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Research Article

On some second maximal subgroups of non-solvable groups

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Abstract

We call a group G belongs to the class of groups S'_p, if for every pd-chief factor A/B of G, $((A/B)_p)' = 1$. In this paper, we investigate the influence of some second maximal subgroups which are related to non-c_p-normal maximal subgroups on the structure of S'_p.

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1. Introduction

All groups considered in this paper will be finite. Our terminology and notation are standard and can be found in [4, 7, 12]. For every $p \in \pi(G)$, $|G|_p$ denotes the p-part of $|G|$ and G' denotes the derived subgroup of G. We write $M \lessdot G$ to express that M is a maximal subgroup of G. $Max(G,H)$ denotes the set of all maximal subgroups M of G such that $H \leq M$. Following Konovalova and Monakhov et al, we let $Max(G)$ denote the set of all maximal subgroups of G and $Max_2(G)$ denote the set of all second maximal subgroups of G. Furthermore, $Max_2^*(G)$ denotes the set of all strictly second maximal subgroups of G, i.e., for any $M \in Max(G,H)$, $H \lessdot M$.

The study of the embedding properties of subgroups in finite groups is one of the most fruitful research areas in the Group Theory. In particular, the embedding properties of maximal and second maximal subgroups tend to give additional information about the group. In terms of normality, we have that G is nilpotent if and only if every maximal subgroup of G is normal([7]). Also, if every second maximal subgroup is normal in G, then G is supersolvable([8]). Various generalizations of normality have been given, and many parallel results obtained on such topics ([10], [13], [15]). Such as, c-normality, cover and avoidance properties. In particular, the concept of a c-normal subgroup was introduced by Wang [15]. And they prove that a group is solvable if and only if every maximal subgroup is c-normal. As an application, Lv and Li [10] localize the above results and they prove that a group G is non-p-solvable if and only if there exists a maximal subgroup is non-c_p-normal in G.

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It is an important trend to extend the study of solvable groups to non-solvable groups, which has attracted the attention of many researchers([3],[9]). In this paper, some characterizations for a non-solvable group are obtained by using properties of the second maximal subgroups which are related to non-c_p-normal maximal subgroups. Consider the following families of subgroups for a given group G.

Definition 1.1. Let G be a group. We define
$Max^{c_p}(G) = \{M|\ M \lessdot G \ and\ M\ is\ c_p - normal\ in\ G\}$
$Max^{\overline{c_p}}(G) = \{M|\ M \lessdot G\ and\ M\ is\ not\ c_p - normal\ in\ G\}$.

Definition 1.2. Let G be a group. We define
$(Max^{\overline{c_p}})'(G) = Max^{\overline{c_p}}(G) \cup \{M|\ M \in Max^{c_p}(G)\ and\ P' \nleq M_p\}$;
$(Max_2^{\overline{c_p}})'(G) = \{H|\ H \lessdot M\ with\ M \in (Max^{\overline{c_p}})'(G)\}$;
$T_1'(G) = \{H|\ H \in (Max_2^{\overline{c_p}})'(G), \forall M \in Max(G,H)\ s.t.\ H_G = M_G\}$;
$T_3'(G) = \{H|\ H \in (Max_2^{\overline{c_p}})'(G), \forall M \in Max(G,H)\ s.t.\ H_G < M_G\}$;
$T_{13}'(G) = T_1'(G) \cup T_3'(G)$;
$X_2'(G) = (Max_2^{\overline{c_p}})'(G) \cap Max_2^*(G)$.

In 2021, Gao and Miao [6] defined a class of non-solvable groups S_p^* containing every group G whose every chief factor A/B satisfies one of the following conditions:

(1) A/B is a p-group; (2) A/B is a p'-group; (3) $|A/B|_p = p$.

As a further and continuation of the above research, we defined a new class of non-solvable groups S_p'.

$$S_p' = \{G|((A/B)_p)' = 1,\ for\ every\ G\ chief\ factor\ A/B\}.$$

For example, $(A_5)_2 \in S_2'$, $(A_6)_3 \in S_3'$. Obviously, S_p' contains S_p^*. Conversely, the reverse containment does not hold in general and $(A_6)_3$ is a counterexample. Furthermore, S_p' is a saturated formation and the subgroup of S_p' is closed.

