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DUS Inverse Weibull Distribution and Parameter Estimation in Regression Model

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Abstract: This paper considers various estimation methods to estimate the unknown parameters of the DUS Inverse Weibull (DIW) distribution using the maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramer-von Mises (CVM) and the Anderson-Darling (AD) estimators. A Monte-Carlo simulation study is conducted to determine the most preferable estimators in terms of their efficiencies. Furthermore, the distribution of the error terms in the simple linear regression is assumed to be DIW to show the implementation of it to the linear models. We also carry out a simulation study for comparing the performances of the estimators of the unknown regression parameters.

DUS Inverse Weibull Dağılımı ve Lineer Regresyonda Parametre Tahmini

Anahtar Kelimeler
DUS dönüşümü,
Inverse Weibull,
Parametre tahmini,
Lineer regresyon,
Monte-Carlo simülasyonu

Öz: Bu çalışma, en çok olabilirlik (ML), en küçük kareler (LS), ağırlıklı en küçük kareler (WLS), Cramer-von Mises (CM) ve Anderson-Darling (AD) tahmin edicilerini kullanarak DUS Inverse Weibull (DIW) dağılımının bilinmeyen parametrelerini tahmin etmek için çeşitli tahmin yöntemlerini ele almaktadır. Etkinlikleri açısından en çok tercih edilen tahmin edicileri belirlemek için bir Monte-Carlo simülasyon çalışması yapılmıştır. Ayrıca, lineer modellere uygulanışını göstermek için basit lineer regresyonda hata terimlerinin dağılımının DIW olduğu varsayılmıştır. Bilinmeyen regresyon parametrelerinin tahmin edicilerinin performanslarının karşılaştırılması için de bir simülasyon çalışması yapılmıştır.

1. Introduction

In recent years, there is a great interest on defining new distributions in order to obtain flexibility for modelling purposes. Therefore, transformations and generalizations depending on the idea of adding a new one or several parameters to the baseline distribution are mostly used in the related literature. Although the additional parameters provide more flexibility to the resulting distribution, computational difficulties arise in the estimation process of the parameters of interests. However, the DUS transformation formulated as follows;

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)} \tag{1}$$

contains the same parameters with the baseline distribution [1]. Here, $f(x)$ and $F(x)$ are the probability density function (pdf) and cumulative distribution function (cdf) of the baseline distribution, respectively. The DUS transformation is not a generalization, so it generates a parsimonious distribution in terms of computation and interpretation, see Kumar et al. [1].

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In this study, we consider the DUS Inverse Weibull (DIW) which is proposed by Gul et al. [2], see also Gul [3]. The DIW distribution is obtained by using DUS transformation, in other words, the pdf and cdf of the well-known Inverse Weibull (also known as the Frechet) distribution are incorporated into the equation (1). Then, the pdf of the DIW distribution is given by

$$g(x;\ \sigma,\beta)=\frac{1}{e-1}\frac{\beta}{\sigma^{-\beta}}x^{-(\beta+1)}exp\left(-\left(\frac{x}{\sigma}\right)^{-\beta}\right)$$
$$\times\, exp\left(exp\left(-\left(\frac{x}{\sigma}\right)^{-\beta}\right)\right), x>0, \beta>0, \sigma>0 \qquad (2)$$

where σ and β are the scale and shape parameters, respectively. It should be realized that there are various extensions of Inverse Weibull distribution including many parameters in the literature, see e.g. Nadarajah and Gupta [4], Nadarajah and Kotz [5], De Gusmao [6], Mahmoud and Mandouh [7] and Krishna et. al. [8]. Unlike the other extensions of Inverse Weibull distribution, the DIW distribution has just two parameters similar to the baseline distribution. The density plots of the DIW distribution for some values of the shape parameter β and $\sigma=1$ are provided in Figure 1. We refer to Gul [3] for further details on the statistical properties of DIW distribution.

