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A-DAVIS-WIELANDT-BEREZIN RADIUS INEQUALITIES

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Abstract. We consider operator V on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel $K_{\mathcal{H},\lambda}(z) = K(z, \lambda)$ and define A-Davis-Wielandt-Berezin radius $\eta_A(V)$ by the formula

$$\eta_A(V) := \sup \left\{ \sqrt{\left|\left\langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \right\rangle_A\right|^2 + \left\| V k_{\mathcal{H},\lambda} \right\|_A^4} : \lambda \in \Omega \right\}$$

and $\widetilde{V}$ is the Berezin symbol of V where any positive operator A-induces a semi-inner product on $\mathcal{H}$ is defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for $x, y \in \mathcal{H}$. We study equality of the lower bounds for A-Wielandt-Berezin radius mentioned above. We establish some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators. In addition, we get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

1. Introduction

Many researchers in mathematics and mathematical physics are interested in the Berezin symbol of an operator defined with the aid of a reproducing kernel Hilbert space. In this context, several mathematicians have conducted substantial research on the Berezin radius inequality (see [4, 14, 16, 20, 21]). In fact, it is of interest to academics to get refinements and extensions of this disparity. We show various inequalities for the A-Davis-Wielandt-Berezin radius of operators on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω in this study. By using A-Berezin transforms, we study some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of some operators. In addition, we get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

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We will now outline the preliminary concepts needed to proceed with the findings of this investigation.

Remember that a reproducing kernel Hilbert space (abbreviated RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω in which:

(a) the evaluation functionals

$$\varphi_\lambda(f) = f(\lambda),\ \lambda \in \Omega,$$

are continuous on $\mathcal{H}$;

(b) for every $\lambda \in \Omega$ there exists a function $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then, via the classical Riesz representation theorem, we know if $\mathcal{H}$ is an RKHS on Ω, there is a unique element $K_{\mathcal{H},\lambda} \in \mathcal{H}$ such that $h(\lambda) = \langle h, K_{\mathcal{H},\lambda}\rangle$ for every $\lambda \in \Omega$ and all $h \in \mathcal{H}$. The reproducing kernel at λ is denoted by the element $K_{\mathcal{H},\lambda}$. Further, we will denote the normalized reproducing kernel at λ as $k_{\mathcal{H},\lambda} := \frac{K_{\mathcal{H},\lambda}}{\|K_{\mathcal{H},\lambda}\|}$. Let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ including the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$.

Linear operators induced by functions are frequently encountered in functional analysis; they include Hankel operators, composition operators, and Toeplitz operators. The inducing function is sometimes referred to as the symbol of the resultant operator. In many circumstances, a linear operator on a Hilbert space $\mathcal{H}$ also gives rise to a function on Ω. Hence, we frequently examine operators induced by functions, and we may similarly research functions induced by operators. The Berezin symbol is an outstanding exemplar of an operator-function link. More accurately, for an operator $V \in \mathcal{L}(\mathcal{H})$, the Berezin symbol (transform) of V, denoted by $\widetilde{V}$, is the complex-valued function on Ω defined by

$$\widetilde{V}(\lambda) := \langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle .$$

For each bounded operator V on $\mathcal{H}$, the Berezin symbol $\widetilde{V}$ is a bounded real-analytic function on Ω. Features of the operator V, are often seen in the features of the Berezin transform $\widetilde{V}$. F. Berezin proposed the Berezin transform in [8] and it has proven to be a fundamental tool in operator theory, since many essential features of significant operators are contained in their Berezin transforms.

The Berezin radius (number) of operator V is defined by

$$\operatorname{ber}(V) := \sup_{\lambda\in\Omega} \left|\widetilde{V}(\lambda)\right| .$$

The Berezin set and the Berezin norm of operator are defined, respectively, by

$$\operatorname{Ber}(V) := \operatorname{Range}\left(\widetilde{V}\right) \text{ and } \|V\|_{\operatorname{Ber}} := \sup_{\lambda\in\Omega} \|V k_{\mathcal{H},\lambda}\| .$$

The Berezin transform and Berezin radius have been studied by many mathematicians over the years (see [3, 4, 14, 26]).

Recall that the Berezin range of an operator V is a subset of the numerical range of V,

$$W(V) = \{\langle Vu, u\rangle : \|u\| = 1\}.$$

It is well knowledge that $\mathrm{Ber}(V) \subseteq W(V)$, $\mathrm{ber}(V) \leq w(V)$ (numerical radius) and $\mathrm{ber}(V) \leq \|V\|_{\mathrm{Ber}}$. See [5, 9, 18, 22, 24, 27] for further details. Two of these generalizations are the Davis-Wielandt radius $dw(V)$ and Davis-Wielandt shell $DW(V)$ of $V \in \mathcal{L}(\mathcal{H})$ defined by

$$dw(V) := \sup\left\{\sqrt{|\langle Vu, u\rangle|^2 + \|Vu\|^4} : u \in \mathcal{H} \text{ and } \|u\| = 1\right\};$$

and

$$DW(V) := \left\{\left(\langle Vu, u\rangle, \|Vu\|^2\right) : u \in \mathcal{H} \text{ and } \|u\| = 1\right\} \subseteq \mathbb{C} \times \mathbb{R}$$

see [5, 10, 25, 28].

