
&\quad + 11n^2 - 5(n-1)^2 + (n-2)^2 - 8(n+1)^2 + 32 + 8n - 8 + (n+1)^2 \\
&= 5P_n - 11P_{n-1} + 15P_{n-2} - 16P_{n-3} + \cdots (-1)^n 16P_2 + 16
\end{aligned}$$

The proof is similar when n is an even integer. Therefore, Equation (2.2) holds for all positive integers $n \geq 2$.

3. Generalized Fibonacci Polynomials, Square and Square Pyramidal Numbers

This section concentrates on the relationship between the generalized Fibonacci polynomials and square and square pyramidal numbers.

MacHenry [14] defined generalized Fibonacci($F_{k,n}(t)$) where t_i $(1 \leq i \leq k)$ are constant coefficients of the core polynomial

$$P(x;t_1,t_2,\ldots,t_k) = x^k - t_1 x^{k-1} - \cdots - t_k$$

which is denoted by the vector $t = (t_1,t_2,\ldots,t_k)$. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0,\ n<0 \\ F_{k,0}(t) &= 1 \\ F_{k,n}(t) &= t_1 F_{k,n-1}(t) + \cdots + t_k F_{k,n-k}(t) \end{aligned} \tag{3.1}$$

In [15], the authors defined matrices $A_{(k)}$ as,

$$A_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_k & t_{k-1} & t_{k-2} & \ldots & t_1 \end{bmatrix}_{k\times k}$$

and showed that the last column of $A_{(k)}^n$ would be obtained as,

$$\begin{bmatrix} F_{k,n-k+1}(t) \\ F_{k,n-k+2}(t) \\ \cdots \\ F_{k,n}(t) \end{bmatrix} \tag{3.2}$$

In addition, in [15], the authors obtained $F_{k,n}(t)$ $(n,k \in \mathbb{N},\ n \geq 1)$ as,

$$F_{k,n}(t) = \sum_{a \vdash n} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k} \tag{3.3}$$

Where the notation $a \vdash n$ and $|a|$ are used instead of $\sum_{j=1}^{k} j a_j = n$ and $\sum_{j=1}^{k} a_j$, respectively.

Some of the important properties of generalized Fibonacci polynomials can be found in [14, 16–18].

Theorem 3.1. For all $n \geq 0$,

$$F_{n,n}(4,-7,8,\cdots,(-1)^{n-1}8) = S_{n+1}$$

and

$$F_{n,n}(5,-11,15,-16,\cdots,(-1)^{n-1}16) = P_{n+1}$$

Proof.

The proof follows directly from Equations (2.1), (2.2), and (3.1).

Corollary 3.2. Let n be any positive integer, then

$$S_{n+1}=\sum_{a\vdash n}\binom{|a|}{a_1,a_2,\cdots,a_n}4^{a_1}(-7)^{a_2}8^{a_3}\ldots((-1)^{n-1}8)^{a_n}$$

Proof.
The proof follows directly from Theorem 3.1 and Equation (3.3).

Corollary 3.3. Let n be any positive integer, then

$$P_{n+1}=\sum_{a\vdash n}\binom{|a|}{a_1,a_2,\cdots,a_n}5^{a_1}(-11)^{a_2}15^{a_3}16^{a_4}\ldots((-1)^{n-1}16)^{a_n}$$

Proof.
The proof follows directly from Theorem 3.1 and Equation (3.3).

Example 3.4. We obtain S_6 and P_6 using Corollary 3.2 and 3.3. First, let's explain the $a\vdash 5$ and $|a|$ as follows:

a_1+	$2a_2+$	$3a_3+$	$4a_4+$	$5a_5=$	5	
5	0	0	0	0	$\longrightarrow$	$\frac{5!}{5!}=1$
3	1	0	0	0	$\longrightarrow$	$\frac{4!}{3!}=4$
2	0	1	0	0	$\longrightarrow$	$\frac{3!}{2!}=3$
1	0	0	1	0	$\longrightarrow$	$\frac{2!}{1!}=2$
0	0	0	0	1	$\longrightarrow$	$\frac{1!}{1!}=1$
0	1	1	0	0	$\longrightarrow$	$\frac{2!}{1!}=2$
1	2	0	0	0	$\longrightarrow$	$\frac{3!}{2!}=3$