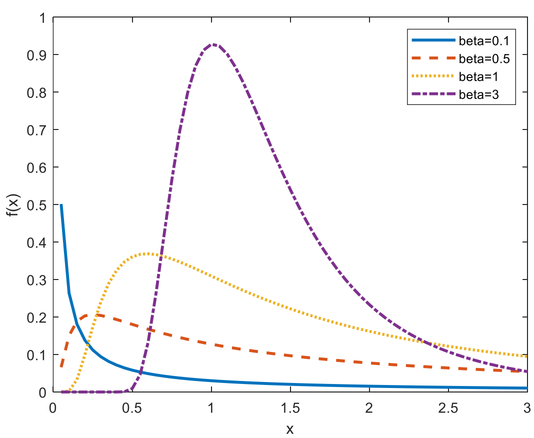

Figure 1. The density plots of DIW distribution for some values of β *(*$\sigma=1$*)*

The aim of this study is to compare the performances of the different estimation methods used for estimating the parameters of DIW distribution. We use the well-known maximum likelihood (ML) estimator along with the least squares (LS), weighted LS (WLS) and some minimum distance estimators such as Cramer-von Mises (CM) and Anderson-Darling (AD) estimators. The efficiencies of these estimators are evaluated via Monte-Carlo simulation study with different parameter settings for the DIW distribution.

In estimating the unknown parameters in a simple linear regression model, it is generally assumed that the distribution of the error terms is Normal with mean 0 and variance σ^2. However, there are many studies in which the distribution of the error terms does not follow a Normal distribution in literature. For example, Tiku et. al. [9] and Islam et. al. [10] considered the distribution of the error terms as non-normal symmetric and skew in the context of simple linear regression model, respectively. Therefore, in this study, the DIW distribution is also used in the context of simple linear regression model to demonstrate the implementation of the proposed distribution to the linear models, see Gul et. al. [11].

The reminder of the article is planned as follows. Section 2 is reserved to brief descriptions of the ML, LS, WLS, CM and AD estimation methods in the context of estimating the unknown parameters of DIW distribution and considered two methods for estimation of the parameters in the simple linear regression model whose error terms is distributed as DIW. Section 3 includes the Monte Carlo simulation study with its results. The paper is ended with conclusion section.

2. Material and Method

In this section, we consider five different estimation methods for estimating the unknown parameters of the proposed DIW distribution. In the rest of the paper, $X_1, X_2, \ldots, X_n$ is a random sample drawn from a DIW distribution and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denotes the corresponding order statistics.

2.1. Parameter estimation methods

2.1.1. Maximum likelihood

There are various parameter estimation methods in the statistical literature. Among them, the ML is the most widely used one due to its properties, e.g., consistency, asymptotic efficiency and invariance. In this subsection, we consider the ML methodology to estimate the parameters of the DIW distribution.

It is well known that the ML estimators are obtained by maximizing the corresponding likelihood function with respect to the parameters of interest.

The corresponding likelihood function is given by

$$L(\beta,\sigma \mid \underline{x}) = \left(\frac{1}{e-1}\right)^{n} \beta^{n}\sigma^{\beta n}\left(\prod_{i=1}^{n} x_i\right)^{-(\beta+1)} \times exp\left(\sum_{i=1}^{n}\left(-\frac{x_i}{\sigma}\right)^{-\beta}\right) \times exp\left(\sum_{i=1}^{n} exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right). \qquad (3)$$

Then the log-likelihood function (lnL) is obtained as follows

$$lnL(\beta,\sigma \mid \underline{x}) = nln\left(\frac{1}{e-1}\right) + nln(\beta) + \beta nln(\sigma) -(\beta+1)\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\beta} + \sum_{i=1}^{n} exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right). \qquad (4)$$