$\mathcal{N}(V)$, its range by $\mathcal{R}(V)$ and adjoint of V by V^* denote the null space of every operator V. If U is a linear subspace of $\mathcal{H}$, then $\overline{U}$ stands for its closure in the norm topology of $\mathcal{H}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called positive, denoted by $A \geq 0$, if $\langle Au, u\rangle \geq 0$ for all $u \in \mathcal{H}$. For $V \in \mathcal{L}(\mathcal{H})$, the absolute value of V, denoted by $|V|$, is defined as $|V| = (V^*V)^{1/2}$. Along with the article, A denotes a non-zero positive operator on $\mathcal{H}$. Notice that any positive operator A induces a semi-inner product on $\mathcal{H}$ defined by

$$\langle u, v\rangle_A := \langle Au, v\rangle_{\mathcal{H}}, \ \forall u, v \in \mathcal{H}.$$

The seminorm induced by $\langle ., .\rangle_A$ is given by $\|u\|_A = \sqrt{\langle u, u\rangle_A} = \left\|A^{1/2}u\right\|$ for all $u \in \mathcal{H}$.

It can be easily verified that $\|.\|_A$ is norm if and only if A is injective and that the seminormed space $(\mathcal{H}, \|.\|_A)$ which is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.

Definition 1. *For $V \in \mathcal{L}(\mathcal{H})$, the A-Berezin set of $\langle Vk_\lambda, k_\lambda\rangle_A$ is defined by*

$$\mathrm{Ber}_A(V) := \{\langle Vk_\lambda, k_\lambda\rangle_A : \lambda \in \Omega\}.$$

$\mathrm{Ber}_A(V)$ is a nonempty subset of $\mathbb{C}$ and it is in general not closed even if $\mathcal{H}$ is finite dimensional are important to be significant.

Definition 2. *(i) A-Berezin transform (also called A-Berezin symbol) $\widetilde{V}^A$ is defined on Ω by*

$$\widetilde{V}^A(\lambda) := \langle Vk_\lambda, k_\lambda\rangle_A \ \ (\lambda \in \Omega),$$

(ii) The supremum modulus of $\mathrm{Ber}_A(V)$*, denoted by* $\mathrm{ber}_A(V)$*, is referred to as the A-Berezin number of V, i.e.,*

$$\mathrm{ber}_A(V) := \sup_{\lambda \in \Omega} |\langle Vk_\lambda, k_\lambda\rangle_A|,$$

(iii) A-Berezin norm of operators $V \in \mathcal{L}(\mathcal{H}(\Omega))$ is defined by

$$\|V\|_{A-\mathrm{Ber}} := \sup_{\lambda \in \Omega} \|AVk_\lambda\|_{\mathcal{H}}.$$

We get the Berezin number if $A = I$. As a result of this new idea, the Berezin number of reproducing kernel Hilbert space operators and the Berezin norm of operators become more generic. See [15, 19] for further information on A-Berezin number inequalities.

Definition 3. *([12]) Let $V \in \mathcal{L}(\mathcal{H})$. An operator $U \in \mathcal{L}(\mathcal{H})$ is called an A-adjoint of V if for every $\lambda, \mu \in \Omega$, identity $\langle Vk_\lambda, k_\mu \rangle_A = \langle k_\lambda, Uk_\mu \rangle_A$ holds.*

Definition 4. *Let $V \in \mathcal{L}(\mathcal{H}(\Omega))$. An operator $U \in \mathcal{L}(\mathcal{H}(\Omega))$ is called (A, r)-adjoint of V if for every $\lambda, \mu \in \Omega$, the identity $\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle_A = \langle k_{\mathcal{H},\lambda}, Uk_{\mathcal{H},\lambda} \rangle_A$ holds.*

Following [12, 13], notice that the existence of an A-adjoint of V is identical to the existence of a solution of the equation $AX = V^*A$. Thanks to the Douglas theorem, these types of equations can be studied and the readers can consult to Moslehian et al. [23]. In summary, Douglas theorem states unequivocally that the operator equation $VX = U$ has a bounded linear solution X if and only if $\mathcal{R}(U) \subseteq \mathcal{R}(V)$. Furthermore, it has just one solution, represented by Q, that satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(V^*)}$ among its numerous solutions. This type of Q is known as the reduced solution or Douglas solution of $VX = U$. $\mathcal{L}_A(\mathcal{H})$ denotes the set of all operators in $\mathcal{L}(\mathcal{H})$ that admit A-adjoint. According to the Douglas theorem,

$$\mathcal{L}_A(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \mathcal{R}(V^*A) \subseteq \mathcal{R}(A)\}.$$

Moreover, $\mathcal{L}_{A^{1/2}}(\mathcal{H})$ denotes the set all operators admitting $A^{1/2}$-adjoints. When we use the Douglas theorem, we get

$$\mathcal{L}_{A^{1/2}}(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \exists \lambda > 0, \ \|Vu\|_A \leq \lambda \|u\|_A, \forall u \in \mathcal{H}\}.$$

A-bounded refers to the operator in $\mathcal{L}_{A^{1/2}}(\mathcal{H})$.