In this paper, some characterizations for a finite group to belong to S_p' are obtained by using the above classes.

2. Preliminaries

Definition 2.1. [10, Definition 1.2] Let H be a subgroup of a finite group G and let p a fixed prime dividing the order of G. A subgroup H of G is said to be c_p-normal in G if there exists a normal subgroup K of G containing H_G such that $G = HK$ and $H \cap K/H_G$ is a p'-group.

Lemma 2.2. [10, Lemma 2.1] *Let G be a group, $H \leq G$ and $p \in \pi(G)$.*
(1) *If H is c-normal in G, then H is c_q-normal in G for each $q \in \pi(G)$;*
(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is c_p-normal in G if and only if H/N is c_p-normal in G/N;*
(3) *Let $H \leq K \leq G$. If H is c_p-normal in G, then H is c_p-normal in K;*
(4) *Every p'-subgroup of G is a c_p-normal subgroup of G.*

Lemma 2.3. [10, Corollary 3.3] *Let G be a group. Then G is p-solvable if and only if every maximal subgroup of G is c_p-normal in G.*

Definition 2.4. [5, Definition 2.1] Let A be a subgroup of a group G and H/K a chief factor of G. We will say that:
(1) A covers H/K if $H \leq KA$;
(2) A avoids H/K if $H \cap A \leq K$;
(3) A has the cover and avoidance properties in G, in brevity, A is a CAP-subgroup of G, if A either covers or avoids every chief factor of G.

Lemma 2.5. [16, Lemma 2.13] *Let H be a second maximal subgroup of a group G and $X \in Max(G, H)$. Assume that N is a normal subgroup of G such that $N \leq X$. If $N \nleq H$, then $X = HN$.*

Lemma 2.6. [1, Lemma 2.3.4] *Let N be a normal subgroup of a group G. A subgroup H of a group G is a minimal supplement of N in G if and only if $HN = G$ and $H \cap N \leq \Phi(H)$.*

Lemma 2.7. [16, Lemma 2.8] *Let N be a normal subgroup of G. A subgroup M of a group G is a minimal p-supplement of N in G if and only if $MN = G$ and $M \cap N \leq \Phi_p(M)$, where $\Phi_p(M) = \cap\{H|\ H \lessdot M,\ P \leq H \text{ and } P \in Syl_p(M)\}$.*

Lemma 2.8. [2, Theorem 2] *Let G be a finite group such that, for all primes p, $N_G(P)$ is nilpotent where P is a Sylow p-subgroup of G. Then G is nilpotent.*

Lemma 2.9. [11, Theorem 2.4] *Let G be a group and H be a second maximal subgroup of G. If $H = 1$, then G is solvable.*

Lemma 2.10. [14, Lemma 4] *If P is a Sylowp-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

3. Main results

Theorem 3.1. *Let G be a group. If $G \notin S_p'$, then $T_{13}'(G) \neq \emptyset$.*

Proof. We may assume that $T_{13}'(G) = \emptyset$. Then for any $H \in (Max_2^{\overline{c_p}})'(G)$, $H \notin T_{13}'(G)$. If G is a simple group, then $T_{13}'(G) = (Max_2^{\overline{c_p}})'(G) \neq \emptyset$, which is absurd. We need only to consider G is not a simple group.

