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Research Article

Sharp inequalities for Toader mean in terms of other bivariate means

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Abstract

In the paper, the author discovers the best constants α_1, α_2, α_3, β_1, β_2 and β_3 for the double inequalities

$$\alpha_1 A\left(\frac{a-b}{a+b}\right)^{2n+2} < T(a,b)-\frac{1}{4}C-\frac{3}{4}A-A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2} < \beta_1 A\left(\frac{a-b}{a+b}\right)^{2n+2}$$

$$\alpha_2 A\left(\frac{a-b}{a+b}\right)^{2n+2} < T(a,b)-\frac{3}{4}\overline{C}-\frac{1}{4}A-A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2} < \beta_2 A\left(\frac{a-b}{a+b}\right)^{2n+2}$$

and

$$\alpha_3 A\left(\frac{a-b}{a+b}\right)^{2n+2} < \frac{4}{5}T(a,b)+\frac{1}{5}H-A-A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{5((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2} < \beta_3 A\left(\frac{a-b}{a+b}\right)^{2n+2}$$

to be valid for all $a,b>0$ with $a\neq b$ and $n=1,2,\cdots$, where

$$C\equiv C(a,b)=\frac{a^2+b^2}{a+b},\quad \overline{C}\equiv\overline{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},\quad A\equiv A(a,b)=\frac{a+b}{2},$$

$$H\equiv H(a,b)=\frac{2ab}{a+b},\quad T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta$$

are respectively the contraharmonic, centroidal, arithmetic, harmonic and Toader means of two positive numbers a and b, $(a,n)=a(a+1)(a+2)(a+3)\cdots(a+n-1)$ denotes the shifted factorial function. As an application of the above inequalities, the author also find a new bounds for the complete elliptic integral of the second kind.

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1. Introduction

In [19], Toader introduced a mean

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\theta=\begin{cases}\dfrac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a>b,\\ \dfrac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a<b,\\ a, & a=b,\end{cases}$$

where

$$\mathcal{E}=\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\theta$$

for $r\in[0,1]$ is the complete elliptic integral of the second kind. The quantities

$$H\equiv H(a,b)=\frac{2ab}{a+b},\quad A\equiv A(a,b)=\frac{a+b}{2},\quad \overline{C}\equiv\overline{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},$$

$$S(a,b)=\sqrt{\frac{a^2+b^2}{2}},\quad C\equiv C(a,b)=\frac{a^2+b^2}{a+b}$$

are called in the literature the harmonic, arithmetic, centroidal, quadratic and contraharmonic means of two positive real numbers a and b with $a\neq b$. For $p\in\mathbb{R}$ and $a,b>0$ with $a\neq b$, the p-th power mean $M_p(a,b)$ is defined by

$$M_p(a,b)=\begin{cases}\left(\dfrac{a^p+a^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases}\tag{1.1}$$

It is well known that

$$H(a,b)=M_{-1}(a,b)<A(a,b)=M_1(a,b)<\overline{C}(a,b)<S(a,b)=M_2(a,b)<C(a,b)$$

for all $a,b>0$ with $a\neq b$.

There are many bounds for the Toader mean in terms of various elementary means, see for example, [6,7,9–15,17,22–28], and recent papers [16,29–32]. In particular, we mention here several interesting results.

In [20], Vuorinen conjectured that

$$M_{3/2}(a,b)<T(a,b)\tag{1.2}$$

for all $a,b>0$ with $a\neq b$. This conjecture was verified by Qiu and Shen [17] and by Barnard, Pearce, and Richards [3].

In [1], Alzer and Qiu presented that

$$T(a,b)<M_{(\ln 2)/\ln(\pi/2)}(a,b)\tag{1.3}$$

for all $a,b>0$ with $a\neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean.

In [8], the authors demonstrated that the double inequality

$$\alpha S(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta S(a,b)+(1-\beta)A(a,b)\tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq\frac{1}{2}$ and $\beta\geq\frac{4-\pi}{(\sqrt{2}-1)\pi}$.

In [18], the authors proved that the double inequalities

$$\alpha_1 C(a,b)+(1-\alpha_1)A(a,b)<T(a,b)<\beta_1 C(a,b)+(1-\beta_1)A(a,b)\tag{1.5}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/4, \beta_1\geq 4/\pi-1$.

