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Amalgamated Rings with *n-UJ*-Properties

Esra MEMİŞOĞULLARINDAN[1*], Tufan ÖZDİN[2]

[1]*Gazi University, Graduate School of Natural and Applied Sciences, 06500, Ankara, Turkey*
[2]*Erzincan Binali Yıldırım University, Science and Art Faculty, 24100, Erzincan, Turkey*

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Abstract

This paper examines the transfer of n-UJ-rings between A and B in an amalgamated duplication of a ring A along some ideal K of a ring B with a ring homomorphism $f: A \to B$ (denoted by $A \bowtie^f K$).

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Author ID (ORCID Number)	
E. Memişoğullarından, 0000-0002-7827-8972	
T. Özdin, 0000-0001-8081-1871	

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1. INTRODUCTION

In this article all rings are associative with nonzero identity elements. For a ring R, the Jacobson radical, the set of nilpotent elements and the set of unit elements of R are denoted by $J(R)$, $Nil(R)$ and $U(R)$ with respectively.

Let A, B be rings, K be an ideal of B and $f: A \to B$ a ring homomorphism. We consider the following subring of $A \times B$:

$$A \bowtie^f K = \{(a, f(a) + k) \mid a \in A, k \in K\}$$

which is called the amalgamation of A with B along K with respect to f (D'Anna et al., 2009).

In Farshad et al. (2021), the authors studied UU-rings (if $U(R) = 1 + Nil(R)$ i.e., rings whose units are unipotent (Calugareanu, 2015) and clean like features of the amalgamation ring $A \bowtie^f K$ as the followings:

(1) If A and $f(A) + K$ are UU-rings, then $A \bowtie^f K$ is UU-ring. (Farshad et al., 2021; Theorem 3.2 (1));

(2) If $A \bowtie^f K$ is a UU-ring, then A is a UU-ring (Farshad et al., 2021; Theorem 3.2 (2)).

(3) If $A \bowtie^f K$ is a nil-clean ring, then A and $f(A) + K$ are nil-clean rings (Farshad et al., 2021; Proposition 2.7 (1)) where a ring R is clean if each element of R is a sum of a unit and an idempotent in R (Nicholson, 1977), and R is nil-clean if each element of R is a sum of a nilpotent and an idempotent in R (Diesl, 2013);

*Corresponding Author, e-mail: esramemisogullarindan@gmail.com

(4) If A is nil-clean ring and $K \subset Nil(B)$, then $A \bowtie^f K$ is a nil-clean ring (Farshad et al., 2021; Theorem 2.10 (1)).

(5) If $A \bowtie^f K$ is a J-clean ring, then A and $f(A) + K$ are J-clean rings (Farshad et al., 2021; Proposition 2.16 (1)) where a ring R is J-clean if each element of R is a sum of an idempotent and an element from $J(R)$ (Chen, 2010).

(6) If A is J-clean ring and $K \subset Nil(B)$, then $A \bowtie^f K$ is J-clean (Farshad et al., 2021; Theorem 2.18 (1)).

The main propose of the president paper is to examine UJ, n-UJ and J-clean properties of the amalgamation ring $A \bowtie^f K$. Here,

- R is said to be a UJ-ring if $U(R) = 1 + J(R)$ (Koşan et al., 2018);
- R is is said to be an n-UJ-ring if $u - u^n \in J(R)$ for each $u \in U(R)$ where $n > 1$ is a fixed integer and R is said to be an ∞-UJ-ring if for each $u \in U(R)$ there exist $n > 1$ such that $u - u^n \in J(R)$ (Koşan et al., 2020)

Notice that every UJ-ring with nil Jacobson radical is UU-rings and R is a UJ-ring if and only if all clean elements of R are J-clean (Koşan et al., 2018; Proposition 4.1); every UJ-rings are n-UJ-rings and hence ∞-UJ-rings.

We denote by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively, the ring of integers and the ring of integers modulo n for a positive integer n.

2. RESULTS

Firstly, we start to with the following useful lemma which is about the general properties of the set of units and Jacobson radical in an amalgamated ring.

Lemma 2.1. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) $U\left(A \bowtie^f K\right) = (u, f(u) + k) \mid u \in U(A), f(u) + k \in U(f(A) + K)$.

(2) $J\left(A \bowtie^f K\right) = (a, f(a) + k) \mid a \in U(A), f(a) + k \in J(f(A) + K)$.