If $V \in \mathcal{L}_A(\mathcal{H})$, then the reduced solution (or Douglas solution) to the equation $AX = V^*A$ is a well-known A-adjoint operator of V, which is represented by V^{*_A}. We observe that

$$V^{*_A} = A^\dagger V^* A,$$

where $A^\dagger$ is the Moore-Penrose inverse of A (see [1, 2]). It is commonly known that the operator V^{*_A} satisfies

$$AV^{*_A} = V^*A, \ \mathcal{R}(V^{*_A}) \subseteq \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(V^{*_A}) = \mathcal{N}(V^*A).$$

Also, note that if $V \in \mathcal{L}_A(\mathcal{H})$, then $V^{*_A} \in \mathcal{L}_A(\mathcal{H})$ and $(V^{*_A})^{*_A} = P_A V P_A$, where P_A represents the ortogonal projection onto $\overline{\mathcal{R}(A)}$. Furthermore, if $V \in \mathcal{L}_A(\mathcal{H})$, then $\|V^{*_A}\| = \|V\|_A$. In order to reach more results and proofs related to these classes of operators, the researchers may want to overview [1, 2].

If AV is selfadjoint, that is, $AV = V^*A$, then an operator $V \in \mathcal{L}(\mathcal{H})$ is called to be A-selfadjoint. Furthermore, an operator V is said to be A-positive if $AV \geq 0$ and we write $V \geq_A 0$.

The Hilbert space $\left(\mathcal{R}\left(A^{1/2}\right), \langle ., . \rangle_{\mathbb{R}\left(A^{1/2}\right)}\right)$ shall be designated simply by $\mathbb{R}\left(A^{1/2}\right)$ in the sequal.

Feki in [12] has found some upper bounds for the A-Davis-Wielandt radius of operators in $\mathcal{L}_A(\mathcal{H})$.

Definition 5. *For any $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$, we define its A-Davis-Wielandt-Berezin shell and A-Davis-Wielandt-Berezin radius, respectively, by the formulas*

$$\mathbf{H}_A(V) := \left\{ \left(\langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle_A, \|A k_{\mathcal{H},\lambda}\|_A^2 \right), \ \lambda \in \Omega \right\}$$

and

$$\eta_A(V) := \sup_{\lambda \in \Omega} \sqrt{\left| \widetilde{V}^A(\lambda) \right|^2 + \|V k_{\mathcal{H},\lambda}\|_A^4}$$

It is apparent that $\eta_A(V) \leq dw_A(V)$. For $V, U \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ one has

(i) $\eta_A(V) \geq 0$ and $\eta_A(V) = 0$ if and only if $V = 0$;

(ii) $\eta_A(\alpha V) \begin{cases} \geq |\alpha| \eta_A(V) & \text{if } |\alpha| > 1 \\ = |\alpha| \eta_A(V) & \text{if } |\alpha| = 1 \\ \leq |\alpha| \eta_A(V) & \text{if } |\alpha| < 1. \end{cases}$

(iii) $\eta_A(V+U) \leq \sqrt{2\left(\eta_A(V) + \eta_A(U) + 4\left(\eta_A(V) + \eta_A(U)\right)^2\right)}$;

therefore $\eta_A(\cdot)$ cannot be a norm on $\mathcal{L}(\mathcal{H}(\Omega))$. The following property of $\eta_A(\cdot)$ is immediate:

$$\max\left\{ \mathrm{ber}_A(V), \|V\|_{A-\mathrm{ber}}^2 \right\} \leq \eta_A(V) \leq \sqrt{\mathrm{ber}_A^2(V) + \|V\|_{A-\mathrm{ber}}^4} \ (V \in \mathcal{L}_{A,r}(\mathcal{H})). \tag{1}$$

Recently, Bhanja et al. in [6] have reached some upper bounds for the A-Davis-Wielandt radius of operators in $\mathcal{L}_A(\mathcal{H}(\Omega))$. The purpose of this article is to find out some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators. For this aim, we employ some well-known inequalities for vectors in inner product spaces (see [6, 7, 11]). We also get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

In particular, for $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ we prove that

$$\eta_A^2(V) \leq \sup_{\theta \in \mathbb{R}} \mathrm{ber}_A^2\left(e^{i\theta} V + V^{*A} V\right) - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}^2(V)$$

and

$$\eta_A^2(V) \leq \inf_{z \in \mathbb{C}} \left\{ \left(2\left\|\mathrm{Re}(z)\,\mathrm{Re}_A(V) + \mathrm{Im}(z)\,\mathrm{Im}_A(V)\right\|_{A-\mathrm{ber}} + \left\|V^{*A} V - 2\,\mathrm{Re}(\overline{z} V)\right\|_{A-\mathrm{ber}} \right)^2 \right. \\ \left. + 2\left\|\mathrm{Re}(\overline{z} V)\right\|_{A-\mathrm{ber}} - |z|^2 + \mathrm{ber}_A^2(V - zI) \right\}.$$

2. Prerequisites

In the present section, we need some auxiliary lemmas including Buzano [7] inequality, Dragomir [11] inequality and Bhanja et al. [6] inequality in order to prove our results.