In [11], the authors proved that the double inequalities

$$\alpha_1\overline{C}(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1\overline{C}(a,b) + (1-\beta_1)A(a,b) \tag{1.6}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 3/4, \beta_1 \geq 12/\pi - 3$.

In [12] it was proved that the double inequality

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b) \tag{1.7}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \frac{\pi}{4}$ and $\beta_1 \geq \frac{4}{5}$.

The main aim of this paper is to give some improvements of (1.5), (1.6) and (1.7).

Theorem 1.1. *The double inequality*

$$\begin{aligned}\alpha_1 A\left(\frac{a-b}{a+b}\right)^{2n+2} &< T(a,b) - \frac{1}{4}C - \frac{3}{4}A - A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}\\ &< \beta_1 A\left(\frac{a-b}{a+b}\right)^{2n+2}\end{aligned} \tag{1.8}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha_1 \leq \frac{\left(\frac{1}{2},n\right)^2}{4((n+1)!)^2} \quad \textit{and} \quad \beta_1 \geq \frac{16-5\pi}{4\pi} - \sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}.$$

Theorem 1.2. *The double inequality*

$$\begin{aligned}\alpha_2 A\left(\frac{a-b}{a+b}\right)^{2n+2} &< T(a,b) - \frac{3}{4}\overline{C} - \frac{1}{4}A - A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}\\ &< \beta_2 A\left(\frac{a-b}{a+b}\right)^{2n+2}\end{aligned} \tag{1.9}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha_2 \leq \frac{\left(\frac{1}{2},n\right)^2}{4((n+1)!)^2} \quad \textit{and} \quad \beta_2 \geq \frac{16-5\pi}{4\pi} - \sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}.$$

Theorem 1.3. *The double inequality*

$$\begin{aligned}\alpha_3 A\left(\frac{a-b}{a+b}\right)^{2n+2} &< \frac{4}{5}T(a,b) + \frac{1}{5}H - A - A\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{5((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}\\ &< \beta_3 A\left(\frac{a-b}{a+b}\right)^{2n+2}\end{aligned} \tag{1.10}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha_3 \leq \frac{\left(\frac{1}{2},n\right)^2}{5((n+1)!)^2} \quad \textit{and} \quad \beta_3 \geq \frac{16-5\pi}{5\pi} - \sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{5((k+1)!)^2}.$$

2. Basic knowledge and lemmas

In order to establish our main results we need some basic knowledge and lemmas, which we present in this section.

For real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, \quad |x| < 1. \tag{2.1}$$

Here, $(a,0)=1$ for $a\neq 0$ and (a,b) denotes the shifted factorial function

$$(a,n)=a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n=1,2\ldots$.

For $0<r<1$, denote $r'=\sqrt{1-r^2}$. It is known that Legendre's complete elliptic integrals of the first and second kind are defined respectively by

$$\begin{cases}\mathcal{K}=\mathcal{K}(r)=\int_0^{\pi/2}\frac{1}{\sqrt{1-r^2\sin^2\theta}}\,\mathrm{d}\theta=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right),\\ \mathcal{K}'=\mathcal{K}'(r)=\mathcal{K}(r'),\\ \mathcal{K}(0)=\frac{\pi}{2},\\ \mathcal{K}(1)=\infty\end{cases}$$

and

$$\begin{cases}\mathcal{E}=\mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\theta=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right),\\ \mathcal{E}'=\mathcal{E}'(r)=\mathcal{E}(r'),\\ \mathcal{E}(0)=\frac{\pi}{2},\\ \mathcal{E}(1)=1.\end{cases}$$

See [4, 5]. For $0<r<1$, the following formulas were presented in [2, Appendix E, pp. 474–475]:

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r}=\frac{\mathcal{E}-(r')^2\mathcal{K}}{r(r')^2},\quad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r}=\frac{\mathcal{E}-\mathcal{K}}{r},\quad \frac{\mathrm{d}(\mathcal{E}-(r')^2\mathcal{K})}{\mathrm{d}r}=r\mathcal{K},$$
$$\frac{\mathrm{d}(\mathcal{K}-\mathcal{E})}{\mathrm{d}r}=\frac{r\mathcal{E}}{(r')^2},\quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}-(r')^2\mathcal{K}}{1+r}.$$