(3) $\frac{A \bowtie^f K}{0 \times K} \backsimeq A$ and $\frac{A \bowtie^f K}{f^{-1}(K) \times 0} \backsimeq f(A) + K$.

(4) $Nil\left(A \bowtie^f K\right) = (a, f(a) + k) \mid a \in Nil(A), f(a) + k \in J(f(A) + K)$.

Proof: (1), (2) and (4) are Farshad et al. (2021) (Lemma 2.5 and Lemma 2.15).

(3) We have the natural projections $\pi_A: A \bowtie^f K \to A$ defined by $\pi_A(a, f(a) + k) = a$ and $\pi_B: A \bowtie^f K \to B$ defined by $\pi_B(a, f(a) + k) = f(a) + k$ by D'Anna et al. (2009) (Proposition 5.1). Hence desired canonical isomorphisms hold.■

For $n \in \mathbb{Z}$, we consider the following notions adapted by Koşan et al. (2020).

$$\mathcal{U}_n(R) = \{u^{n-1} : u \in U(R)\} \subseteq U(R)$$

$$\mathcal{V}_n(R) = \{u \in U(R) : u^{n-1} \in 1 + J(R)\}$$

By using above notation, we have

$$\mathcal{U}_n(A \bowtie^f K) = \{(u^{n-1}, (f(u)+k)^{n-1}) : u \in U(A), f(u)+k \in U(f(A)+K)\}$$

and

$$\mathcal{V}_n(A \bowtie^f K) = \{(u, f(u)+k) \in U(A \bowtie^f K) : (u^{n-1}, (f(u)+k)^{n-1}) \in (1,1) + J(A \bowtie^f K)\}.$$

Lemma 2.2. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) $\mathcal{U}_n(A \bowtie^f K) \subseteq U(A \bowtie^f K)$

(2) If A and B are commutative rings, then $\mathcal{U}_n(A \bowtie^f K)$ and $\mathcal{V}_n(A \bowtie^f K)$ are subgroups of $U(A \bowtie^f K)$.

(3) If $A \bowtie^f K$ is an n-UJ-ring, then $\mathcal{V}_n(A \bowtie^f K) = U(A \bowtie^f K)$ and hence $\mathcal{U}_n(A \bowtie^f K) \subseteq (1,1) + J(A \bowtie^f K)$.

(4) An amalgamated ring $A \bowtie^f K$ is ∞-UJ-ring if and only if $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n(A \bowtie^f K) = U(A \bowtie^f K)$.

(5) If the amalgamated ring $A \bowtie^f K$ is n-UJ-ring such that $(n-1, n-1) \in U(A \bowtie^f K)$ then $Nil(A \bowtie^f K) \subseteq J(A \bowtie^f K)$.

Proof: They are obvious.■

Proposition 2.3. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B, and $n, m \in \mathbb{N}$, $n, m > 1$.

(1) If the amalgamated ring $A \bowtie^f K$ is n-UJ-ring, then $(2,2) \in J(A \bowtie^f K)$ if n is an even number.

(2) If the amalgamated ring $A \bowtie^f K$ is n-UJ-ring and $n-1$ divides $m-1$, then $A \bowtie^f K$ is an m-UJ-ring.

Proof: (1) We assume that n is an even number and $A \bowtie^f K$ is an n-UJ-ring. Then

$$(-1,-1) = ((-1)^{(n-1)}, (-1)^{(n-1)}) \in (1,1) + J(A \bowtie^f K)$$

which implies that $(2,2) \in J(A \bowtie^f K)$.

(2) Since $n-1$ divides $m-1$, we get $\mathcal{V}_n \subseteq \mathcal{V}_m$ which implies that $A \bowtie^f K$ is an m-UJ-ring by Lemma 2.2 (3). ■

Theorem 2.4. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) If A and $f(A)+K$ are n-UJ-rings (respectively, ∞-UJ-rings), then $A \bowtie^f K$ is an n-UJ-ring (respectively, ∞-UJ-ring).

(2) Let $f: A \to B$ be a ring monomorphism and $f^{-1}(K) \subseteq J(A)$ an ideal. If $A \bowtie^f K$ is an n-UJ-ring, then A and $f(A)+K$ are n-UJ-rings.