Buzano [7] made an extension of the Cauchy-Schwarz inequality which states that for any $a_1, a_2, a_3 \in \mathcal{H}$ with $\|a_3\| = 1$

$$|\langle a_1, a_3\rangle \langle a_3, a_2\rangle| \leq \frac{1}{2}\left(|\langle a_1, a_2\rangle| + \|a_1\| \|a_2\|\right). \tag{2}$$

Dragomir [11] proved the following inequalities.

Lemma 1. *Let u_1, $u_2 \in \mathcal{H}$ and $z \in \mathbb{C}$. Then the following equality holds:*

$$\|u_1\|^2 \|u_2\|^2 - |\langle u_1, u_2\rangle|^2 = \|u_1 - zu_2\|^2 \|u_2\|^2 - |\langle u_1 - zu_2, u_2\rangle|^2.$$

We need the following lemmas, given in [6].

Lemma 2. *Let $u_1, u_2, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$|\langle u_1, e\rangle_A \langle e, u_2\rangle_A| \leq \frac{1}{2}\left(|\langle u_1, u_2\rangle_A| + \|u_1\|_A \|u_2\|_A\right). \tag{3}$$

Lemma 3. *Let $u_1, u_2, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$\|u_1\|_A^2 \|u_2\|_A^2 - |\langle u_1, u_2\rangle_A|^2 \geq 2\,|\langle u_1, e\rangle_A \langle e, u_2\rangle_A| \left(\|u_1\|_A \|u_2\|_A - |\langle u_1, u_2\rangle_A|\right).$$

Lemma 4. *Let $u_1, u_2, e \in \mathcal{H}$ and $z \in \mathbb{C}$. Then we have the following equality:*

$$\|u_1\|_A^2 \|u_2\|_A^2 - |\langle u_1, u_2\rangle_A|^2 = \|u_1 - zu_2\|_A^2 \|u_2\|_A^2 - |\langle u_1 - zu_2, u_2\rangle_A|^2.$$

3. Main results

We use the lemmas from the preceding section to derive additional inequalities for the A-Davis-Wielandt-Berezin radius of operators on $\mathcal{H} = \mathcal{H}(\Omega)$.

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. The A-Berezin symbol of operator $V \in \mathcal{L}(\mathcal{H}(\Omega))$ is naturally defined the by the formula

$$\widetilde{V}^A(\lambda) := \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A = \langle AVk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle, \ \lambda \in \Omega.$$

Therefore, $\mathcal{L}_{A,r}(\mathcal{H}) := \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ denotes the set of all operators in $\mathcal{L}(\mathcal{H}(\Omega))$ admitting (A, r)-adjoints.

For $V \in \mathcal{L}_{A,r}(\mathcal{H})$, its Crawford number $c_A(V)$ is defined by

$$c_A(V) := \inf\{|\langle Vu, u\rangle_A| : u \in \mathcal{H}, \ \|u\|_A = 1\}$$

(see [27]). We also introduce the number $\widetilde{c}_A(V) := \inf_{\lambda\in\Omega} \left|\widetilde{V}^A(\lambda)\right|$. It is clear that

$$c_A(V) \leq \widetilde{c}_A(V) \leq \mathrm{ber}_A(V).$$

Our first result in this paper reads as follows.

Theorem 1. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then, the following inequalities hold.*

(i) $\eta_A^2(V) \geq \max\left\{\mathrm{ber}_A^2(V) + \widetilde{c}_A^2\left(V^{*A}V\right), \|V\|_{A\text{-Ber}}^4 + \widetilde{c}_A^2(V)\right\}$,

(ii) $\eta_A^2(V) \geq 2\max\left\{\mathrm{ber}_A(V)\,\widetilde{c}_A\left(V^{*A}V\right), \widetilde{c}_A(V)\,\|V\|_{A-\mathrm{Ber}}^2\right\}$.

Proof. For any $\lambda \in \Omega$, we have

$$\begin{aligned}\eta_A^2(V) &\geq \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\ &= \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2 \\ &\geq \left|\widetilde{V}^A(\lambda)\right|^2 + \inf_{\lambda\in\Omega}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2,\end{aligned}$$

hence, taking supremum over $\lambda \in \Omega$ gives

$$\eta_A^2(V) \geq \mathrm{ber}_A^2(V) + \widetilde{c}_A^2\left(V^{*A}V\right).$$

Moreover, by taking into consideration $\eta_A^2(V) \geq \left|\widetilde{V}^A(\lambda)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4$, we see that

$$\eta_A^2(V) \geq \widetilde{c}_A^2(V) + \|Vk_{\mathcal{H},\lambda}\|_A^4.$$

Hence, on taking the supremum over $\lambda \in \Omega$, we obtain

$$\eta_A^2(V) \geq \widetilde{c}_A^2(V) + \|V\|_{A-\mathrm{Ber}}^4,$$

which proves (i).