Lemma 2.1 ([2, pp. 70, Exercises,13(a)]). *$f(r)=2\mathcal{E}(r)-r'^2\mathcal{K}(r)$ is increasing and log-convex from $(0,1)$ onto $(\pi/2,2)$.*

Lemma 2.2. *The function*

$$F_n(r)=\frac{\frac{2}{\pi}\left(2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right)-\left(1+\frac{1}{4}r^2\right)-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2k+2}}{r^{2n+2}} \tag{2.2}$$

is strictly increasing from $(0,1)$ onto (λ_n,μ_n), where

$$\lambda_n=\frac{\left(\frac{1}{2},n\right)^2}{4((n+1)!)^2},\quad \textit{and}\quad \mu_n=\frac{16-5\pi}{4\pi}-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}.$$

Proof. Making use of series expansion we have

$$\begin{aligned}
&\frac{2}{\pi}\left(2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right)-\left(1+\frac{1}{4}r^2\right)\\
&=2\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k}-(1-r^2)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k}-\left(1+\frac{1}{4}r^2\right)\\
&=2\sum_{k=2}^{\infty}\frac{\left(-\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k}-\sum_{k=2}^{\infty}\frac{\left(\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k}+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k+2}\\
&=\sum_{k=2}^{\infty}\frac{-\left(k+\frac{3}{2}\right)\left(\frac{1}{2},k-1\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k}+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2},k\right)\left(\frac{1}{2},k\right)}{(k!)^2}r^{2k+2}\\
&=\sum_{k=1}^{\infty}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2k+2}. \qquad (2.3)
\end{aligned}$$

it follows from (2.3) that the function $F_n(r)$ can be rewritten as

$$F_n(r)=\sum_{k=n}^{\infty}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2(k-n)} \qquad (2.4)$$

So the function $F_n(x)$ is strictly increasing on $(0,1)$. Moreover, it is easy to obtain $\lambda_n=\lim_{r\to0^+}F_n(r)=\frac{\left(\frac{1}{2},n\right)^2}{4((n+1)!)^2}$ and by Lemma 2.1, one can get $\mu_n=\lim_{r\to1^-}F_n(r)=\frac{16-5\pi}{4\pi}-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}$. the proof of Lemma 2.2 is complete. ☐

3. Proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.1. Since $A(a,b), C(a,b)$ and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $r=\frac{a-b}{a+b}\in(0,1)$. Then

$$T(a,b)=\frac{2}{\pi}A(a,b)[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)], \qquad (3.1)$$

$$C(a,b)=A(a,b)(1+r^2). \qquad (3.2)$$

we clearly see that inequality (1.8) is equivalent to

$$\begin{aligned}
\alpha_1&<\frac{T(a,b)-\frac{1}{4}C(a,b)-\frac{3}{4}A(a,b)-A(a,b)\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}}{A(a,b)\left(\frac{a-b}{a+b}\right)^{2n+2}}\\
&=\frac{\frac{2}{\pi}\left(2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right)-\left(1+\frac{1}{4}r^2\right)-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2k+2}}{r^{2n+2}}\\
&=F_n(r)<\beta_1. \qquad (3.3)
\end{aligned}$$

Therefore, Theorem 1.1 follows easily from (3.3) and Lemma 2.1. ☐

Proof of Theorem 1.2. Since $A(a,b), \overline{C}(a,b)$ and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $r=\frac{a-b}{a+b}\in(0,1)$. Then

$$\overline{C}(a,b)=A(a,b)(1+\frac{1}{3}r^2). \qquad (3.4)$$

we clearly see that inequality (1.8) is equivalent to

$$\begin{aligned}\alpha_2 &< \frac{T(a,b)-\frac{3}{4}\overline{C}(a,b)-\frac{1}{4}A(a,b)-A(a,b)\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}}{A(a,b)\left(\frac{a-b}{a+b}\right)^{2n+2}}\\ &= \frac{\frac{2}{\pi}\left(2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right)-\left(1+\frac{1}{4}r^2\right)-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2k+2}}{r^{2n+2}}\\ &= F_n(r) < \beta_2. \end{aligned} \tag{3.5}$$

