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Equivalents of various maximum principles

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Abstract

Certain maximum principles can be reformulated to various types of fixed point theorems and conversely, based on Metatheorem due to ourselves. Such principles are Zorn's lemma, Banach contraction principle, Nadler's fixed point theorem, Brézis-Browder principle, Caristi's fixed point theorem, Ekeland's variational principle, Takahashi's nonconvex minimization theorem, some others and their variants, generalizations or equivalent formulations. Consequently, we have many new theorems equivalent to known results on fixed point, common fixed point, stationary point, common stationary point, and others. We show that such points are all maximal elements of certain ordered sets. Further we introduce our earlier related works as a history of our Metatheorem.

Keywords: Banach, Nadler, Zorn, Brézis-Browder, Caristi, Kirk, Ekeland, Takahashi, pre-order, quasi-metric space, fixed point, stationary point.
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1. Introduction

The celebrated existence principles like Zorn's lemma, the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, Takahashi's nonconvex minimization theorem, and many others are forceful tools in nonlinear analysis, control theory, economic theory, global analysis, and many other mathematical sciences. These theorems are extended by a large number of authors since their birth.

In 1982-2000, we published several articles mainly related to the Caristi fixed point theorem, the Ekeland variational principle for approximate solutions of minimization problems, and their equivalent formulations with some applications; see [2], [11]-[25]. From the beginning of such study, we obtained a Metatheorem for some equivalent statements on maximality, fixed points, stationary points, common fixed points, common

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stationary points, and others. We applied the Metatheorem for various occasions. However, for a long period, the Metatheorem was not attracted any attention from others.

Our basic philosophy of the Metatheorem is that certain order theoretic maximal elements are corresponding fixed elements or stationary elements of maps or multimaps and common fixed elements or common stationary elements of a family of maps or multimaps.

In the present article, we add up some statements to the previous versions of the Metatheorem and, by applying new Metatheorem, we obtain logically equivalent formulations of existence of maximal elements of pre-ordered set, Zorn's lemma, Banach contraction principle, Nadler's fixed point theorem, Brézis-Browder principle, Caristi's fixed point theorem, Ekeland's variational principle, Takahashi's nonconvex minimization theorem, and other various results.

Consequently, we have many new theorems equivalent to known results on fixed point, common fixed point, stationary point, common stationary point, and others. We show that such points are all maximal elements of certain ordered sets in a series of many works.

We organize this article as follows: Section 2 concerns with an extended new version of our Metatheorem given in [20]-[22].

In Section 3, as direct consequences of Metatheorem, we give the particular versions for pre-ordered sets, equivalent forms of Zorn's Lemma, Banach contraction principle, Nadler's fixed point theorem, Kirk's fixed point theorem, Brézis-Browder principle, Caristi's fixed point theorem, the Ekeland variational principle, Turinici's fixed point theorem, Takahashi's minimization theorem, a generalization of the Ekeland principle due to Kada et al., and an extension of a Caristi type theorem on vector valued metric spaces due to Agarwal-Khamsi.

Section 4 deals with some history of related matters to Metatheorem given in Section 2. In fact , we introduce the contents of our previous works [2] and [11]-[27], which were mainly concerned with extensions or applications of the Caristi fixed point theorem, the Ekeland variational principle and related results.

Finally in Section 5, we deal with some final remarks.

2. A metatheorem related to the Ekeland principle

The well-known central result of I. Ekeland [5]-[7] on the variational principle for approximate solutions of problems runs as follows:

Theorem E. ([5]) *Let V be a complete metric space, and $F : V \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ be given, and a point $u \in V$ such that $F(u) \leqq \inf_V F + \varepsilon$. Then for every $\lambda > 0$, there exists a point $v \in \overline{\mathrm{B}}(u, \lambda)$ such that $F(v) \leqq F(u)$ and $F(w) > F(v) - \varepsilon\lambda^{-1}d(v, w)$ for any $w \in V$, $w \neq v$.*

When $\lambda = 1$, this is called the ε-variational principle. In order to obtain some equivalents of this principle, we obtained a Metatheorem in [20]-[22]. Later we found two additional conditions and, consequently, we obtain a new extended version of Metatheorem. Now we add its simplified proof for the completeness.

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following seven statements are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v, w)$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

(iv) *If $T : A \multimap X$ is a multimap such that $\neg G(x, y)$ holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(v) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $\neg G(x, fx)$ *for all* $x \in A$ *with* $x \neq fx$*, then* $\mathcal{F}$ *has a common fixed element* $v \in A$*, that is,* $v = fv$ *for all* $f \in \mathfrak{F}$*.*

(vi) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $\neg G(x, y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(vii) *If* Y *is a subset of* X *such that for each* $x \in A\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $\neg G(x, z)$*, then there exists a* $v \in A \cap Y$*.*

Here, multimaps have nonempty values and $\neg$ denotes the negation. Note that the element $v \in A$ is the same for all (i)–(vii).

For the completeness, we give the proof of (i)–(v) as in [20] and add for (vi) and (vii).

Proof. (i)$\Longrightarrow$(ii): Suppose $v \notin T(v)$. Then there exists a $y \in X\backslash\{v\}$ satisfying $\neg G(v, y)$. This is a contradiction.

(ii)$\Longrightarrow$(iii): Clear.

(iii)$\Longrightarrow$(iv): Suppose T has no stationary element, that is, $T(x)\backslash\{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{T(x)\backslash\{x\} : x \in A\}$. Then f has no fixed element by its definition. However, for any $x \in A$, we have $x \neq fx$ and there exists a $y \in T(x)\backslash\{x\}$ satisfying $\neg G(x, y)$. Therefore, by (iii), f has a fixed element, a contradiction.

(iv)$\Longrightarrow$(v): Define a multimap $T : A \multimap X$ by $T(x) := \{fx : f \in \mathcal{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, fx)$ for any $x \in A$ and any $f \in \mathcal{F}$, by (iv), T has a stationary element $v \in A$, which is a common fixed element of $\mathcal{F}$.