Let $\lambda \in \Omega$ be arbitrary. It can be observed that

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\|Vk_{\mathcal{H},\lambda}\|_A^2 \tag{4}$$

and

$$\begin{aligned}\eta_A^2(V) &\geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\ &\geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\inf_{\lambda\in\Omega}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\ &= 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\widetilde{c}_A\left(V^{*A}V\right).\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$\eta_A^2(V) \geq 2\mathrm{ber}_A(V)\,\widetilde{c}_A\left(V^{*A}V\right).$$

From the inequality (4), we get

$$\eta_A^2(V) \geq 2\widetilde{c}_A(V)\,\|Vk_{\mathcal{H},\lambda}\|_A^2.$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$\eta_A^2(V) \geq 2\widetilde{c}_A(V)\,\|V\|_{A\text{-Ber}}^2.$$

Hence the proof is complete. □

Remark 1. *It is clear that the lower bound obtained in Theorem 1 (i) is more solid than that in (1). Also, both of inequalities in ([17], Th. 1) follow from Theorem 1 by considering $A = I$.*

For $A \in \mathcal{L}(\mathcal{H}(\Omega))$, we define

$$m_{A-\mathrm{ber}}^2(V) := \inf_{\lambda\in\Omega} \|Vk_{\mathcal{H},\lambda}\|_A^2 .$$

We get an upper bound for the A-Davis-Wielandt-Berezin radius of bounded linear operators on RKHS in the following result.

Theorem 2. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then*

$$\eta_A^2(V) \le \sup_{\theta\in\mathbb{R}} \mathrm{ber}_A^2\left(e^{i\theta}V + V^{*A}V\right) - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}^2(V).$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Then there exists $\theta \in \mathbb{R}$ such that

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| = e^{i\theta}\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A .$$

Now,

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\\
&= \left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2 + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2\\
&= \left(\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right)^2\\
&\quad - 2\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A .
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&2\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\\
&= \left\langle \left(e^{i\theta}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2\\
&\le \mathrm{ber}_A^2\left(e^{i\theta}V + V^{*A}V\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left|\langle V_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|^4\\
&\le \sup_{\theta\in\mathbb{R}} \mathrm{ber}_A^2\left(e^{i\theta}V + V^{*A}V\right)
\end{aligned}$$

and so,

$$2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}^2(V) + \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \le \sup_{\theta\in\mathbb{R}} \mathrm{ber}_A^2\left(e^{i\theta}V + |V|_A^2\right).$$

Hence, taking supremum over $\lambda \in \Omega$ gives

$$\eta_A^2(V) \le \sup_{\theta\in\mathbb{R}} \mathrm{ber}_A^2\left(e^{i\theta}V + V^{*A}V\right) - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}^2(V).$$

This completes the proof. □

Remark 2. *According to the inequality in ([17], Th. 2),*

$$\eta^2(V) \le \sup_{\theta\in\mathbb{R}} \mathrm{ber}^2\left(e^{i\theta}V + V^*V\right) - 2\widetilde{c}(V)\, m_{\mathrm{ber}}^2(V).$$

This shows that the inequality in ([17], Th. 2) follows from Theorem 2 by considering $A = I$.

We can now show the following inequality for the A-Davis-Wielandt-Berezin radius of bounded linear operators.

Theorem 3. *Let* $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. *Then*

$$\frac{1}{2}\left\{\mathrm{ber}_A^2\left(V + V^{*A}V\right) + \widetilde{c}_A^2\left(V - V^{*A}V\right)\right\} \leq \eta_A^2(V)$$
$$\leq \frac{1}{2}\left\{\mathrm{ber}_A^2\left(V + V^{*A}V\right) + \mathrm{ber}_A^2\left(V - V^{*A}V\right)\right\}.$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Then

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&= \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&+ \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&+ \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&= \frac{1}{2}\left|\left\langle\left(V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \frac{1}{2}\left|\left\langle\left(V - V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\geq \frac{1}{2}\left\{\left|\left\langle\left(V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \widetilde{c}_A^2\left(V - V^{*A}V\right)\right\}
\end{aligned}$$

Therefore, taking supremum over $\lambda \in \Omega$, we get

$$\eta_A^2(V) \geq \frac{1}{2}\left\{\mathrm{ber}_A^2\left(V + V^{*A}V\right) + \widetilde{c}_A^2\left(V - V^{*A}V\right)\right\}.$$

Similarly,

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&= \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&+ \frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \frac{1}{2}\left|\left\langle\left(V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \frac{1}{2}\left|\left\langle\left(V - V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2.
\end{aligned}$$

Therefore, taking supremum over $\lambda \in \Omega$, we get

$$\eta_A^2(V) \leq \frac{1}{2}\left\{\mathrm{ber}_A^2\left(V + V^{*A}V\right) + \mathrm{ber}_A^2\left(V - V^{*A}V\right)\right\}.$$

Hence completes the proof. □

Now we give upper bounds for the A-Davis-Wielandt-Berezin radius of $V \in \mathcal{L}_{A,r}(\mathcal{H})$.