Therefore, Theorem 1.2 follows easily from (3.5) and Lemma 2.2. □

Proof of Theorem 1.3. Since $A(a,b), H(a,b)$ and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $r = \frac{a-b}{a+b} \in (0,1)$. Then

$$H(a,b) = A(a,b)(1-r^2). \tag{3.6}$$

we clearly see that inequality (1.10) is equivalent to

$$\begin{aligned}\alpha_3 &< \frac{\frac{4}{5}T(a,b)+\frac{1}{5}H(a,b)-A(a,b)-A(a,b)\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{5((k+1)!)^2}\left(\frac{a-b}{a+b}\right)^{2k+2}}{A(a,b)\left(\frac{a-b}{a+b}\right)^{2n+2}}\\ &= \frac{4}{5}\frac{\frac{2}{\pi}\left(2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right)-\left(1+\frac{1}{4}r^2\right)-\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}r^{2k+2}}{r^{2n+2}}\\ &= \frac{4}{5}F_n(r) < \beta_3. \end{aligned} \tag{3.7}$$

Therefore, Theorem 1.3 follows easily from (3.7) and Lemma 2.2. □

4. New bounds for the complete elliptic integral of the second kind

By the virtue of Theorem 1.1, new lower and upper bounds for the complete elliptic integral $\mathcal{E}(r)$ of the second kind are given as follows.

Theorem 4.1. *For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, we have*

$$\begin{aligned}&\frac{\pi}{2}\left[\alpha\frac{1+r'}{2}\left(\frac{1-r'}{1+r'}\right)^{2n+2}+\frac{1}{4}\frac{1+r'^2}{1+r'}+\frac{3(1+r')}{8}+\frac{1+r'}{2}\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{1-r'}{1+r'}\right)^{2k+2}\right]\\ &< \mathcal{E}(r)\\ &< \frac{\pi}{2}\left[\beta\frac{1+r'}{2}\left(\frac{1-r'}{1+r'}\right)^{2n+2}+\frac{1}{4}\frac{1+r'^2}{1+r'}+\frac{3(1+r')}{8}\right.\\ &\qquad\left.+\frac{1+r'}{2}\sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}\left(\frac{1-r'}{1+r'}\right)^{2k+2}\right]. \end{aligned} \tag{4.1}$$

where

$$\alpha = \frac{\left(\frac{1}{2},n\right)^2}{4((n+1)!)^2} \quad \text{and} \quad \beta = \frac{16-5\pi}{4\pi} - \sum_{k=1}^{n-1}\frac{(\frac{1}{2},k)^2}{4((k+1)!)^2}.$$

Let $n=1$ and $n=2$ in (4.1), one get

Corollary 4.2. *For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, we have*

$$\frac{\pi}{2}\left[\frac{(9r'^2+14r'+9)^2}{128(1+r')^3}\right] < \mathcal{E}(r) < \frac{\pi}{2}\left[\frac{16-5\pi}{4\pi}\frac{1+r'}{2}\left(\frac{1-r'}{1+r'}\right)^4+\frac{5r'^2+6r'+5}{8(1+r')}\right]. \tag{4.2}$$

Corollary 4.3. *For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, we have*

$$\frac{\pi}{2}\left[\frac{1+r'}{512}\left(\frac{1-r'}{1+r'}\right)^6 + \frac{(9r'^2+14r'+9)^2}{128(1+r')^3}\right] < \mathcal{E}(r)$$
$$< \frac{\pi}{2}\left[\frac{256-81\pi}{64\pi}\frac{1+r'}{2}\left(\frac{1-r'}{1+r'}\right)^6 + \frac{(9r'^2+14r'+9)^2}{128(1+r')^3}\right]. \tag{4.3}$$

Remark 4.4. In [31, Corollary 3.4], it was presented that

$$\frac{\pi}{2}\left(\frac{1+r'^{7/4}}{1+r'^{-1/4}}\right)^{1/2} < \mathcal{E}(r) < \frac{\pi}{2}\left(\frac{1+r'^{13/12}}{1+r'^{5/12}}\right)^{3/2} \tag{4.4}$$

for all $r \in (0,1)$.