Let L be a minimal normal subgroup of G. We consider the quotient group G/L. Then the quotient group G/L plainly satisfies the hypotheses of our theorem and L is the unique minimal normal subgroup of G. By the Frattini argument, we have $G = LN_G(L_p)$. If $N_G(L_p) = G$, then $L_p = L$, a contradiction. Hence we may assume that $N_G(L_p) < G$ and thus there exists a maximal subgroup M of G such that $N_G(L_p) \leq M$ and $M_G = 1$. We claim that $M \in (Max^{\overline{c_p}})'(G)$. If not, by Definition 2.1, there exists a normal subgroup K of G such that $G = MK$ and $M \cap K$ is a p'-group, i.e., $L_p \leq M \cap L \leq M \cap K$ is a p'-group, a contradiction. By hypothesis, $H \notin T_{13}'(G)$ for any maximal subgroup H of M and so there exists a maximal subgroup $M_1 \in Max(G, H)$ such that $H_G < (M_1)_G$. Thus $M_1 = LH < G$ by Lemma 2.5. It follows from Lemma 2.6 that $M \cap L \leq \Phi(M)$. Thereby $N_L(L_p) = N_G(L_p) \cap L \leq M \cap L \leq \Phi(M)$ is nilpotent.

For any $q \in \pi(L) \setminus \{p\}$, we have $G = LN_G(L_q)$ and there exists a maximal subgroup M_2 of G such that $N_G(L_q) \leq M_2$ and $(M_2)_G = 1$. If $M_2 \notin (Max^{\overline{c_p}})'(G)$, then $M_2 \in Max^{c_p}(G)$ and $P' \leq (M_2)_p$. Based on Definition 2.1 and the fact that $(L_p)' \leq P' \leq (M_2)_p$, we deduce that $(L_p)'$ is a p'-group, a contradiction. If $M_2 \in (Max^{\overline{c_p}})'(G)$, then $H_1 \notin T_{13}(G)$ for any $H_1 \lessdot M_2$. With the similar discussion as above and Lemma 2.5, we can obtain $H_1L < G$. By Lemma 2.6, $M_2 \cap L \leq \Phi(M_2)$. And so $N_L(L_q)$is nilpotent. By Lemma 2.8, L is solvable. Further, $G \in S_p'$, again a contradiction. □

Corollary 3.2. *Let G be a group. If $T_{13}'(G) = \emptyset$, then $G \in S_p'$.*

Theorem 3.3. *Let G be a group. If every subgroup $H \in T_{13}'(G)$ is a CAP-subgroup of G, then $G \in S_p'$.*

Proof. We assume that the result is false and let G be a counterexample with minimal order. By Corollary 3.2, $T_{13}'(G) \neq \emptyset$. In particular, G is not a simple group by Definition 2.4 and Lemma 2.9. Let L be a minimal normal subgroup of G and consider the quotient group G/L. Then it is easy to see that our hypotheses is quotient closed, and so by the choice of G, we see that $G/L \in S_p'$. We can also obtain $G = LM$, where M is a maximal subgroup of G such that $N_G(L_p) \leq M$ and $M_G = 1$. Obviously, $M \in (Max^{\overline{c_p}})'(G)$. Otherwise, we conclude that $(L_p)' = 1$ by Definition 2.1 and $(L_p)' \leq P' \leq M_p$, a contradiction. Now we consider the following cases separately.

(1) $L_p = P < M$. For any maximal subgroup H of M with $P \leq H$, if $H \in T'_{13}(G)$, then H is a CAP-subgroup of G. Since $M_G = 1$, it follows that $H \cap L = 1$ by Definition 2.4, which implies that $|L|_p = 1$, a contradiction. If $H \notin T'_{13}(G)$, then there exists a maximal subgroup $M_1 \in Max(G, H)$ such that $H_G < (M_1)_G$. By Lemma 2.5, $M_1 = HL < G$. Based on Lemma 2.7 and the fact that $\Phi_p(M)$ is p-closed, we see that $M \cap L \leq \Phi_p(M)$ and $L_p\ char\ M \cap L \trianglelefteq M$, i.e., $L_p \trianglelefteq M$. We claim that $(L_p)' = 1$. If not, we note that $M_{p'}$ is a Hall p'-subgroup of M and $M_{p'}(L_p)' < M$. Hence we may pick a maximal subgroup H_1 of M such that $M_{p'}(L_p)' \leq H_1$. With the similar discussion as above, we get $H_1 \notin T'_{13}(G)$. By Lemma 2.5, $H_1 L = G$, again a contradiction. Therefore, $G \in S'_p$, which is absurd.