Theorem 4. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then the inequalities listed below are true.*
(i) $\eta_A^2(V) \leq \left\| V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V \right\|_{A-\mathrm{ber}}$,
(ii) $\eta_A^2(V) \leq \frac{1}{2}\left(\mathrm{ber}_A\left(V^2\right) + \|V\|_A^2\right) + \|V\|_{A-\mathrm{Ber}}^4$.

Proof. Let $\lambda \in \Omega$ be arbitrary. Applying (3) for $u_1 = Vk_{\mathcal{H},\lambda}$, $e = k_{\mathcal{H},\lambda}$ and $u_2 = Vk_{\mathcal{H},\lambda}$, we have that

$$\begin{aligned}\left|\widetilde{V}(\lambda)\right|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 &= \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle k_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right| \\ &\quad + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \left\langle k_{\mathcal{H},\lambda}, V^{*A}Vk_{\mathcal{H},\lambda}\right\rangle_A \\ &\leq \frac{1}{2}\left(\|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right) \\ &\quad + \frac{1}{2}\left(\left\|V^{*A}Vk_{\mathcal{H},\lambda}\right\|_A^2 + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, V^{*A}Vk_{\mathcal{H},\lambda}\right\rangle_A\right) \\ &= \left\langle \left(V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.\end{aligned}$$

taking the supremum over $\lambda \in \Omega$, we have

$$\sup_{\lambda\in\Omega}\left\{\left|\widetilde{V}(\lambda)\right|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\right\} \leq \sup_{\lambda\in\Omega}\left\langle \left(V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.$$

This proves (i). The proof of (ii) is immediate from

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 = \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \left\langle k_{\mathcal{H},\lambda}, V^{*A}k_{\mathcal{H},\lambda}\right\rangle_A\right| \tag{5}$$

by applying (3) for $u = Vk_{\mathcal{H},\lambda}$, $e = k_{\mathcal{H},\lambda}$, $v = V^*k_{\mathcal{H},\lambda}$ in (5). The theorem is proved. □

It is widely known that if V is A-normaloid then $\left\|V^2\right\|_A = \|V\|_A^2$. Hence, both the inequalities in Theorem 4 becomes equality if V is A-normaloid can be observed easily.

We now obtain another upper bounds for the Davis-Wielandt-Berezin radius of bounded linear operators.

Theorem 5. *If $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$, then we have*

$$\begin{aligned}\eta_A^2(V) \leq 3\left\|\left(V^{*A}V\right)^{*A}V^{*A}V + V^{*A}V\right\|_{A-\mathrm{ber}} &- \widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\ &- \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).\end{aligned} \tag{6}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. It follows from Lemmas 2-3 that

$$\begin{aligned}&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\ &\leq \|Vk_{\mathcal{H},\lambda}\|_A^2 \|k_{\mathcal{H},\lambda}\|_A^2 \\ &- 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \left(\|Vk_{\mathcal{H},\lambda}\|_A \|k_{\mathcal{H},\lambda}\|_A - \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\right)\end{aligned}$$

$$
\begin{aligned}
&= \|Vk_{\mathcal{H},\lambda}\|_A^2 + 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \left|\langle k_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right| - 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \|Vk_{\mathcal{H},\lambda}\|_A \\
&\leq \|Vk_{\mathcal{H},\lambda}\|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A - 2\widetilde{c}_A(V)\|Vk_{\mathcal{H},\lambda}\|_A \\
&\leq 3\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}(V).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&= \frac{1}{2}\left(\left|\|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \left|\|Vk_{\mathcal{H},\lambda}\|_A^2 - \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2\right) \\
&= \frac{1}{2}\left(\left|\left\langle\left(V^{*A}V + V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle\left(V^{*A}V - V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2\right) \\
&\leq \frac{1}{2}\Big(3\left\langle\left|V^{*A}V + V\right|_A^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A\left(V^{*A}V + V\right)m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad +3\left\langle\left|V^{*A}V - V\right|_A^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A\left(V^{*A}V - V\right)m_{A-\mathrm{ber}}\left(V^{*A}V - V\right)\Big) \\
&= \frac{3}{2}\left\langle\left(\left|V^{*A}V + V\right|_A^2 + \left|V^{*A}V - V\right|_A^2\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \widetilde{c}_A\left(V^{*A}V + V\right)m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad - \widetilde{c}_A\left(V^{*A}V - V\right)m_{A-\mathrm{ber}}\left(V^{*A}V - V\right) \\
&= 3\left\langle\left(\left(V^{*A}V\right)^{*A}V^{*A}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \widetilde{c}_A\left(V^{*A}V + V\right)m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) - \widetilde{c}_A\left(V^{*A}V - V\right)m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).
\end{aligned}
$$

Thus, by taking supremum over $\lambda \in \Omega$, we obtain

$$
\begin{aligned}
\sup_{\lambda\in\Omega}\left(\left|\widetilde{V}^A(\lambda)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\right) &\leq 3\sup_{\lambda\in\Omega}\left\langle\left(\left(V^{*A}V\right)^{*A}V^{*A}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \sup_{\lambda\in\Omega}\widetilde{c}_A\left(V^{*A}V + V\right)m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad - \widetilde{c}_A\left(V^{*A}V - V\right)m_{A-\mathrm{ber}}\left(V^{*A}V - V\right)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\eta_A^2(V) \leq 3&\left\|\left(V^{*A}V\right)^{*A}V^{*A}V + V^{*A}V\right\|_{A-\mathrm{ber}} \\
&- \widetilde{c}_A\left(V^{*A}V + V\right)m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&- \widetilde{c}_A\left(V^{*A}V - V\right)m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).
\end{aligned}
$$

This immediately proves (6) as required. □

We are now able to establish the following theorem.