The lower bound in (4.2) for $\mathcal{E}$(r) is better taht the lower bound in (4.4). Indeed, Let $x = (r')^{1/4} \in (0,1)$

$$\left[\frac{(9r'^2+14r'+9)^2}{128(1+r')^3}\right]^2 - \left[\left(\frac{1+r'^{7/4}}{1+r'^{-1/4}}\right)^{1/2}\right]^2$$
$$= \left[\frac{(9x^8+14x^4+9)^2}{128(1+x^4)^3}\right]^2 - \left[\left(\frac{1+x^7}{1+x^{-1}}\right)^{1/2}\right]^2$$
$$= \frac{(1+x^2)(1-x)^4}{16384(1+x^4)^6}P(x) > 0,$$

for $x \in (0,1)$, where

$$P(x) = 6561x^{26} + 9860x^{25} + 9897x^{24} + 6672x^{23} + 47570x^{22} + 65288x^{21} + 59826x^{20}$$
$$+ 31184x^{19} + 131863x^{18} + 170300x^{17} + 146495x^{16} + 60448x^{15} + 187612x^{14}$$
$$+ 230000x^{13} + 187612x^{12} + 60448x^{11} + 146495x^{10} + 170300x^9 + 131863x^8$$
$$+ 31184x^7 + 59826x^6 + 65288x^5 + 47570x^4 + 6672x^3 + 9897x^2 + 9860x + 6561.$$

Remark 4.5. In [16, corollary 3.5], it was established that

$$\frac{\pi}{2}\frac{\sqrt{6(1+r'^2)+4r'}}{4} < \mathcal{E}(r) < \sqrt{(1+r'^2) + (\frac{\pi^2}{2}-4)r'} \tag{4.5}$$

The following equivalence relations show that the lower bound in (4.2) for $\mathcal{E}(r)$ is better than the lower bound in (4.5):

$$\left[\frac{(9x^2+14x+9)^2}{128(1+x)^3}\right]^2 - (\frac{\sqrt{6(1+x^2)+4x}}{4})^2$$
$$= \frac{(417x^4+1532x^3+2246x^2+1532x+417)(1-x)^4}{16384(1+x)^6} > 0,$$

for $x \in (0,1)$.

Remark 4.6. In [21, theorem 4.4], the following double inequality are proved.

For all $r \in (0,1)$, we have

$$\frac{\pi}{2}\sqrt{\alpha\left[\frac{7r'^4+18r'^2+7}{16(1+r'^2)}\right] + (1-\alpha)\left[\frac{3r'^2+2r'+3}{8}\right]}$$
$$< \mathcal{E}(r) < \frac{\pi}{2}\sqrt{\beta\left[\frac{7r'^4+18r'^2+7}{16(1+r'^2)}\right] + (1-\beta)\left[\frac{3r'^2+2r'+3}{8}\right]} \tag{4.6}$$

with the best possible constants $\alpha = 3/16, \beta = 4(16/\pi^2 - 3/2)$.

The following equivalence relations show that the lower bound in (4.2) for $\mathcal{E}(r)$ is better than the lower bound in (4.6)

$$\left[\frac{1}{128}\frac{(9+14x+9x^2)^2}{(1+x)^3}\right]^2-\left[\frac{3}{16}\left(\frac{7x^4+18x^2+7}{16(1+x^2)}\right)+\frac{13}{16}\left(\frac{(3x^2+2x+3)}{8}\right)\right]$$
$$=\frac{(225x^4+830x^3+1218x^2+830x+225)(1-x)^6}{16384(1+x)^6(1+x^2)}>0,$$

for $x\in(0,1)$.

Remark 4.7. The following double inequality was derived by Zhang et al. in [30, Corollary 3.1] : Let $l(r)=(1+r)/2$ and $u(r)=(3+r^2)/4$, then

$$\frac{\pi}{2}\left[\frac{u(r')}{4}+\frac{3l(r')}{4}\right]<\mathcal{E}(r)<\frac{\pi}{2}\left[\sigma u(r')+(1-\sigma)l(r')\right] \tag{4.7}$$

holds for all $r\in(0,1)$, where $\sigma=2(4/\pi-1)$.

The following equivalence relations show that the lower bound in (4.2) for $\mathcal{E}(r)$ is better than the lower bound in (4.7)

$$\frac{1}{128}\frac{(9+14x+9x^2)^2}{(1+x)^3}-\left(\frac{3(1+x^2)}{16}+\frac{3(1+x)}{8}\right)$$
$$=\frac{(8x^2+15x+9)(1-x)^3}{128(1+x)^3}>0,$$

for $x\in(0,1)$.

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