(v)$\Longrightarrow$(i): Suppose that for any $x \in A$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$. Choose fx to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $\neg G(x, fx)$ for all $x \in A$. Let $\mathcal{F} = \{f\}$. By (v), f has a fixed element, a contradiction.

(i)+(iv) $\Longrightarrow$ (vi): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \in X\backslash\{v\}$. For each $i \in I$, by (iv), we have a $v_i \in A$ such that $\{v_i\} = T_i(v_i)$. Suppose $v \neq v_i$. Then $G(v, v_1)$ holds by (i) and $\neg G(v, v_1)$ holds by assumption on (vi). This is a contradiction. Therefore $v = v_i$ for all $i \in I$.

(vi) $\Longrightarrow$ (iv): Clear.

(i) $\Longrightarrow$ (vii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

(vii) $\Longrightarrow$ (i): For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ \neg G(x, y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the bypothesis of (vi) is satisfied. Therefore, by (vi), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v, w)$ for all $w \neq v$. Hence (i) holds.

This completes our proof. □

Note that the element v is the same throughout (i)–(vii).

3. Applications of Metatheorem

In this section, we give several examples or applications or direct consequences of Metatheorem.

(I) Pre-ordered Set. Let $(X, \preceq)$ be a *pre-ordered set*; that is, X is a nonempty set and $\preceq$ is reflexive and transitive. For each $x \in X$, we denote $S(x) = \{y \in X : x \preceq y\}$ and $G(x, y)$ means $x \not\preceq y$.

Theorem 1.1. *Let* $x_0 \in X$ *and* $A = S(x_0)$*. Then the following seven statements are equivalent:*

(i) *There exists a maximal element* $v \in A$ *such that* $v \not\preceq w$ *for any* $w \in X \backslash \{v\}$.

(ii) *If* $T : A \multimap X$ *is a multimap such that for any* $x \in A \backslash T(x)$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $x \preceq y$*, then* T *has a fixed element* $v \in A$*, that is,* $v \in T(v)$.

(iii) *If* $f : A \to X$ *is a map such that for any* $x \in A$ *with* $x \neq fx$*, there exists a* $y \in X \backslash \{x\}$ *satisfying* $x \preceq y$*, then* f *has a fixed element* $v \in A$*, that is,* $v = fv$.

(iv) *If* $T : A \multimap X$ *is a multimap such that* $x \preceq y$ *holds for any* $x \in A$ *and any* $y \in T(x) \backslash \{x\}$*, then* T *has a stationary element* $v \in A$*, that is,* $\{v\} = T(v)$.

(v) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $x \preceq fx$ *for all* $x \in A$ *with* $x \neq fx$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(vi) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $x \preceq y$ *holds for any* $x \in A$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(vii) *If* Y *is a subset of* X *such that for each* $x \in A \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *such that* $x \preceq z$*, then there exists an element* $v \in A \cap Y$.

Proof. In Metatheorem, put $A = S(x_0)$ and let $G(v, w)$ be the statement $v \not\preceq w$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem.

This completes our proof. □

Note that we claimed that (i)–(vii) are equivalent in Theorem 1.1 and did not claim that they are true. From now on, in most cases, we are going to give examples that they are true based on their original sources.

(II) Zorn's Lemma. For an application of Theorem 1.1, we recall our earlier work [22] in 1987 as follows:

A nonempty poset (partially ordered set) is said to be *inductively ordered* if

(A) every simply ordered subset has an upper bound.

We adopt the following form of Zorn's lemma:

Lemma 2.1. *Let* $(P, \preceq)$ *be an inductively ordered set. Then for any* $a \in P$*, there exists a maximal element* $v \in P$ *such that* $v \in S(a) = \{x \in P : a \preceq x\}$.

It is well-known that an equivalent form of Zorn's lemma is obtained by replacing (A) by the following:

(B) every nonempty well-ordered subset has an upper bound.

As an application of Theorem 1.1, we have the following:

Theorem 2.2. *Let* $(P, \preceq)$ *be a poset satisfying (A) or (B),* $x_0 \in P$*, and* $A = S(x_0)$*. Then the equivalent conditions* (i)–(viii) *of Theorem 1.1 hold.*

Proof. By Lemma 2.1, $(P, \preceq)$ has a maximal element, that is, Theorem 1.1 holds. Therefore, Theorem 2.2 follows from Theorem 1.1. □

The 29 references of [22] show the origins, variants, consequences, applications of Theorem 2.2, and we will only indicate names of their authors like Abian (1971), Ekeland [7](1979), Bishop-Phelps (1961), Turinici (1980-1984), Smithon (1971, 1973), Höft-Höft (1976), Tuy (1981), Kasahara (1975), Maschler-Peleg (1976), Phelps (1964), Caristi (1976), Banach (1922), and others.

Recall that Taskovic [30] in 1986 showed that Zorn's lemma is equivalent to the following:

(v′) *Let* $\mathfrak{F}$ *be a family of selfmaps* $f : X \to X$ *on a poset* X *satisfying* $x \preceq fx$ *(resp.* $fx \preceq x$*) for all* $x \in X$ *and all* $f \in \mathfrak{F}$*. If each chain in* X *has an upper bound (resp. lower bound), then* $\mathfrak{F}$ *has a common fixed point.*

This follows from Theorem 2.2(v).

(III) Nadler's fixed point theorem. Let (X, d) be a metric space and $\mathrm{CB}(X)$ be the class of all nonempty bounded closed subsets of X. Let H be the *Hausdorff metric* with respect to d, that is,

$$H(A, B) = \max\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\}$$

for every $A, B \in \mathrm{CB}(X)$. Then a map $T : X \to \mathrm{CB}(X)$ is called a *k-contraction* if there exists $k < 1$ such that $H(Tx, Ty) \leq kd(x, y)$ for all $x,\ y \in X$.