Proof: (1) Let $(u, f(u)+k) \in U(A \bowtie^f K)$. By Lemma 2.1 (1), we have $u \in U(A)$ and $f(u)+k \in U(f(A)+K)$. Since A and $f(A)+K$ are n-UJ-rings (respectively, ∞-UJ-rings), we obtain that $1-u^{n-1} \in J(A)$ which implies $u-u^n \in J(A)$ and $f(u^{n-1})+k-1 \in J(f(A)+K)$. Hence $(u^{n-1}, f(u^{n-1})+k) - (1,1) \in J(A \bowtie^f K)$. Therefore $A \bowtie^f K$ is an n-UJ-ring.

(2) Let $A \bowtie^f K$ be an n-UJ-ring. By Koşan et al. (2020) (Proposition 2.9 (1)), A and $f(A)+K$ are n-UJ-rings because $\frac{A\bowtie^f K}{0\times K} \backsimeq A$ and $\frac{A\bowtie^f K}{f^{-1}(K)\times 0} \backsimeq f(A)+K$ by Lemma 2.1 (3). ■

Let $I \subseteq J(R)$ be an ideal of R. By Koşan et al. (2020) (Proposition 2.9 (1)), R is an n-UJ-ring if and only if R/I is an n-UJ-ring.

Corollary 2.5. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) If $B = K$ or $f: A \to B$ is an epimorphism, then $A \bowtie^f K$ is an n-UJ-ring if and only if A and B are n-UJ-rings, since in this case $A \bowtie^f K = A \times B$.

(2) If $f^{-1}(K) = 0$, then $A \bowtie^f K$ is an n-UJ-ring if and only if $f(A)+K$ is an n-UJ-ring (by Lemma 2.1 (3) and Koşan et al., 2020; Proposition 2.9 (1)).

(3) If $K = 0$, then $A \bowtie^f K$ is an n-UJ-ring if and only if A is an n-UJ-ring (by Lemma 2.1 (3) and Koşan et al., 2020; Proposition 2.9 (1)).

Theorem 2.6. Let R be a ring and let M be an (R,R) bimodule. R is an n-UJ-ring if and only if the trivial extension $T(R,M)$ is an n-UJ-ring.

Proof: This is proven in Koşan et al. (2020) (Theorem 3.1). ■

Since UJ-rings are n-UJ-rings, we have following theorem.

Theorem 2.7. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) If A and $f(A)+K$ are UJ-rings, then $A \bowtie^f K$ is a UJ-ring.

(2) Let $f: A \to B$ be a ring monomorphism and $f^{-1}(K) \subseteq J(A)$ an ideal. If $A \bowtie^f K$ is a UJ-ring, then A and $f(A)+K$ are UJ-rings.

In Theorem 2.7, the assumption "$f^{-1}(K) \subseteq J(A)$" is not superflous because if we can take $A = B = \mathbb{Z}_4 \times \mathbb{Z}_4$, $K = 0 \times 2\mathbb{Z}_4 \subseteq J(B) = 2\mathbb{Z}_4 \times 2\mathbb{Z}_4$ and a ring homomorphism $f: A \to B$ defined by $f\big((a,b)\big) = (0,a)$. Then $f^{-1}(K) = 2\mathbb{Z}_4 \times \mathbb{Z}_4 \nsubseteq J(A) = 2\mathbb{Z}_4 \times 2\mathbb{Z}_4$.

Corollary 2.7. We have the following statements for the amalgamated ring $A \bowtie^f K$ of the rings A and B.

(1) If $B = K$ or $f: A \to B$ is an epimorphism, then $A \bowtie^f K$ is a UJ-ring if and only if A and B are UJ-rings, since in this case $A \bowtie^f K = A \times B$.

(2) If $f^{-1}(K) = 0$, then $A \bowtie^f K$ is a UJ-ring if and only if $f(A)+K$ is a UJ-ring (by Lemma 2.1 (3) and Koşan et al., 2018; Proposition 2.3 (5)).

If $K = 0$, then $A \bowtie^f K$ is a UJ-ring if and only if A is a UJ-ring (by Lemma 2.1 (3) and Koşan et al., 2018; Proposition 2.3 (5)).