Differentiating (4) with respect to β and σ and equating them to zero, the following likelihood equations are obtained

$$\frac{dlnL(\beta,\sigma \mid \underline{x})}{d\beta} = \frac{n}{\beta} + nln(\sigma) - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\beta} ln\left(\frac{x_i}{\sigma}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\beta} ln\left(\frac{x_i}{\sigma}\right) exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right) = 0 \qquad (5)$$

$$\frac{dlnL(\beta,\sigma \mid \underline{x})}{d\sigma} = \frac{n\beta}{\sigma} - \beta\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\beta-1}\left(\frac{x_i}{\sigma^2}\right) -\beta\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\beta-1}\left(\frac{x_i}{\sigma^2}\right) exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right) = 0. \qquad (6)$$

Simultaneous solutions of these equations give the ML estimators of the parameters β and σ.

2.1.2. Least squares and weighted least squares

The LS estimators and WLS estimators are first suggested by Swain et. al. [12] in the context of estimating the parameters of Beta distribution.

LS method

The LS estimators of β and σ can be obtained by minimizing the following expression

$$LS(\beta,\sigma \mid \underline{x}) = \sum_{i=1}^{n}\left(G\left(X_{(i)}\right) - \frac{i}{n+1}\right)^{2} \qquad (7)$$

with respect to unknown parameters. Here and in the rest of the parameter $G(.)$ denotes the cdf of the DIW distribution which is formulated as follows:

$$G(x,\sigma,\beta) = \frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x}{\sigma}\right)^{-\beta}\right)\right) - 1\right). \qquad (8)$$

Then, in this case, the LS estimators of β and σ are obtained by minimizing

$$LS(\beta,\sigma \mid \underline{x}) = \sum_{i=1}^{n}\left(\frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right) - 1\right) - \frac{i}{n+1}\right)^{2}, \qquad (9)$$

with respect to the parameters β and σ, respectively. Resulting estimators are denoted as $\hat{\beta}_{LS}$ and $\hat{\sigma}_{LS}$.

WLS method

The WLS estimators of β and σ are defined as solution of the following minimization problem

$$(\hat{\beta},\hat{\sigma}) = argmin_{(\beta,\sigma)}\sum_{i=1}^{n}\frac{1}{V(G(X_{(i)}))} \times \left(G\left(X_{(i)}\right)\frac{i}{n+1}\right)^{2} \qquad (10)$$

where

$$V\left(G\left(X_{(i)}\right)\right) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Therefore, in this case, the WLS estimators of β and σ are minimizers of the following objective function

$$WLS(\beta,\sigma \mid \underline{x}) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left(\frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right) - 1\right) - \frac{i}{n+1}\right)^{2} \qquad (11)$$

Resulting estimators are denoted as $\hat{\beta}_{WLS}$ and $\hat{\sigma}_{WLS}$.

2.1.3. The Cramer-von Mises and the Anderson-Darling estimators

In this subsection, we obtain the CM and AD estimators of the parameters of the DIW distribution. These estimators were first considered by Wolfowitz [13,14]. They are also known as minimum distance estimators, see e.g. Luceno [15].

CM estimators

The solution of the following minimization problem are the CM estimators of β and σ

$$(\hat{\beta},\hat{\sigma}) = argmin_{(\beta,\sigma)}\left\{\frac{1}{12n}+\sum_{i=1}^{n}\left(G\left(X_{(i)},\beta,\sigma\right)-\frac{2i-1}{2n}\right)^2\right\} \quad (12)$$

Therefore, in this case, the CM estimators of β and σ are minimizers of the following objective function

$$CM(\beta,\sigma \mid \underline{x}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right)-1\right)-\frac{2i-1}{2n}\right)^2 \quad (13)$$

The CM estimators are shortly denoted by $\hat{\beta}_{CM}$ and $\hat{\sigma}_{CM}$.