The following is the well-known fixed point theorem due to Nadler:

Theorem 3.1. *Let X be a complete metric space and $T : X \to \mathrm{CB}(X)$ a k-contraction. Then there exists an $x_0 \in X$ with $x_0 \in T(x_0)$.*

Motivated by Theorem 3.1 and Metatheorem, we have the following:

Theorem 3.2. *Let X be a complete metric space, $T : X \to \mathrm{CB}(X)$ be a multimap, and $0 < k < 1$. Then the following equivalent statements hold:*

(i) *There exists an element $v \in X$ such that $H(Tv, Tw) > kd(v, w)$ for any $w \in X\backslash\{v\}$.*

(ii) *If for any $x \in X\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $H(Tx, Ty) \leq kd(x, y)$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that for any $x \in X$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $d(fx, fy) \leq kd(x, y)$, then f has a fixed element $v \in X$, that is, $v = fv$.*

(iv) *If $H(Tx, Ty) \leq kd(x, y)$ holds for any $x \in X$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

(v) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $d(fx, fy) \leq kd(x, y)$ for all $x \in X$ with $x \neq fx$, then $\mathcal{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(vi) *If $\mathcal{F}$ is a family of multimaps $T_i : X \to X$, $i \in I$, satisfying $H(Tx, Ty) \leq kd(x, y)$ for all $x \in X$ and any $y \in T_i(x)\backslash\{x\}$, $i \in I$, then $\mathcal{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(vii) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $H(Tx, Tz) \leq kd(x, z)$, then there exists a $v \in X \cap Y = Y$.*

Proof. Note that, in Metatheorem, put $A = X$ and let $G(v, w)$ be the statement $H(Tv, Tw) > kd(v, w)$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem and (iii) holds by [28]. This completes our proof. □

Note that (ii) or (iv) imply Nadler's theorem and (iii) implies the Banach contraction principle. Therefore, in some sense, these two theorems are equivalent. Moreover, (iii) shows that the fixed point v throughout Theorem 3.3 is unique.

Mizoguchi and Takahashi [10] extended Nadler's theorem as follows:

Theorem 3.4. ([10]) *Let (X, d) be a complete metric space and let $T : X \to \mathrm{CB}(X)$. Assume*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $x,\ y \in X$, where $\alpha : [0, \infty) \to [0, 1)$ is a function satisfying $\limsup_{s\to t+0} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then there exists $z \in X$ such that $z \in Tz$.

For this theorem, we can formulate equivalents as for Theorem 3.2.

(IV) Kirk's fixed point theorem. In 1965, Kirk [9] obtained the following:

Theorem 4.1. *Let X be a Banach space and suppose that C is a nonempty weakly compact convex subset of X which has the normal structure property. Then, any nonexpansive mapping $f : C \to C$ has a fixed point.*

From this and our Metatheorem, we have the following equivalency:

Theorem 4.2. *Let C be as in Theorem 4.1 and $\phi : C \to C$ is a function. Then the following statements are equivalent:*

(i) *There exists an element $v \in C$ such that $||\phi(v) - \phi(w)|| > ||v - w||$ for any $w \in C \backslash \{v\}$.*

(ii) *If $T : C \multimap C$ is a multimap such that for any $x \in C \backslash T(x)$ there exists a $y \in C \backslash \{x\}$ satisfying $||\phi(x) - \phi(y)|| \leq ||x - y||$, then T has a fixed element $v \in C$, that is, $v \in T(v)$.*

(iii) *If $f : C \to C$ is a map such that for any $x \in C$ with $x \neq f(x)$, there exists a $y \in C \backslash \{x\}$ satisfying $||\phi(x) - \phi(y)|| \leq ||x - y||$, then f has a fixed element $v \in C$, that is, $v = fv$.*

(iv) *If $T : C \multimap C$ is a multimap such that $||\phi(x) - \phi(y)|| \leq ||x - y||$ holds for any $x \in C$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in C$, that is, $\{v\} = T(v)$.*

(v) *If $\mathcal{F}$ is a family of maps $f : C \to C$ satisfying $||\phi(x) - \phi(fx)|| \leq ||x - fx||$ for all $x \in C$ with $x \neq fx$, then $\mathcal{F}$ has a common fixed element $v \in C$, that is, $v = fv$ for all $f \in \mathcal{F}$.*

(vi) *If $\mathfrak{F}$ is a family of multimaps $T_i : C \multimap C$ for $i \in I$ with an index set I such that $||\phi(x) - \phi(y)|| \leq ||x - y||$ holds for any $x \in C$ and any $y \in T_i(x) \backslash \{x\}$ for all $i \in I$, then T_i has a common stationary element $v \in C$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(vii) *If Y is a subset of C such that for each $x \in C \backslash Y$ there exists a $z \in C \backslash \{x\}$ satisfying $||\phi(x) - \phi(z)|| \leq ||x - z||$, then there exists a $v \in C \cap Y = Y$.*

Proof. In Metatheorem, let $G(v, w)$ be the statement $||\phi(v) - \phi(w)|| > ||v - w||$ and let $A = X = C$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem. This completes our proof. $\square$

We are still unable to show whether any of (i)–(vii) is true.

(V) Brézis-Browder Principle. In [3], by considering an pre-ordered set $(X, \preceq)$ and using the notation

$$S(x) = \{y \in X : x \preceq y\},$$

Brézis and Browder proved the following important maximality principle (that is, (i) in below) as an immediate consequence of a more general result. It can be reformulated by our Metatheorem as follows:

Theorem 5.1. *Let $(X, \preceq)$ be a pre-ordered set, $x_0 \in X$, and $A = \{x \in X : x_0 \preceq x\}$.*
Let $\phi : X \to \mathbb{R}$ be a function satisfying

(1) *ϕ is bounded above;*

(2) *$x \preceq y$ and $x \neq y$ imply $\phi(x) < \phi(y)$; and*

(3) *For any $\preceq$-increasing sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ (i.e. $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$), there exists some $y \in X$ such that $x_n \preceq y$ for all $n \in \mathbb{N}$.*

Then the following equivalent statements hold:

(i) *There exists a maximal element $v \in A$ such that*

$$v \preceq w \ \text{ implies } \ \phi(v) = \phi(w)$$

for any $w \in X \backslash \{v\}$, that is, $S(v) = \{v\}$.