3. EXAMPLES

Example 3.1. Let $A = \mathbb{Z}_6$, $B = \mathbb{Z}_3 \times \mathbb{Z}_3$, $K = 0 \times \mathbb{Z}_3$ and $f: A \to B$ defined by

$$f(0) = f(3) = (0,0),\ f(1) = f(4) = (1,1), \text{ and } f(2) = f(5) = (2,2).$$

Clearly,

$$f(A) + K = \{(0,0), (0,1), (0,2), (1,1), (1,2), (1,0), (2,2), (2,0), (2,1)\},$$

$$A \bowtie^f K = \{(0,(0,0)), (0,(0,1)), (0,(0,2)), (1,(1,1)), (1,(1,2)), (1,(1,0)),$$

$$(2,(2,2)), (2,(2,0)), (2,(2,1)), (3,(0,0)), (3,(0,1)), (3,(0,2)),$$

$$(4,(1,1)), (4,(1,2)), (4,(1,0)), (5,(2,2)), (5,(2,0)), (5,(2,1))\}$$

and

$$U(A \bowtie^f K) = \{(1,(1,1)), (1,(1,2)), (5,(2,2)), (5,(2,1))\},$$

$$J(A \bowtie^f K) = \{(0,(0,0)), (0,(0,1)), (4,(1,1)), (4,(1,0))\}.$$

If we compute $u - u^n$ for $n = 2, 3, 4$;

$$u - u^2 = (1,(1,1)) - (1,(1,1)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^2 = (1,(1,2)) - (1,(1,1)) = (0,(0,1)) \in J(A \bowtie^f K)$$

$$u - u^2 = (5,(2,2)) - (1,(1,1)) = (4,(1,1)) \in J(A \bowtie^f K)$$

$$u - u^2 = (5,(2,1)) - (1,(1,1)) = (4,(1,0)) \in J(A \bowtie^f K)$$

$$u - u^3 = (1,(1,1)) - (1,(1,1)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^3 = (1,(1,2)) - (1,(1,2)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^3 = (5,(2,2)) - (5,(2,2)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^3 = (5,(2,1)) - (5,(2,1)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^4 = (1,(1,1)) - (1,(1,1)) = (0,(0,0)) \in J(A \bowtie^f K)$$

$$u - u^4 = (1,(1,2)) - (1,(1,1)) = (0,(0,1)) \in J(A \bowtie^f K)$$

$$u - u^4 = (5,(2,2)) - (1,(1,1)) = (4,(1,1)) \in J(A \bowtie^f K)$$

$$u - u^4 = (5,(2,1)) - (1,(1,1)) = (4,(1,0)) \in J(A \bowtie^f K)$$

If we continue in a similar way, we get $(u - u^n) \in J(A \bowtie^f K)$.

Example 3.2. Let $A = \mathbb{Z}_6$, $B = \mathbb{Z}_3 \times \mathbb{Z}_3$, $K = 0 \times \mathbb{Z}_3$ and $f: A \to B$ defined as follow,

$$f(0) = f(3) = (0,0),\ f(1) = f(4) = (1,1), \text{ and } f(2) = f(5) = (2,2).$$

Clearly, $U(A) = \{1,5\}$, $Id(A) = \{0,1,3,4\}$. Let $K = \{(0,0),(0,1),(0,2)\}$. Then,

$$R = A \bowtie^f K = \{(0,(0,0)),(0,(0,1)),(0,(0,2)),(1,(1,1)),(1,(1,2)),(1,(1,0)),$$

$$(2,(2,2)),(2,(2,0)),(2,(2,1)),(3,(0,0)),(3,(0,1)),(3,(0,2)),$$

$$(4,(1,1)),(4,(1,2)),(4,(1,0)),(5,(2,2)),(5,(2,0)),(5,(2,1))\}.$$

So,

$$U(f(A)) + K = \{(1,1),(1,2),(2,2),(2,1)\}$$

$$U(A \bowtie^f K) = \{(1,(1,1)),(1,(1,2)),(5,(2,2)),(5,(2,1))\}$$

$$Id(A \bowtie^f K) = \{(0,(0,0)),(0,(0,1)),(1,(1,1)),(1,(1,0)),$$

$$(3,(0,0)),(3,(0,1)),(4,(1,1)),(4,(1,0))\}$$

$$f(A) + K = \{(0,0),(0,1),(0,2),(1,1),(1,2),(1,0),(2,2),(2,0),(2,1)\}$$

$$f(U(A)) + K = \{(1,1),(1,2),(1,0),(2,2),(2,0),(2,1)\}$$

and $x(f(A)+K) = y(f(A)+K)$ then $(2,0)(f(A)+K) = (1,0)(f(A)+K)$. However,

$$(2,0) \neq (1,0).(1,2)$$

(i.e $x \neq yu$). By Koşan et al. (2018) (Corollary 2.5), $A = \mathbb{Z}_6$ is a UJ-ring but we can easily see that $f(A)+K$ is not a UJ-ring, and so $R = A \bowtie^f K$ is not a UJ-ring.