(2) $L_p < P = M$. Notice that P possesses a maximal subgroup P_1 such that $L_p \nleq P_1$ by Lemma 2.10. If $P_1 \in T'_{13}(G)$, then P_1 is a CAP-subgroup of G. We have either $P_1 L = P_1$ or $P_1 \cap L = 1$. However, the former case is impossible and the latter case gives that $|L_p| \leq p$, a contradiction. If $P_1 \notin T'_{13}(G)$, then there exists a maximal subgroup $M_2 \in Max(G, P_1)$ such that $(P_1)_G < (M_2)_G$. By Lemma 2.5, $M_2 = P_1 L = G$, again a contradiction.

(3) $L_p < P < M$. In this case, we also obtain $M \cap L \leq \Phi_p(M)$ and $L_p \trianglelefteq M$. Notice that $(L_p)' \leq H_2$ for any $H_2 \lessdot M$ since $(L_p)' \leq \Phi(L_p) \leq \Phi(M)$. If $H_2 \in T'_{13}(G)$, then H_2 is a CAP-subgroup of G. It follows from $M_G = 1$ and Definition 2.4 that $(L_p)' = 1$, a contradiction. If $H_2 \notin T'_{13}(G)$, then $H_2 L < G$ by Lemma 2.5. In view of Lemma 2.6, $M \cap L \leq \Phi(M)$ and so $N_L(L_p)$ is nilpotent.

(4) The final contradiction.

For any $q \in \pi(L) \setminus \{p\}$, we have $G = LN_G(L_q)$. Then there exists a maximal subgroup M_3 of G such that $N_G(L_q) \leq M_3$ and $(M_3)_G = 1$. Clearly, $M_3 \in (Max^{\overline{c_p}})'(G)$. For any maximal subgroup H_3 of M_3, if $Q \leq H_3$, with the similar discussion as above, we can obtain $H_3 L < G$, $M_3 \cap L \leq \Phi_q(M_3)$ and $L_q \trianglelefteq M_3$. We need only to consider $Q \nleq H_3$. If $H_3 \notin T'_{13}(G)$, then $H_3 L < G$ by Lemma 2.5. If $H_3 \in T'_{13}(G)$, then we have either $H_3 L = H_3$ or $H_3 \cap L = 1$ by hypothesis. However, the fact taht $(M_3)_G = 1$ implies that the former case is impossible. For the latter case, we have $H_3 L = G$ or $H_3 L < G$. If $H_3 L = G$, then $M_3 = M_3 \cap H_3 L = H_3(M_3 \cap L)$. In view of $M_3 \cap L \trianglelefteq M_3$ and $H_3 \cap (M_3 \cap L) = 1$, $M_3 \cap L$ is a minimal normal subgroup of M_3. We also obtain $L_q = M_3 \cap L$ since $L_q \trianglelefteq M_3$. Then $N_L(L_q)$ is nilpotent, a contradiction. Thus $H_3 L < G$. Based on the discussion as above, for any $H_3 \lessdot M_3$, $H_3 L < G$. By Lemma 2.6, $M_3 \cap L \leq \Phi(M_3)$ is nilpotent, which implies that $N_L(L_q)$ is nilpotent. And so L is solvable by Lemma 2.8. Further, $G \in S'_p$, a contradiction. □

Theorem 3.4. *Let G be a group. If $G \notin S'_p$, then $X'_2(G) \neq \emptyset$.*

Proof. We assume that $X'_2(G) = \emptyset$. Then $H \in Max_2(G) \backslash Max^*_2(G)$ for any $H \in (Max_2^{\overline{c_p}})'(G)$. Now we consider the following cases:

Case 1: G is a simple group.