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying*

$$x \preceq y \ \text{ and } \ \phi(x) \neq \phi(y),$$

then T has a fixed element $v \in A$, that is, $v \in T(v)$.

(iii) *If* $f : A \to X$ *is a map such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X\backslash\{x\}$ *satisfying*

$$x \preceq y \quad and \quad \phi(x) \neq \phi(y),$$

then f *has a fixed element* $v \in A$, *that is,* $v = fv$.

(iv) *If* $T : A \multimap X$ *is a multimap such that*

$$x \preceq y \quad and \quad \phi(x) \neq \phi(y)$$

holds for any $x \in A$ *and any* $y \in T(x)\backslash\{x\}$, *then* T *has a stationary element* $v \in A$, *that is,* $\{v\} = T(v)$.

(v) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying*

$$x \preceq fx \quad and \quad \phi(x) \neq \phi(fx),$$

for all $x \in A$, *then* $\mathcal{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(vi) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that*

$$x \preceq y \quad and \quad \phi(x) \neq \phi(y)$$

holds for any $x \in A$ *and any* $y \in T_i(x)\backslash\{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(vii) *If* Y *is a subset of* X *such that for each* $x \in A\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying*

$$x \preceq z \quad and \quad \phi(x) \neq \phi(z),$$

then there exists a $v \in A \cap Y$.

(VI) Caristi's theorem. In [22], the present author obtained several consequences of Metatheorem. One of them is the following equivalent formulations of the Caristi fixed point theorem by applying our Metatheorem; see [[22], Theorem 6].

Theorem 6.1. *Let* (X_0, d) *be a metric space, and* $\phi : X_0 \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function bounded from below. Define a partial order* $\preceq$ *on* $X = \{x \in X_0 : \phi(x) < +\infty\}$ *by*

$$x \preceq y \quad iff \quad d(x, y) \leq \phi(x) - \phi(y).$$

Let $x_0 \in X$ *and suppose* $A = S(x_0) = \{y \in X : x_0 \preceq y\}$ *is* $\preceq$*-complete.*

Then the following equivalent statements hold.

(i) *There exists a maximal element* $v \in A$ *such that* $v \not\preceq w$ *or* $d(v, w) > \phi(v) - \phi(w)$ *for any* $w \in X\backslash\{v\}$.

(ii) *If* $T : A \multimap X$ *is a multimap such that for any* $x \in A\backslash T(x)$ *there exists a* $y \in X\backslash\{x\}$ *satisfying* $d(x, y) \leq \phi(x) - \phi(y)$, *then* T *has a fixed element* $v \in A$, *that is,* $v \in T(v)$.

(iii) *If* $f : A \to X$ *is a map such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X\backslash\{x\}$ *satisfying* $d(x, y) \leq \phi(x) - \phi(y)$, *then* f *has a fixed element* $v \in A$, *that is,* $v = fv$.

(iv) *If* $T : A \multimap X$ *is a multimap such that* $d(x, y) \leq \phi(x) - \phi(y)$ *holds for any* $x \in A$ *and any* $y \in T(x)\backslash\{x\}$, *then* T *has a stationary element* $v \in A$, *that is,* $\{v\} = T(v)$.

(v) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $d(x, fx) \leq \phi(x) - \phi(fx)$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathcal{F}$.

(vi) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $d(x, y) \leq \phi(x) - \phi(y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x)\backslash\{x\}$, *then* T_i *has a common stationary element* $v \in A$, *that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(vii) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ such that $d(x,z) \leq \phi(x) - \phi(z)$, then there exists an element $v \in A \cap Y$.*

Proof. In Theorem 6.1, (i) is proved in [22]. Since ϕ is l.s.c., $S(x_0)$ is closed. Therefore, if (X_0, d) is complete, so is $S(x_0)$ and $\preceq$-complete. Now, Theorem 6.1 follows from Metatheorem. $\square$

Theorem 6.1(i)–(v) was given in Park [19], [20]. Actually, Theorem 6.1(i) is the variational principle of Ekeland [7](1979), (ii) essentially due to Tuy (1981), (v) to Kasahara (1975), (iv) to Mascher-Peleg (1976), and (iii) to Caristi (1976), which implies the Banach contraction principle. Classical applications of Theorem 6.1 are numerous in vast fields of mathematical sciences; see, e.g. [7], [20], [21].

From (iii) of Theorem 6.1 we have Caristi's theorem (1976) as follows:

Corollary 6.2. *Let X be a complete metric space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function bounded from below. If $f : X \to X$ is a map such that*

$$d(x, fx) \leq \phi(x) - \phi(fx) \quad \text{or} \quad x \preceq fx$$

for every $x \in X$. Then there exists an $x_0 \in X$ with $fx_0 = x_0$.

Recall that, from Caristi's theorem, Mizoguchi-Takahashi [10] deduced a particular form of Theorem 6.1(ii) and applied it to obtain Ekeland's ε-variational principle, generalizations of Nadler's and Reich's theorems.

Similarly, any of (iv) and (v) of Theorem 6.1, we can obtain the following for a complete metric space (X, d), resp.

Corollary 6.3. *If $T : X \multimap X$ is a multimap such that $d(x,y) \leq \phi(x) - \phi(y)$ holds for any $x \in X$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

Corollary 6.4. *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $d(x, fx) \leq \phi(x) - \phi(fx)$ for all $x \in X$, then $\mathcal{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathcal{F}$.*

These seem to be new generalizations of the Caristi theorem.

(VII) Ekeland principle. In [23], we gave a simple proof of the Ekeland ε-variational principle. Now based on our new Metatheorem, we improve [[23], Theorem 5] as follows:

Theorem 7.1. *Let X be a complete metric space, and $F : X \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ be given, a point $u \in X$ such that $F(u) \leq \inf_X F + \varepsilon$, and $A = \{x \in X : F(x) \leq F(u) - \varepsilon d(u,x)\}$.*

Then the following equivalent statements hold:

(i) *There exists a point $v \in A$ such that*

$$\forall w \neq v, \;\; F(w) > F(v) - \varepsilon d(v, w).$$

(ii) *If $T : A \multimap X$ satisfies the condition*

$$\forall x \in A \backslash T(x) \; \exists y \in X \backslash \{x\} \text{ such that } F(y) \leq F(x) - \varepsilon d(x,y),$$

then T has a fixed point $v \in A$, that is, $v \in T(v)$.