Theorem 6. *Let* $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. *Then the inequalities listed below are true.*

(i)

$$\eta_A^2(V) \le \inf_{r\in\mathbb{R}} \sup_{\theta\in\mathbb{R}} \Big\{ 2|r| \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI \right\|_A + \frac{1}{2} \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI \right\|_A^2 + \frac{1}{2} \left\| \cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 \Big\}.$$

(ii)

$$\eta_A^2(V) \le \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\{ \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 + \left\| \cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 \right\}.$$

Proof. (i) Let $\lambda \in \Omega$ be arbitrary. Then there exists $\theta \in \mathbb{R}$ such that $\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| = e^{-i\theta}\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A$. By applying the Cartesian decomposition of V, we see that

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| &= \left\langle e^{-i\theta} Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&= \langle ((\cos\theta - i\sin\theta)(\operatorname{Re}_A(V) + i\operatorname{Im}_A(V)))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&= \langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + i\,\langle (\cos\theta \operatorname{Im}_A(V) - \sin\theta \operatorname{Re}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A .
\end{aligned}$$

So, by $\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \in \mathbb{R}$ we get

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| = \langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A .$$

Thus, by using Lemma 4, we get for any $r \in \mathbb{R}$,

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 &= \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \|(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}\|_A^2 \\
&\quad - \|(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda} - rk_{\mathcal{H},\lambda}\|_A^2 \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda} - rk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|_A^2 \\
&= \left\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))^2\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \left\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)^2\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \Big\langle \Big\{ (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))^2 \\
&\quad - (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)^2 \Big\}\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \Big\rangle_A \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2
\end{aligned}$$

$$= \left\langle \left(2r\left(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V)\right) - r^2 I\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \right\rangle_A$$
$$+ \left|\left\langle \left(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \right\rangle_A\right|^2.$$

By using Lemma 4, we obtain

$$\begin{aligned}\|Vk_{\mathcal{H},\lambda}\|_A^4 &= \left|\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &= \left\langle \left(2rV^{*A}V - r^2 I\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left|\left\langle \left(V^{*A}V - rI\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2.\end{aligned}$$

Now,

$$\begin{aligned}&\left|\left\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&= \left\langle 2r\left\{\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\} k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2r^2 \\
&\quad + \frac{1}{2}\left|\left\langle \left(\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\quad + \frac{1}{2}\left|\left\langle \left(\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\
&\le 2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A \\
&\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\
&\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - |V|_A^2 + \sin\theta \operatorname{Im}_A(V)\right\|_A^2 \\
&\le \sup_{\theta\in\mathbb{R}}\left\{2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A\right. \\
&\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\
&\quad \left. + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we get

$$\begin{aligned}\eta_A^2(V) \le \sup_{\theta\in\mathbb{R}}&\left\{2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A\right. \\
&+ \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\
&\left. + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

Because this inequality holds for every $r \in \mathbb{R}$, we have the required inequality.

(ii) If we pick $r = 0$, for example,

$$\begin{aligned}\eta_A^2(V) \le \frac{1}{2}\sup_{\theta\in\mathbb{R}}&\left\{\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right. \\
&\left. + \left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

□

Following so, we find the inequality shown below.

Theorem 7. *Let* $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$*. Then the inequalities listed below are true.*

(i)

$$\eta_A^2(V) \le \inf_{z\in\mathbb{C}} \Big\{ \Big(2\left\| \operatorname{Re}(z) \operatorname{Re}_A(V) + \operatorname{Im}(z) \operatorname{Im}_A(V) \right\|_{A-\mathrm{ber}} + \left\| V^{*A}V - 2\operatorname{Re}(\overline{z}V) \right\|_{A-\mathrm{ber}} \Big)^2 + 2\left\| \operatorname{Re}(\overline{z}V) \right\|_{A-\mathrm{ber}} - |z|^2 + \mathrm{ber}_A^2(V - zI) \Big\}.$$

(ii) $\eta_A^2(V) \le \mathrm{ber}_A^2(V) + \|V\|_{A-\mathrm{ber}}^4$.