AD estimators

The solution of the following minimization problem are called as AD estimators of β and σ:

$$AD(\beta,\sigma \mid \underline{x}) = -n-\frac{1}{n}\sum_{i=1}^{n}(2i-1) \times log\left[G\left(X_{(i)},\beta,\sigma\right)\left(1-G\left(X_{(n-i+1)},\beta,\sigma\right)\right)\right] \quad (14)$$

Therefore, in this case, the AD estimators of β and σ are obtained by minimizing

$$AD(\beta,\sigma \mid \underline{x}) = -n-\frac{1}{n}\sum_{i=1}^{n}(2-1)log \left(\frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right)-1\right)\right) \times\left(1-\frac{1}{e-1}\left(exp\left(exp\left(-\left(\frac{x_i}{\sigma}\right)^{-\beta}\right)\right)-1\right)\right) \quad (15)$$

with respect to β and σ. Resulting estimators are denoted as $\hat{\beta}_{AD}$ and $\hat{\sigma}_{AD}$.

It should be mentioned that we use *fminsearch* function which is available in MATLAB software to calculate the ML, LS, WLS, CM and AD estimates of β and σ since their explicit solutions cannot be obtained.

2.2. Parameter estimation for simple linear regression

In this section, we consider the following simple linear regression model

$$y_i = \beta_0+\beta_1 x_i+\varepsilon_i, \qquad i = 1,2,\ldots,n. \quad (16)$$

Where x_i is the explanatory variable, y_i is the response variable, β_0 is the intercept and β_1 is the slope parameter. Traditionally, in simple linear regression model, the error terms ε_i are assumed to be independent and identically distributed (iid) Normal $N(0,\sigma^2)$. However, we here assume that the random error terms ε_i follow the DIW distribution. We consider two methods of estimation for obtaining the estimators of the model parameters under this assumption.

2.2.1. ML estimation

L and lnL functions for the simple linear regression model for which the distribution of the error terms is assumed to be DIW can be obtained as follows

$$L(\beta_0,\beta_1,\sigma) = \left(\frac{\gamma\sigma^\gamma}{e-1}\right)^n\prod_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)^{-(\gamma+1)} \times exp\left(-\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^{-\gamma}\right) \times exp\left(exp\left(-\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^{-\gamma}\right)\right) \quad (17)$$

and

$$lnL(\beta_0,\beta_1,\sigma) = -nln(e-1)+nln(\gamma)+n\gamma ln(\sigma) -(\gamma+1)\sum_{i=1}^{n}\ln\left(y_i-\beta_0-\beta_1x_i\right) -\sum_{i=1}^{n}\left(\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^{-\gamma}\right)\sum_{i=1}^{n}exp\left(-\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^{-\gamma}\right) \quad (18)$$

respectively. Differentiating (18) with respect to the parameters β_0,β_1 and σ and setting them equal to zero, we get the likelihood equations given below,

$$\frac{dlnL}{d\beta_0} = (\gamma+1)\sum_{i=1}^{n}\frac{1}{y_i-\beta_0-\beta_1 x_i} - \frac{\gamma}{\sigma}\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1} - \frac{\gamma}{\sigma}\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1} \times exp\left(-\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma}\right) = 0 \quad (19)$$

$$\frac{dlnL}{d\beta_1} = (\gamma+1)\sum_{i=1}^{n}\frac{x_i}{y_i-\beta_0-\beta_1 x_i} - \frac{\gamma}{\sigma}\sum_{i=1}^{n}x_i\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1} - \frac{\gamma}{\sigma}\sum_{i=1}^{n}x_i\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1} \times exp\left(-\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma}\right) = 0 \quad (20)$$

and

$$\frac{dlnL}{d\sigma} = \frac{n\gamma}{\sigma} - \gamma\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1}\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma^2}\right) - \gamma\sum_{i=1}^{n}\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma-1} \times \left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma^2}\right) exp\left(-\left(\frac{y_i-\beta_0-\beta_1 x_i}{\sigma}\right)^{-\gamma}\right) = 0 \quad (21)$$

Since these likelihood equations are non-linear, they can be routinely solved using the *fminsearch* function.