(iii) *If $f : A \to X$ is a function satisfying*

$$F(fx) \leq F(x) - \varepsilon d(x, fx)$$

for all $x \in A$, then f has a fixed point.

(iv) *If $T : A \multimap X$ satisfies the condition*

$$\forall x \in A \ \forall y \in T(x)\backslash\{x\}, \ F(y) \leq F(x) - \varepsilon d(x,y),$$

then T has a stationary point $v \in A$.

(v) *A family $\mathfrak{F}$ of functions $f : A \to X$ satisfying*

$$F(y) \leq F(x) - \varepsilon d(x,y) \ for \ all \ x \in A \ with \ x \neq fx$$

has a common fixed point $v \in A$, that is $v = fv$ for all $f \in \mathcal{F}$.

(vi) *If $\mathfrak{F}$ is a family of multimaps $T_i : A \multimap X$ for $i \in I$ with an index set I such that*

$$F(y) \leq F(x) - \varepsilon d(x,y)$$

holds for any $x \in A$ and any $y \in T_i(x)\backslash\{x\}$, then T_i has a common stationary element $v \in A$, that is, $\{v\} = T_i(v)$ for all $i \in I$.

(vii) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying*

$$F(z) \leq F(x) - \varepsilon d(x,z),$$

then there exists an element $v \in A \cap Y$.

Proof. Note that (i) is the ε-variational principle. In Metatheorem, let $G(v,w)$ be the statement $F(w) > F(v) - \varepsilon d(v,w)$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem. This completes our proof. □

In Theorem 7.1, if A is complete, we do not need to assume the completeness of X.

Actually, Theorem 7.1 is a little weaker than Ekeland's principle [7] in 1979, for

$$A \subset \{x \in X : F(x) \leq F(u), d(u,x) \leq 1\} \subset \overline{B}(u,1).$$

Theorem 7.1(iii) for $\varepsilon = 1$ is known as the Caristi fixed point theorem which characterizes metric completeness.

(VIII) Turinici's order. In 1980, Turinici [31] obtained a maximal element result of Brøndstep type in a class of order complete metric spaces extending Caristi's theorem:

Theorem 8.1. ([31]) *Let (X,d) be a metric space, $\leq$ a partial order on X such that*

(1) *$\leq$ is a closed order on X,*
(2) *(X,d) is $\leq$-asymptotic metric space, and*
(3) *(X,d) is $\leq$-complete metric space.*

Then, for any $x \in X$, the following equivalent statements hold:

(i) *There is a $z \in X$ such that $x \leq z$ and $X(z,\leq) = \{z\}$ (or, in other words, for every $x \in X$, there is a maximal element $z \in X$ such that $x \leq z$).*

(iii) *A map $f : X \to X$ satisfies*

$$y \leq fy \ for \ all \ y \in X.$$

Then there is a $z \in X$ such that $x \leq z$, $X(z,\leq) = \{z\}$, and $z = fz$.

For the terminology, see [31]. Here $X(z,\leq) = \{x \in X : z \leq x\}$.

Proof. Note that (i) and (iii) are Theorems 3.1 and 3.2 of [31], resp. Equivalence of them follow from Metatheorem or Theorem 1. □

Moreover, by Theorem 8.1, we also have the corresponding (ii)–(vii) hold. Turinici also noted that (i) and (iii) gives results of Brøndsted and Caristi, resp., under certain case.

(IX) Takahashi's minimization theorem. In this part, we obtain equivalent formulations of Takahashi's nonconvex minimization theorem [29] in 1991.

Theorem 9.1. *Let X be a complete metric space and let $\phi : X \to (-\infty, \infty]$ be a proper l.s.c. function, bounded from below. Suppose that, for each $u \in X$ with $\inf_{x\in X} \phi(x) < \phi(u)$, there exists a $v \in X\backslash\{u\}$ such that $\phi(v) + d(u, v) \leq \phi(u)$.*

Then the following equivalent statements hold:

(i) *There exists an element $v \in X$ such that $\phi(v) \leq \phi(w)$ for any $w \in X\backslash\{v\}$, that is, $\phi(v) = \inf_{x\in X} \phi(x)$.*

(ii) *If $T : X \multimap X$ is a multimap such that for any $x \in X\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $\phi(x) > \phi(y)$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that for any $x \in X$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $\phi(x) > \phi(y)$, then f has a fixed element $v \in X$, that is, $v = fv$.*

(iv) *If $T : X \multimap X$ is a multimap such that $\phi(x) > \phi(y)$ holds for any $x \in X$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

(v) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $\phi(x) > \phi(fx)$ for all $x \in X$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathcal{F}$.*

(vi) *If $\mathfrak{F}$ is a family of multimaps $T_i : A \multimap X$ for $i \in I$ with an index set I such that $\phi(x) > \phi(y)$ holds for any $x \in A$ and any $y \in T_i(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(vii) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $\phi(x) > \phi(z)$, then there exists a $v \in X \cap Y = Y$.*

Proof. Recall that (i) is the Takahashi's theorem. In Metatheorem, let $G(v, w)$ be the statement $\phi(v) \leq \phi(w)$ and let $A = X$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem. This completes our proof. □

Note that Takahashi [29] used (i) to obtain Caristi's fixed point theorem, Ekeland's ε-variational principle, and Nadler's fixed point theorem. Similarly, we can use any of (ii)–(vii) for them.

Proposition 9.2. *Ekeland's ε-variational principle is equivalent to Takahashi's theorem,*

Proof. Let V be a complete metric space. From Ekeland's ε-variational principle, there exists $v \in V$ such that

$$\varepsilon d(v, x) > F(v) - F(x) \quad \forall x \in V \text{ with } x \neq v.$$

We claim that $F(v) = \inf_{x\in V} F(x)$.