There exists a maximal subgroup M such that $P \leq M$, where $P \in Syl_p(G)$. If $M \notin (Max^{\overline{c_p}})'(G)$, then $M \in Max^{c_p}(G)$ and $P' \leq M_p$. Since $M_G = 1$, it follows that M is a p'-group by Definition 2.1, which contradicts $P \leq M$. Hence we may assume that $M \in (Max^{\overline{c_p}})'(G)$. Next we divide into the following two cases.

(1) $P = M$. Then for any $P_1 \lessdot P$, $P_1 \in Max_2(G) \backslash Max^*_2(G)$. Let $\tau = \{P_1^{M_1} | P_1 < \cdots < P_1^{M_1} \lessdot M_1 \lessdot G\}$. It follows from $P_1 \in Max_2(G) \backslash Max^*_2(G)$ that τ is a non-empty set and we may choose an element $P_1^{M_1} \in \tau$ of maximal order in τ. We get $P_1^{M_1} \in Max^*_2(G)$. Clearly, $M_1 \notin (Max^{\overline{c_p}})'(G)$. Otherwise, $P_1^{M_1} \in X'_2(G)$, a contradiction. So $M_1 \in Max^{c_p}(G)$ and $P' \leq (M_1)_p$. Then M_1 is a p'-group by Definition 2.1, a contradiction.

(2) $P < M$. In this case, there exists a maximal subgroup H of M such that $P \leq H$. Since $H \in Max_2(G) \backslash Max_2^*(G)$, there exists a second maximal subgroup $H^{M_2} \in Max_2^*(G)$ such that $H < \cdots < H^{M_2} \lessdot M_2 \lessdot G$. Obviously, $M_2 \notin (Max^{\overline{c_p}})'(G)$. And Definition 2.1 implies that M_2 is a p'-group, a contradiction.

Case 2: G is not a simple group.

Let L be a minimal normal subgroup of G and we consider the quotient group G/L. Then the quotient group G/L plainly satisfies the hypotheses of our theorem. By induction, we see that $G/L \in S_p'$. By the Frattini argument, we have $G = LN_G(L_p)$ and there exists a maximal subgroup M of G such that $N_G(L_p) \leq M$ and $M_G = 1$. We claim that $M \in (Max^{\overline{c_p}})'(G)$. If not, L_p is a p'-group by Definition 2.1, a contradiction. Then we consider the following cases separately.

(1) $L_p = P$. In this case, $L_p = P < M$. For any maximal subgroup H of M with $P \leq H$, by hypothesis, $H \in Max_2(G) \backslash Max_2^*(G)$. Thus there exists a second maximal subgroup $H^{M_3} \in Max_2^*(G)$ such that $H < \cdots < H^{M_3} \lessdot M_3 \lessdot G$. If $(M_3)_G = 1$, then $M_3 \notin (Max^{\overline{c_p}})'(G)$ and so we see immediately that L_p is a p'-group by Definition 2.1, a contradiction. Thereby, $(M_3)_G \neq 1$. Lemma 2.5 gives that $M_3 = LH < G$. By Lemma 2.7 and the fact that $\Phi_p(M)$ is p-closed, it follows that $M \cap L \leq \Phi_p(M)$ and L_p *char* $M \cap L \trianglelefteq M$, i.e., $L_p \trianglelefteq M$. We claim that $(L_p)' = 1$. If not, we note that $M_{p'}$ is a Hall p'-subgroup of M and $M_{p'}(L_p)' < M$. Hence we may pick a maximal subgroup H_1 of M such that $M_{p'}(L_p)' \leq H_1$. Since $H_1 \in Max_2(G) \backslash Max_2^*(G)$, there exists a second maximal subgroup $H^{M_4} \in Max_2^*(G)$ such that $H_1 < \cdots < H^{M_4} \lessdot M_4 \lessdot G$. If $(M_4)_G = 1$, then $M_4 \notin (Max^{\overline{c_p}})'(G)$ and so L_p is a p'-group by Definition 2.1, a contradiction. If $(M_4)_G \neq 1$, then $M_4 = LH_1 = G$ by Lemma 2.5, again a contradiction. Therefore $G \in S_p'$, a contradiction.