2.2.2. LS estimation

In the simple linear regression model, LS is the most widely used estimation method. Here, it is assumed that the random error terms ε_i have the DIW distribution. The estimators of the parameters β_0 and β_1 are the values which minimizing the following function

$$\sum_{i=1}^{n}\varepsilon_i^2 = \sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2$$

LS estimators of the parameters β_0 and β_1 are obtained as follows

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} - E(\hat{\varepsilon})$$

where $\bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}$, $\bar{y} = \sum_{i=1}^{n}\frac{y_i}{n}$ and $E(\hat{\varepsilon}) = E(\hat{z})\hat{\sigma}$ is the bias correction term. $E(\hat{z})$ and $\hat{\sigma}$ are as follows

$$E(\hat{z}) = \frac{\hat{\sigma}}{e-1}\Gamma\left(1-\frac{1}{\hat{\beta}}\right)T_1$$

and

$$\hat{\sigma} = \sqrt{\frac{s^2}{\frac{1}{e-1}\left(\Gamma\left(1-\frac{2}{\hat{\beta}}\right)T_2 - \frac{1}{e-1}\Gamma^2\left(1-\frac{1}{\hat{\beta}}\right)T_1^2\right)}}$$

respectively. Here, $s^2 = \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n-2}$ and $T_r = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{(1+m)^{1-(r/\beta)}}$, see Gul [3] for further details.

3. Results

In this paper, two different Monte Carlo simulation studies are performed. First is done to compare the efficiencies of the estimators described in Section 2. Second is conducted to compare the efficiencies of ML and LS estimators of the unknown parameters of the simple linear regression model in which the random error terms follow the DIW distribution.

Case I

In this part of the Monte Carlo simulation study, we compare the performances of the ML, LS, WLS, CM and AD estimators with respect to their means, mean-squared errors (MSEs) and deficiencies (DEFs) for different parameter values and sample sizes. The scale parameter σ is fixed at 1.0 without loss of generality. We consider β= 0.5, 1.0, 1.5, 2.0, 3.0 and 4.0 and n= 30, 50 and 100. The means, MSEs and DEFs of the estimators are computed based on 10,000 Monte Carlo runs. DEF is the natural measure of the joint efficiency of the pair $(\hat{\beta},\hat{\sigma})$ and is formulated as $DEF(\widehat{\beta,\sigma}) = MSE(\hat{\beta}) + MSE(\hat{\sigma})$, see e.g. Kantar and Senoglu [16]. The simulation results are reported in Table 1.

Table 1. Simulated Mean, MSE and DEF values for the estimators $\hat{\beta}$ and $\hat{\sigma}$