Suppose to the contrary that $F(v) > \inf_{x\in V} F(x)$. By our assumption, there exists $y = y(v) \in V$ with $y \neq v$ such that

$$\varepsilon d(v, y) \leq F(v) - F(y).$$

Then we have

$$\varepsilon d(v, y) \leq F(v) - F(y) < \varepsilon d(v, y)$$

which is a contradiction.

Conversely, suppose that, in Ekeland's ε-variational principle, for each $x \in V$, there exists $y \in V$ with $y \neq x$ such that $F(y) \leq F(x) - \varepsilon d(x, y)$. Then, by Theorem 5, there exists $v \in V$ such that $F(v) = \inf_{x\in V} F(x)$. By our supposition, there exists $w \in V$ with $w \neq v$ such that $\varepsilon d(v, w) \leq F(v) - F(w) \leq 0$. Hence $d(v, w) = 0$ and $v = w$, which leads to a contradiction. □

(X) Quasi-metric space. Our most recent generalization of the Ekeland principle is as follows in 2000 [25].

Kada, Suzuki, and Takahashi [8] introduced the concept of W-distances for a metric space (X, d) as follows:

A function $\omega : X \times X \to [0, \infty)$ is called a *W-distance* on X if the following are satisfied:

(1) $\omega(x, z) \leq \omega(x, y) + \omega(y; z)$ for any $x, y, z \in X$;
(2) $\omega(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous for any $x \in X$; and
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

In [8], many examples and properties of W-distances were given.

Let X be a non-empty set and $\preccurlyeq$ a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on X. Let $S(x) = \{y \in X : x \preccurlyeq y\}$ for $x \in X$, and $\leq$ be the usual order in the extended real number system $[-\infty, \infty]$.

Let d be a quasi-metric (that is, not necessarily symmetric) on X. Then for the quasi-metric space (X, d), the concepts of W-distances, Cauchy sequences, completeness, and Banach contractions can be defined.

In a quasi-metric space (X, d) with a quasi-order $\preccurlyeq$, a set $S(u)$ for some $u \in X$ is said to be *$\preccurlyeq$-complete* if every nondecreasing Cauchy sequence in $S(u)$ converges. For details, see [8] and references therein.

Throughout this part, let $\phi : X \times X \to (-\infty, \infty]$ be a function such that

(4) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ for any $x, y, z \in X$;
(5) $\phi(x, \cdot) : X \to (-\infty, \infty]$ is lower semicontinuous for any $x \in X$; and
(6) there exists an $x_0 \in X$ such that $\inf_{y \in X} \phi(x_0, y) > -\infty$.

Based on Metatheorem, we can extend the main result in our previous work [27] in 2000:

Theorem 10.1. *Let (X, d) be a quasi-metric space. Let $\omega : X \times X \to [0, \infty)$ be a W-distance on X and $\phi : X \times X \to (-\infty, \infty]$ a function satisfying conditions (4)–(6). Define a quasi-order $\preccurlyeq$ on X by*

$$x \preccurlyeq y \quad iff \quad x = y \text{ or } \phi(x, y) + \omega(x, y) \leq 0.$$

Suppose that there exists a $u \in X$ such that $\inf_{y \in X} \phi(u, y) > -\infty$ and $S(u) = \{y \in X : u \preccurlyeq y\}$ is $\preccurlyeq$-complete. Then the following equivalent statements hold:

(i) *There exists a maximal point $v \in S(u)$; that is,*

$$\forall w \in X \backslash \{v\}, \quad \phi(v, w) + \omega(v, w) > 0.$$

(ii) *If $T : S(u) \multimap X$ satisfies the condition*

$$\forall x \in S(u) \backslash T(x) \;\; \exists y \in X \backslash \{x\} \text{ such that } x \preccurlyeq y,$$

then T has a fixed point $v \in S(u)$; that is, $v \in T(v)$.

(iii) *A function $f : S(u) \to X$ satisfying $x \preccurlyeq f(x)$ for all $x \in S(u)$ has a fixed point.*

(iv) *If $T : S(u) \multimap X$ satisfies the condition*

$$\forall x \in S(u), \; \forall y \in T(x), \; x \preccurlyeq y \text{ holds},$$

then T has a stationary point $v \in S(u)$; that is, $T(v) = \{v\}$.

(v) *A family $\mathcal{F}$ of functions $f : S(u) \to X$ satisfying $x \preccurlyeq f(x)$ for all $x \in S(u)$ has a common fixed point $v \in S(u)$.*

(vi) *If $\mathfrak{F}$ is a family of multimaps $T_i : A \multimap X$ for $i \in I$ with an index set I such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in T_i(x) \backslash \{x\}$, then $\{T_i\}$ has a common stationary element $v \in A$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(vii) *If Y is a subset of X such that for each $x \in S(u)\backslash Y$ there exists a $z \in X\backslash\{x\}$ such that $x \preccurlyeq z$, then there exists a $v \in S(u) \cap Y$.*

Proof. The statement (i) was proved in [27] in 2000. In Metatheorem, put $A := S(u)$ and let $G(v, w)$ be the statement $\phi(v, w) + \omega(v, w) > 0$. Then each of (i)–(vii) follows from the corresponding ones in Metatheorem. This completes our proof. □

Theorem 10.2. *Under the hypothesis of Theorem 10.1, the following also holds:*

(viii) *If, for each $v \in S(u)$ with $\inf_{y\in X} \phi(v, y) < 0$, there exists a $w \in S(v)\backslash\{v\}$, then there exists an $x_0 \in S(u)$ such that $\inf_{y\in X} \phi(x_0, y) \geq 0$.*

In fact, any of Theorem 10.1(i)–(vii) implies (viii). Conversely, (viii) implies any of (i)–(vii) whenever either (a) $\omega(x, y) = 0$ implies $x = y$; or (b) $\omega(x, x) = 0$ for all $x \in X$.