(2) $L_p < P$. For any $q \in \pi(L)$, we have $G = LN_G(L_q)$ and there exists a maximal subgroup M_5 of G such that $N_G(L_q) \leq M_5$ and $(M_5)_G = 1$. In view of Definition 2.1 and the fact that $(L_p)' \leq P' \leq (M_5)_p$, we obtain $M_5 \in (Max^{\overline{c_p}})'(G)$. By hypotheses, $H_2 \in Max_2(G) \backslash Max_2^*(G)$ for any $H_2 \lessdot M_5$. Then there exists a second maximal subgroup $H^{M_6} \in Max_2^*(G)$ such that $H_2 < \cdots < H^{M_6} \lessdot M_6 \lessdot G$. If $(M_6)_G = 1$, then $M_6 \notin (Max^{\overline{c_p}})'(G)$ by $H^{M_6} \in Max_2^*(G)$. Moreover, Definition 2.1 and the fact that $(L_p)' \leq P' \leq (M_6)_p$ indicate that $(L_p)' = 1$, again a contradiction. If $(M_6)_G \neq 1$, then $M_6 = LH_2 < G$ by Lemma 2.5. Lemma 2.6 implies that $M_5 \cap L \leq \Phi(M_5)$. Thus $N_L(L_q)$ is nilpotent. In view of Lemma 2.8, L is solvable. Further, $G \in S_p'$, again a contradiction. □

Corollary 3.5. *Let G be a group. If $X_2'(G) = \emptyset$, then $G \in S_p'$.*

Theorem 3.6. *Let G be a group. If every subgroup $H \in X_2'(G)$ is a CAP-subgroup of G, then $G \in S_p^{'}$.*

Proof. We assume that the result is false and let G be a counterexample with minimal order. Obviously, $X_2'(G) \neq \emptyset$ by Corollary 3.5. Moreover, by Definition 2.4 and Lemma 2.9, G is not a simple group. Let L be a minimal normal subgroup of G and we consider the quotient group G/L. It is clear that the hypotheses of the theorem are satisfied for the quotient group G/L of G. A trivial argument shows that G has the unique minimal normal subgroup L. By the Frattini argument, we have $G = LN_G(L_p)$. Then there exists a maximal subgroup M of G such that $N_G(L_p) \leq M$ and $M_G = 1$. We may assume that $M \in (Max^{\overline{c_p}})'(G)$ by Definition 2.1. Now we consider the following cases separately.

(1) $L_p = P$. For any maximal subgroup H of M with $P \leq H$, if $H \in Max_2^*(G)$, then we have either $HL = H$ or $H \cap L = 1$ by hypothesis and Definition 2.4. However, the former case is impossible and so $H \cap L = 1$. Since $P \leq H$, we get $|L_p| = 1$, a contradiction. Hence we may assume that $H \in Max_2(G) \backslash Max_2^*(G)$ and there exists a second maximal subgroup $H^{M_1} \in Max_2^*(G)$ such that $H < \cdots < H^{M_1} \lessdot M_1 \lessdot G$. If $(M_1)_G = 1$, then $M_1 \in (Max^{\overline{c_p}})'(G)$ by Definition 2.1. Thus, $H^{M_1} \in X_2'(G)$. Moreover, Definition 2.4 and

$(M_1)_G = 1$ indicate that $H^{M_1} \cap L = 1$. This gives $L_p = 1$, a contradiction. If $(M_1)_G \neq 1$, then Lemma 2.5 shows that $M_1 = HL < G$. Since Lemma 2.7 and $\Phi_p(M)$ is p-closed, it follows that $M \cap L \leq \Phi_p(M)$ and L_p *char* $M \cap L \trianglelefteq M$, i.e., $L_p \trianglelefteq M$. Using similar arguments as in the proof of Theorem 3.4, we also obtain $(L_p)' = 1$. Therefore, $G \in S'_p$, again a contradiction.