			Mean		*MSE*		*DEF*
β	n		$\hat{\beta}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\sigma}$	
0.5	30	*ML*	0.523	1.142	0.006	0.285	0.291
		LS	0.498	1.079	0.008	0.298	0.306
		WLS	0.502	1.086	0.007	0.272	0.279
		AD	0.508	1.097	0.006	0.263	0.269
		CM	0.525	1.152	0.010	0.357	0.367
	50	*ML*	0.515	1.078	0.003	0.134	0.137
		LS	0.499	1.047	0.004	0.154	0.158
		WLS	0.503	1.053	0.004	0.138	0.142
		AD	0.504	1.052	0.003	0.132	0.135
		CM	0.514	1.082	0.005	0.168	0.173
	100	*ML*	0.507	1.036	0.001	0.057	0.058
		LS	0.499	1.020	0.002	0.068	0.070
		WLS	0.503	1.030	0.001	0.060	0.061
		AD	0.501	1.035	0.001	0.060	0.061
		CM	0.506	1.042	0.002	0.072	0.074
1	30	*ML*	1.050	1.048	0.026	0.051	0.077
		LS	0.997	1.010	0.032	0.055	0.087
		WLS	1.009	1.017	0.029	0.051	0.080
		AD	1.014	1.025	0.024	0.051	0.075
		CM	1.046	1.044	0.039	0.062	0.101
	50	*ML*	1.028	1.027	0.014	0.028	0.042
		LS	0.996	1.008	0.018	0.032	0.050
		WLS	1.006	1.011	0.015	0.028	0.043
		AD	1.009	1.014	0.014	0.029	0.043
		CM	1.030	1.030	0.021	0.034	0.055
	100	*ML*	1.012	1.012	0.006	0.013	0.019
		LS	0.998	1.002	0.008	0.015	0.023
		WLS	1.003	1.009	0.007	0.014	0.021
		AD	1.003	1.006	0.007	0.014	0.021
		CM	1.014	1.014	0.009	0.016	0.025
1.5	30	*ML*	1.570	1.025	0.057	0.020	0.077
		LS	1.498	1.001	0.073	0.023	0.096
		WLS	1.511	1.005	0.063	0.021	0.084
		AD	1.525	1.009	0.056	0.020	0.076
		CM	1.579	1.026	0.093	0.025	0.118
	50	*ML*	1.541	1.016	0.031	0.012	0.043
		LS	1.499	1.002	0.042	0.013	0.055
		WLS	1.506	1.004	0.035	0.012	0.047
		AD	1.510	1.006	0.031	0.012	0.043
		CM	1.541	1.016	0.046	0.014	0.060
	100	*ML*	1.522	1.007	0.014	0.006	0.020
		LS	1.497	0.999	0.020	0.006	0.026
		WLS	1.506	1.003	0.016	0.006	0.022
		AD	1.504	1.002	0.015	0.006	0.021
		CM	1.520	1.007	0.021	0.007	0.028

Table 1 (continued)

			Mean		MSE		DEF
β	n		$\hat{\beta}$	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\sigma}$	
2	30	ML	2.100	1.017	0.107	0.011	0.118
		LS	1.999	1.001	0.134	0.013	0.147
		WLS	2.013	1.003	0.116	0.012	0.128
		AD	2.027	1.007	0.098	0.011	0.109
		CM	2.098	1.017	0.160	0.014	0.174
	50	ML	2.058	1.009	0.054	0.006	0.060
		LS	1.996	0.999	0.076	0.007	0.083
		WLS	2.014	1.003	0.062	0.007	0.069
		AD	2.016	1.004	0.055	0.007	0.062
		CM	2.058	1.009	0.085	0.007	0.092
	100	ML	2.026	1.004	0.024	0.003	0.027
		LS	1.999	0.999	0.036	0.003	0.039
		WLS	2.007	1.001	0.030	0.003	0.033
		AD	2.010	1.002	0.027	0.003	0.030
		CM	2.029	1.005	0.037	0.003	0.040
3	30	ML	3.144	1.010	0.239	0.004	0.243
		LS	2.995	0.997	0.299	0.005	0.304
		WLS	3.027	1.001	0.258	0.005	0.263
		AD	3.042	1.004	0.220	0.005	0.225
		CM	3.150	1.008	0.367	0.005	0.372
	50	ML	3.083	1.005	0.124	0.002	0.126
		LS	2.990	0.999	0.166	0.003	0.169
		WLS	3.020	1.001	0.144	0.003	0.147
		AD	3.019	1.002	0.127	0.003	0.130
		CM	3.101	1.005	0.192	0.003	0.195
	100	ML	3.042	1.002	0.056	0.001	0.057
		LS	2.995	0.999	0.080	0.001	0.081
		WLS	3.013	1.001	0.066	0.001	0.067
		AD	3.008	1.001	0.062	0.001	0.063
		CM	3.042	1.002	0.088	0.001	0.089
4	30	ML	4.194	1.008	0.427	0.002	0.429
		LS	3.990	0.997	0.525	0.003	0.528
		WLS	4.022	1.001	0.442	0.003	0.445
		AD	4.057	1.001	0.398	0.002	0.400
		CM	4.191	1.005	0.630	0.003	0.633
	50	ML	4.119	1.004	0.223	0.001	0.224
		LS	3.994	0.998	0.297	0.001	0.298
		WLS	4.025	1.001	0.254	0.001	0.255
		AD	4.032	1.001	0.225	0.001	0.226
		CM	4.115	1.004	0.334	0.002	0.336
	100	ML	4.052	1.002	0.100	0.001	0.101
		LS	3.995	0.998	0.143	0.001	0.144
		WLS	4.019	1.001	0.118	0.001	0.119
		AD	4.019	1.001	0.111	0.001	0.112
		CM	4.055	1.001	0.150	0.001	0.151