This was given as Theorem 1′ in our previous work [27] and here we add (vi) and (vii). Moreover, in [27], we gave the following consequences of Theorems 10.1 and 10.2:

Theorem 2: A simplified form of Theorems 10.1 and 10.2.
Theorem 3: The central result of Ekeland [7] on the variational principle for approximate solutions of the minimization problem.
Theorem 4: Extension of a fixed point theorem of Downing-Kirk [4] as a simple application of Theorem 10.1(iii).

Note that, in Section 3 of [27], we stated that the primitive versions of Theorem 2 for a very particular case contain a large number of previous works of other authors.

(XI) Vector valued metric spaces. As an another application of Metatheorem, we recall the work of Agarwal and Khamsi [1] in 2011 on extension of Caristi's theorem to vector valued metric space. We follow [1] as follows:

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V} : \theta \preceq v\}$, where θ is the zero-vector of $\mathcal{V}$, satisfies the usual properties:
(1) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$,
(2) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$,
(3) $\alpha\mathcal{V}_+ \subset \mathcal{V}_+$ for all $\alpha \geq 0$.

The concept of vector-valued metric spaces relies on the following definition:

Definition 11.1. Let M be a set. A map $d : M \times M \to \mathcal{V}$ defines a distance if:

(i) $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for any $x, y \in M$,
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for any $x, y, z \in M$.

The pair (M, d) is called a *vector valued metric space* (vvms for short).

We need the following (Theorem 2 in [1]):

Theorem 11.2. *Let (M, d) be a complete vvms over an order complete and order continuous Banach lattice $\mathcal{V}$. Let $F : M \to \mathcal{V}_+$ be a l.s.c. map. Then any $f : M \to M$ such that*

$$d(x, fx) \preceq F(x) - F(fx)$$

for any $x \in M$ has a fixed point.

Now we have the following from Theorem 11.2 and Metatheorem.

Theorem 11.3. *Let (M, d) be a complete vvms over an order complete and order continuous Banach lattice $\mathcal{V}$, and $F : M \to \mathcal{V}_+$ be a l.s.c. map.*

Then the following equivalent statements hold:

(i) *There exists a point $v \in M$ such that*

$$\forall w \neq v, \quad d(v, w) \npreceq F(v) - F(w).$$

(ii) *If $T : M \multimap M$ satisfies the condition*

$$\forall M \backslash T(x) \quad \exists y \in M \backslash \{x\} \text{ such that } d(x, y) \preceq F(x) - F(y),$$

then T has a fixed point $v \in M$, that is, $v \in T(v)$.

(iii) *If $f : M \to M$ is a function satisfying*

$$d(x, fx) \preceq F(x) - F(fx)$$

for all $x \in M$, then f has a fixed point.

(iv) *If $T : M \multimap M$ satisfies the condition*

$$\forall x \in M \quad \forall y \in T(x) \backslash \{x\}, \; d(x, y) \preceq F(x) - F(y),$$

then T has a stationary point $v \in M$.

(v) *A family $\mathfrak{F}$ of functions $f : M \to M$ satisfying*

$$d(x, fx) \preceq F(x) - F(fx) \text{ for all } x \in M \text{ with } x \neq fx$$

has a common fixed point $v \in M$, that is $v = fv$ for all $f \in \mathfrak{F}$.

(vi) *If $\mathfrak{F}$ is a family of multimaps $T_i : M \multimap M$ for $i \in I$ with an index set I such that*

$$\forall x \in M, \; \forall y \in T_i(x) \backslash \{x\}, d(x, y) \preceq F(x) - F(y),$$

then T_i has a common stationary element $v \in M$, that is, $\{v\} = T_i(v)$ for all $i \in I$.

(vii) *If Y is a subset of M such that for each $x \in M \backslash Y$ there exists a $z \in M \backslash \{x\}$ satisfying $d(x, z) \preceq F(x) - F(z)$, then there exists a $v \in M \cap Y = Y$.*

Proof. In Metatheorem, let $G(v, w)$ be the statement $d(v, w) \npreceq F(v) - F(w)$, and $A = M$. Then (iii) holds by Theorem 11.2. Therefore (i)–(vii) hold by Metatheorem. This completes the proof. □

Note that other results in [1] also follow from Theorem 11.2.

4. History of Metatheorem

We recall some history of Metatheorem given in the previous sections. In fact, the following previous works of ours are mainly concerned with Metatheorem related to extensions of the Ekeland principle.

(1) [11] in 1982: Kasahara's extension of the Caristi-Kirk fixed point theorem with respect to a family of selfmaps of an L-space is generalized. Our generalization includes the Downing-Kirk fixed point theorem.

(2) [12] in 1982: Let $f : X \to Y$ be a closed map between complete metric spaces, $\phi : f(X) \to \mathbb{R}^+$ a l.s.c. function, and $c > 0$. Then there exists a point $p \in X$ such that

$$\phi(fp) - \phi(fx) < \max\{d(p, x), cd(fp, fx)\}$$

for each $x \in X$ other than p.

This is an equivalent formulation of Theorems of Caristi, Ekeland, and Downing-Kirk.

(3) [13] in 1983: We show that Siegel's theorem includes the results of Downing-Kirk and Kasahara in the metric version, and we provide constructive proofs of results of Downing-Kirk and Kasahara. Also, we note that Downing-Kirk's generalization is actually equivalent to the Caristi-Kirk theorem. Simultaneously, we give a number of other equivalent formulations of the Caristi-Kirk theorem and some of their applications.

(4) [2] in 1983: In attempting to improve the Caristi-Kirk fixed point theorem, Kirk has raised the question of whether f continues to have a fixed point if we replace $d(x, fx)$ by $d(x, fx)^p$ where $p > 1$ in the theorem. In this paper, we first give an example which shows that Kirk's problem is not affirmative even if ϕ and f are continuous. We consider certain circumstances where Kirk's problem is valid, and, consequently obtain generalization of Theorems of Caristi, Ekeland, and Park.

(5) [14] in 1984: We give some necessary and sufficient conditions for a metric space to be complete. Such characterizations of metric completeness are given mainly by results relevant to Caristi's fixed point theorem. Works of Cantor, Kuratowski, Ekeland, Caristi, Kirk, Wong, Weston, Ćirić, Hu, Reich, Subrahmanyam, and others are combined.