4. *J*-CLEANESS

Let $M = (M_i)_{i=1}^{n}$ be a family of R-modules and $\varphi = \{\varphi_{i,j}\}_{\substack{i+j\leq n \\ 1\leq i,j\leq n-1}}$ be a family of bilinear maps such that each $\varphi_{i,j}$ is written multiplicatively:

$$\varphi_{i,j}: M_i \times M_j \longrightarrow M_{i+j}$$

$$(m_i, m_j) \longmapsto \varphi_{i,j}(m_i, m_j) := m_i m_j$$

In particular, if all M_i are submodules of the same R-algebra L, then the bilinear maps, if they are not specified, are just the multiplication of L (see examples in Anderson et al. (2017) (Section 2)). The n-φ-trivial extension of R by M is the set denoted by $R \ltimes_\varphi M_1 \ltimes \cdots \ltimes M_n$ or simply $R \ltimes_\varphi M$ whose underlying additive group is $R \oplus M_1 \oplus \cdots \oplus M_n$ with multiplication given by

$$(m_0, \dots, m_n)(m'_0, \dots, m'_n) = (\sum_{j+k=i} m_j m_k{}')$$

for all $(m_i), (m_i{}') \in R \ltimes_\varphi M$.

We could also define the product $\varphi_{i,j}: M_i \times M_j \longrightarrow M_{i+j}$ as an R-bimodule homomorphism $\tilde{\varphi}_{i,j}: M_i \otimes M_j \to M_{i+j}$; see Anderson et al. (2017) (Section 2) for details.

For the sake of simplicity, it is convenient to set $M_0 = R$. In what follows, if no ambiguity arises, the n-φ-trivial extension of R by M will be simply called an n-φ-trivial extension of R by M and denoted by $R \ltimes_n M_1 \ltimes \ldots \ltimes M_n$ or simply $R \ltimes_n M$. Morever, $R \ltimes_\varphi M$ is naturally isomorphic to the subring of the generalized triangular matrix ring

$$\begin{pmatrix} R & M_1 & M_2 & \cdots & M_n \\ 0 & R & M_1 & \cdots & M_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_1 \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$$

consisting of matrices

$$\begin{pmatrix} r & m_1 & m_2 & \cdots & m_n \\ 0 & r & m_1 & \cdots & m_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_1 \\ 0 & 0 & 0 & \cdots & r \end{pmatrix}$$

where $r \in R$ and $m_i \in M_i$ for every $i \in \{1, \ldots, n\}$. Any n-φ-trivial extension $R \ltimes_\varphi M_1 \ltimes \ldots \ltimes M_n$ can be seen as the amalgamation of R with $R \ltimes_\varphi M_1 \ltimes \ldots \ltimes M_n$ along $0 \ltimes_\varphi M_1 \ltimes \ldots \ltimes M_n$ with respect to the canonical injection (Anderson et al., 2017).

Proposition 4.1. If all clean element of A and $f(A) + K$ are J-clean, then $A \bowtie^f K$ is a UJ-ring.

Proof: $x = (u, f(u) + k)$ which implies $u \in U(A)$ and $f(u) + k \in U(f(A) + K)$. $x = (u, f(u) + k)$ is a clean element of $A \bowtie^f K$. Again u is a clean element of A and $f(u) + k$ is a clean element of $f(A) + K$ by Farshad et al. (2021) (Lemma 2.5 (2)). By hypothesis u and $f(u) + k$ are J-clean of A and $f(A) + K$ respectively. Hence $u = e_1 + j_1$ where $e_1 \in Id(A)$ and $j_1 \in J(A)$. $f(u) + k = e_2 + j_2$ where $e_2 \in Id(f(A) + K)$ and $j_2 \in J(f(A) + K)$. Since $1 = e_1 u^{-1} + j_1 u^{-1}$ we obtain that $e_1 u^{-1} = 1 - j_1 u^{-1}$ is a unit of A. Hence $e_1 = 1$ which implies that $u = e_1 + j_1 = 1 + j_1$. Similarly, we get that $(f(u) + k)^{-1}$ is a unit of $f(A) + K$ and hence $e_2 = 1$. $x = (u, f(u) + k) = (e_1 + j_1, e_2 + j_2) = (1 + j_1, 1 + j_2)$. ■

Corollary 4.2. If A and $f(A) + K$ are J-clean rings, then $A \bowtie^f K$ is a clean UJ-ring.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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