Proof. Let $z \in \mathbb{C}$. Choosing in Lemma 4 $u_1 = Vk_{\mathcal{H},\lambda}$ and $u_2 = k_{\mathcal{H},\lambda}$, we have for all $\lambda \in \Omega$

$$\|Vk_{\mathcal{H},\lambda}\|_A^2 \|k_{\mathcal{H},\lambda}\|_A^2 - \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 = \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \|k_{\mathcal{H},\lambda}\|_A^2 - \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2.$$

Then by using the Cartesian decomposition of V we have that

$$\begin{aligned}
\|Vk_{\mathcal{H},\lambda}\|_A^2 &= \left(\langle \operatorname{Re}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 - \left(\langle \operatorname{Re}_A(V - zI) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 \\
&\quad + \left(\langle \operatorname{Im}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 - \left(\langle \operatorname{Im}_A(V - zI) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 \\
&\quad + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= \langle (2\operatorname{Re}_A(V) - \operatorname{Re}(z) I) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle \operatorname{Re}(z) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + \langle (2\operatorname{Im}_A(V) - \operatorname{Im}(z) I) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle \operatorname{Im}(z) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= 2\operatorname{Re}(z) \langle \operatorname{Re}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + 2\operatorname{Im}(z) \langle \operatorname{Im}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad - (\operatorname{Re}(z))^2 - (\operatorname{Im}(z))^2 + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= 2\left(\operatorname{Re}(z) \langle \operatorname{Re}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \operatorname{Im}(z) \langle \operatorname{Im}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right) \\
&\quad - |z|^2 + \langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\rangle_A \\
&= 2\left(\operatorname{Re}(z) \langle \operatorname{Re}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \operatorname{Im}(z) \langle \operatorname{Im}_A(V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right) \\
&\quad + \left\langle \left(V^{*A}V - 2\operatorname{Re}_A(\overline{z}V)\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.
\end{aligned}$$

Again by using Lemma 4, we get

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 &= \|Vk_{\mathcal{H},\lambda}\|_A^2 - \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= 2\langle \operatorname{Re}(\overline{z}V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - |z|^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
&\left|\widetilde{V}^A(z)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&\le 2\langle \operatorname{Re}(\overline{z}V) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - |z|^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+2\left\langle\left(\operatorname{Re}(z)\operatorname{Re}_A(V)+\operatorname{Im}(z)\operatorname{Im}_A(V)\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\\
&+\left\langle\left(V^{*A}V-2\operatorname{Re}_A(\overline{z}V)\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A)^2.
\end{aligned}
$$

for all $\lambda\in\Omega$. Hence, taking supremum over $\lambda\in\Omega$, and infimum over all $z\in\mathbb{C}$, we have

$$
\begin{aligned}
\eta_A^2(V)\leq \inf_{z\in\mathbb{C}}\Big\{&\left(2\left\|\operatorname{Re}(z)\operatorname{Re}_A(V)+\operatorname{Im}(z)\operatorname{Im}_A(V)\right\|_{A-\mathrm{ber}}+\left\|V^{*A}V-2\operatorname{Re}_A(\overline{z}V)\right\|_{A-\mathrm{ber}}\right)^2\\
&+2\left\|\operatorname{Re}_A(\overline{z}V)\right\|_{A-\mathrm{ber}}-|z|^2+\mathrm{ber}_A^2(V-zI)\Big\}.
\end{aligned}
$$

(ii) Taking $z=0$, we get $\eta_A^2(V)\leq \mathrm{ber}_A^2(V)+\|V\|_{A-\mathrm{ber}}^4$. This proves the required result. □

Then, we have an upper bound on the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

Theorem 8. *Let U, $V\in\mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then the inequalities listed below are true.*
(i) $\eta_A(U+V)\leq\eta_A(U)+\eta_A(V)+\mathrm{ber}_A\left(U^{*A}V+V^{*A}U\right)$;
(ii) If $U^{*A}V+V^{*A}U=0$, *then* $\eta_A(U+V)\leq\eta_A(U)+\eta(V)$.

Proof. (i) It follows from Definition 5 that

$$
\begin{aligned}
\mathbf{H}_A(U+V)&=\left\{\left(\left\langle(U+V)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A,\left\langle(U+V)k_{\mathcal{H},\lambda},(U+V)k_{\mathcal{H},\lambda}\right\rangle_A\right),\ \lambda\in\Omega\right\}\\
&=\left\{\left(\left\langle Uk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A,\left\langle Uk_{\mathcal{H},\lambda},Uk_{\mathcal{H},\lambda}\right\rangle_A\right)\right.\\
&\quad+\left(\left\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A,\left\langle Vk_{\mathcal{H},\lambda},Vk_{\mathcal{H},\lambda}\right\rangle_A\right)\\
&\quad\left.+\left(0,\left\langle\left(U^{*A}V+V^{*A}U\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right):\lambda\in\Omega\right\}.
\end{aligned}
$$

So, $\mathbf{H}_A(U+V)\subseteq\mathbf{H}_A(U)+\mathbf{H}_A(V)+X$, where

$$
X=\left\{\left(0,\left\langle\left(U^{*A}V+V^{*A}U\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right):\lambda\in\Omega\right\}.
$$

This demonstrates (i). The evidence of (ii) is obvious from (i) and $A\left(U^{*A}V+V^{*A}U\right)=O$, and the proof of theorem is completed. □

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