An eminent editor of a journal wrote me that "Who dare use this kind of things to check the completeness of a metric space?" and rejected our manuscript to accept to publish.

(6) [15] in 1984: D. Downing and W. A. Kirk (1977) obtained a number of fixed point theorems for multimaps satisfying certain "inwardness" conditions in metric and Banach spaces. A key aspect of their approach is the application of Caristi's theorem. On the other hand, in [20], we obtained a number of equivalent formulations of Ekeland's principle. Some of such formulations include sharpened forms of the Caristi theorem. In this paper, using one of such formulations, we show that main results of Downing-Kirk are substantially improved by giving geometric estimations of fixed points.

(7) [16] in 1985: Some fixed point theorems for multimaps are obtained as consequence of an equivalent formulation [x] of Ekeland's variational principle. The main results of Husain-Sehgal (1980) and Kirk-Ray (1977) on directional contractions are substantially improved by giving geometric estimations of fixed points.

(8) [17] in 1985: Ray-Walker (1982) derived mapping theorems for nonlinear operators on Banach spaces satisfying two different kinds of local assumptions–differentiability and monotonicity–by using basic technique involving applications of their extended version of the Caristi-Kirk fixed points theorem. In this short paper, we show that the Ray-Walker theorem is actually equivalent to the Caristi-Kirk theorem.

The Ray-Walker theorem can be reformulated by our Metatheorem.

(9) [19] in 1985: This is a survey of some recent applications of Ekeland's variational principle. The Metatheorem for (i)–(iv) was first given in [19] and to deduce the main theorem of [20], Several applications of the Ekeland principle were introduced. For example. results of Downing-Kirk (1977), Kirk-Ray (1977), Husain-Sehgal (1980). Sehgal (1980), Goebel et al. (1970-1983). Finally, it is suggested that, for the problems of variational inequalities, quasi-variational inequalities, minimax inequalities, optimizations, and so on, a certain metatheorem similar to ours can be established so that so that those problems can be converted to fixed point results, and vice versa.

(10) [20] in 1986: We obtain several equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and applications to fixed point results. Our applications include geometric estimations of fixed points of Lipschitzian maps, localizations of the Banach contraction principle, and the Caristi-Kirk fixed point theorem, metrically inward maps, and dissipative maps. Consequently, earlier works of Aubin-Siegel (1980), Caristi (1976), Clarke (1976), Edelstein (1961), Lee-Tan (1977), Maschiler-Peleg (1976), Nadler (1969), Penot (1979), Reich (1971, 1978), Robinson (1973), Tuy (1981), Williamson (1975), Wong (1976, 1976) and others are substantially improved or extended.

The main theorem is the reformulation of Ekeland's theorem according to our Metatheorem with (i)–(iv).

(11) [22] in 1987: We give a result that maximum principles including Zorn's lemma can be regarded as various types of fixed point theorems. Our main application is that the well-known ordering principles in nonlinear analysis including the Bishop-Phelps argument and a number of its generalizations can be converted

to fixed point theorems and vice versa. Consequently, we obtain new results and unify many known results. The main theorem is given as Theorem 2 in this paper.

(12) [JKMS] in 1986: In our previous works [13], [14], [15] we showed that certain maximum principles were formulated equivalently to fixed point results, and obtained their applications. In the present paper, such formulations are applied to characterize countably compact topological spaces. Consequently, some new fixed point results are obtained, and some of them include most of the extensions of the Furi-Vignoli type fixed point theorems on densifying maps.

(13) [23] in 1987: We obtain characterizations of metric completeness using certain partial orders and, by applying them, we improve various formulations of Ekeland's celebrated variational principle given by Caristi-Kirk-Browder (1975 by Kirk), Kasahara (1975), Siegel (1977), Park (1986), Dancs-Hegedüs-Medvegyev (1983) are unified and generalized.

(14) [24] in 1993: Extensions or equivalent formulations of Ekeland's variational principle are unified. In fact, recent works of Conserva (1991), Guo (1991), Hicks (1989), Kang-Park (1990), Kasahara (1975), Le Van Hot (1982), Mizoguchi-Takahashi (1989), Takahashi (1991), and others are generalized and unified. Their proofs are simplified or replaced by constructive ones. Finally, some characterizations of metric completeness are given.

(15) [xx] in 1995e: from fixed point theorems recently due to the present author (1992-1993), we deduce some existence theorems for two variable functions on topological vector spaces. Such existence theorems are shown to be equivalent to known fixed point theorems and extend previously known results of Park (1988), Takahashi (1991), and Im-Kim (1991).

(16) [26] in 1997: We extend and unify some equivalent formulations of the Ekeland variational principle due to Oettli-Théra (1993), Blum-Oettli (1994), Kada-Suzuki-Takahashi (1996), and Park-Kang (1993).

(17) [27] in 2000: The aim of this paper is to unify the results of Blum-Oettli (1994), Kada-Suzuki-Takahashi (1996), and Oettli-Théra (1993) along the lines of Park-Kang (1993) and to improve the equivalent formulations of Ekeland's principle in various aspects. In fact, we obtain far-reaching generalized forms of Ekeland's principle and its six equivalents (Theorems 1, 10, and 2). We also show that one of our formulations readily implies the principle (Theorem 3). Moreover, as a simple application, we give an extended form (Theorem 4) of a fixed point theorem of Downing and Kirk (1997). Finally, we add historical remarks.

5. Conclusion

In this article, we extend our earlier Metatheorem and Theorem 1.1 by adding more equivalent statements. We showed that the maximal elements in certain pre-ordered sets can be reformulated to fixed points or stationary points of maps or multimaps and to common fixed points or common stationary points of a family of maps or multimaps, and conversely. Actually such points are same as we have seen in the proof of Metatheorem. Therefore, if we have a theorem on any of such points, it can be converted to at least six equivalent theorems on other types of points without any serious argument.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works and the present article. Therefore, a metatheorem like Theorem 1.1 is a machine to expand our knowledge easily. In this article we presented relatively old and well-known examples.

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