(2) $L_p < P$. For any $q \in \pi(L) \setminus \{p\}$, we have $G = LN_G(L_q)$ and there exists a maximal subgroup M_2 of G such that $N_G(L_q) \leq M_2$ and $(M_2)_G = 1$. It is easy to check that $M_2 \in (Max^{\overline{c_p}})'(G)$. Thereby, for $H \lessdot M_2$, if $Q \leq H$, then we also obtain $HL < G$, which implies that $M_2 \cap L \leq \Phi_q(M_2)$ and $L_q \trianglelefteq M_2$. We now turn to the case $Q \nleq H$. If $H \in Max_2^*(G)$, then H is a CAP-subgroup of G. Since $(M_2)_G = 1$, we have $H \cap L = 1$ by Definition 2.4. Thus $HL = G$ or $HL < G$. For the former case, using similar arguments as in the proof of Theorem 3.3, we have $N_L(L_q)$ is nilpotent, a contradiction. And so $HL < G$. If $H \in Max_2(G) \backslash Max_2^*(G)$, then there exists a second maximal subgroup $H^{M_3} \in Max_2^*(G)$ such that $H < \cdots < H^{M_3} \lessdot M_3 \lessdot G$. If $(M_3)_G = 1$, then we can also conclude that $N_L(L_q)$ is nilpotent or $HL < G$. For the former case, we have nothing to prove. Thereby, $HL < G$. If $(M_3)_G \neq 1$, then $M_3 = LH < G$ by Lemma 2.5. Based on the discussion as above, we always have $HL < G$ for any $H \lessdot M_2$. In view of Lemma 2.6, $M_2 \cap L \leq \Phi(M_2)$, which implies that $N_L(L_q)$ is nilpotent. By Lemma 2.8, L is solvable, a contradiction. □

Theorem 3.7. *Let G be a group. If $G \notin S'_p$, then $T'_{13}(G) \cap X'_2(G) \neq \emptyset$.*

Proof. In view of Theorem 3.4, we only need to consider that G is not simple. Clearly, $G/L \in S'_p$, where L is a unique minimal normal subgroup of G. Let $G = LN_G(L_q) = LM$, where $M \in (Max^{\overline{c_p}})'(G)$. We may claim that for any $H \lessdot M$, $H \notin T'_{13}(G)$. Therefore, $HL < G$ by Lemma 2.5. Otherwise, first suppose that $H_1 \in T'_{13}(G)$. There exists a chain of subgroups $H_1 < \cdots < H^{M_1} \lessdot M_1 \lessdot G$. Also we may assume that $M_1 \in (Max^{\overline{c_p}})'(G)$. On the other hand, note that $H^{M_1} \in Max_2^*(G)$, then $H^{M_1} \notin T'_{13}(G)$. This shows that $HL \lessdot G$, which contradicts with the fact $H_1 \in T'_{13}(G)$. Now we can conclude that L is solvable since for any $H \lessdot M$, $HL < G$, a final contradiction. □

Corollary 3.8. *Let G be a group. If $T'_{13}(G) \cap X'_2(G) = \emptyset$, then $G \in S'_p$.*

Theorem 3.9. *Let G be a group. If every subgroup $H \in T'_{13}(G) \cap X'_2(G)$ is a CAP-subgroup in G, then $G \in S'_p$.*

Proof. Similarly, suppose that $G/L \in S'_p$, where L is a unique minimal normal subgroup of G. Further, $G = LN_G(L_p) = LM$, where $M \in (Max^{\overline{c_p}})'(G)$ and $L_p < P$. For any $q \in \pi(L)$, we have $G = LN_G(L_q) = LM_1$, where $M_1 \in (Max^{\overline{c_p}})'(G)$. Let $H_1 \lessdot M_1$, if $Q \leq H_1$, we see that $H_1L < G$, $M_1 \cap L \leq \Phi_q(M_1)$ and then $L_q \trianglelefteq M$. For the case of $Q \nleq H_1$, by using similar arguments as in the proof of Theorem 3.6, we also obtain that $H_1L < G$, which implies that $N_L(L_q)$ is nilpotent. By Lemma 2.8, L is solvable, a contradiction. □

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