Then,

$$\sum_{a\vdash 5}\binom{|a|}{a_1,a_2,\ldots,a_5}4^{a_1}(-7)^{a_2}8^{a_3}(-8)^{a_4}8^{a_5}=1.4^5+4.4^3.(-7)+3.4^2 8+2.4.(-8)+1.8+2.(-7).8+3.4.(-7)^2$$
$$=36$$

and

$$\sum_{a\vdash 5}\binom{|a|}{a_1,a_2,\ldots,a_5}5^{a_1}(-11)^{a_2}15^{a_3}(-16)^{a_4}16^{a_5}=1.5^5+4.5^3.(-11)+3.5^2 15+2.5.(-16)+1.16+2.(-11).15+3.5.(-11)^2$$
$$=91$$

The two corollary given below are obvious from Equation (3.2) and Theorem 3.1.

Corollary 3.5. Let n be any positive integer and

$$A_S=\begin{bmatrix} 0 & 1 & & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ (-1)^{n-1}8 & (-1)^{n-2}8 & \dots & 8 & -7 & 4 \end{bmatrix}_{n\times n}$$

Then last column of matrix $(A_S)^n$ is

$$[S_2, S_3, \dots, S_{n+1}]^T$$

Corollary 3.6. Let n be any positive integer and

$$A_P=\begin{bmatrix} 0 & 1 & & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ (-1)^{n-1}16 & (-1)^{n-2}16 & \dots & 16 & 15 & -11 & 5 \end{bmatrix}$$

Then last column of matrix $(A_P)^n$ is

$$[P_2, P_3, \dots, P_{n+1}]^T$$

Lemma 3.7. [19] Let A_n be an $n\times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\det(A_0) = 1$. Then, $\det(A_1) = a_{11}$ and for $n \geq 2$

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1}[(-1)^{n-r}a_{n,r}(\prod_{j=r}^{n-1}a_{j,j+1})\det(A_{r-1})] \tag{3.4}$$

In [16], the authors obtained generalized Fibonacci polynomials with the help of determinant. Considering Theorem 3.1 and the determinant representation of generalized Fibonacci polynomials, the following corollaries are obtained.

Corollary 3.8. Let n be any integer such that $n \geq 1$, and let $A_n^S = [a_{i,j}]_{i,j=1,2,\dots,n}$ be an $n\times n$ Hessenberg matrix defined as

$$a_{i,j}=\begin{cases} -1, & \text{if } i=j-1; \\ 4, & \text{if } i=j; \\ -7, & \text{if } i=j+1; \\ (-1)^{i-j}8, & \text{if } i-j=k>1; \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

Then,

$$\det(A_n^S) = S_{n+1}$$

We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than n. We prove it for n. In fact, by using (3.4) we obtain that

$$\begin{aligned}
\det(A_n^S) &= \sum_{k=1}^{n}\left[(-1)^{k-1}a_{k,1}(a_{1,2})^{k-1}\det(A_{n-k}^S)\right]\\
&= 4\det(A_{n-1}^S) - 7A_{n-2}^S + \sum_{k=3}^{n}\left[(-1)^{k-1}(-1)^{k-1}8(-1)^{k-1}\det(A_{n-k}^S)\right]\\
&= 4S_n - 7S_{n-1} + \sum_{k=3}^{n}\left[(-1)^{k-1}8S_{n-k+1}\right]\\
&= 4S_n - 7S_{n-1} + 8S_{n-2} - 8S_{n-3} + \cdots + (-1)^{n-1}8S_1\\
&= S_{n+1}
\end{aligned}$$

Corollary 3.9. Let n be any integer such that $n \geq 1$, and let $A_n^P = [a_{i,j}]_{i,j=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as

$$a_{i,j} = \begin{cases} -1, & \text{if } i = j-1;\\ 5, & \text{if } i = j;\\ -11, & \text{if } i = j+1;\\ 15, & \text{if } i = j+2\\ (-1)^{i-j}16, & \text{if } i-j = k > 2;\\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\det(A_n^P) = P_{n+1}$$

Proof.

The proof is similar to the proof of Corollary 3.8.

Author Contributions

The author read and approved the last version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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