The simulation results given in Table 1 show that the LS, WLS and AD estimators of σ and β are almost unbiased for all values of the shape parameter β and sample size n. The ML and CM estimators of σ have slight biases except β = 0.5 and n = 30. On the other hand, the ML and CM estimators of β have larger bias values especially for small and moderate sample sizes. According to the MSE criterion, performances of all estimators of σ are quite close to each other for all values of the shape parameter β and sample size n, except β = 0.5. For estimating the shape parameter β, it is observed that the ML and AD estimators have the smallest MSE values and they are followed by WLS estimator. In view of the DEF values, ML estimator is the best for n = 50 and 100, AD estimator is the most preferable among the others for n = 30. They are followed by WLS estimator. According to the MSE and DEF values of β, LS and CM estimators did not perform well for all sample sizes and shape parameters.

Case II

Here, we compare the performances of the ML and LS estimators of the unknown regression parameters with respect to the mean, MSE and DEF criteria for different sample sizes and parameter values when the error terms ε_i have the DIW distribution.

We consider β as 3.0, 4.0, 5.0 and 6.0 and sample size n as 25, 50, 100, 300 and 500. Without loss of generality β_0, β_1 and σ are taken to be 0, 1 and 1, respectively. The Monte Carlo simulation is repeated for 10.000 times to evaluate the mean, MSE and DEF values of the estimators. The simulation results are reported in Table 2.

Table 2. Simulated Mean, MSE and DEF values for the estimators $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\sigma}$

				Mean			*MSE*		*DEF*
β	n		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\sigma}$	
3	25	*ML*	0.068	1.016	0.933	0.038	0.006	0.043	0.087
		LS	0.255	0.988	0.835	0.555	0.051	0.343	0.951
	50	*ML*	0.049	1.011	0.953	0.016	0.005	0.020	0.041
		LS	0.180	0.998	0.883	0.586	0.025	0.328	0.939
	100	*ML*	0.014	0.996	0.975	0.007	0.001	0.009	0.017
		LS	0.156	0.999	0.898	0.298	0.011	0.162	0.472
	300	*ML*	0.001	0.999	1.006	0.003	0.000	0.003	0.006
		LS	0.080	0.999	0.948	0.210	0.004	0.101	0.315
	500	*ML*	0.001	1.000	0.996	0.001	0.000	0.002	0.004
		LS	0.074	1.000	0.951	0.177	0.002	0.085	0.265
4	25	*ML*	0.080	0.997	0.934	0.035	0.004	0.038	0.077
		LS	0.119	0.998	0.911	0.233	0.016	0.174	0.424
	50	*ML*	0.057	0.998	0.948	0.019	0.002	0.021	0.042
		LS	0.077	0.999	0.943	0.202	0.007	0.140	0.350
	100	*ML*	0.010	1.002	0.999	0.007	0.001	0.007	0.015
		LS	0.047	1.000	0.964	0.102	0.001	0.062	0.165
	300	*ML*	0.005	1.003	0.997	0.002	0.000	0.003	0.005
		LS	0.029	0.999	0.977	0.056	0.001	0.036	0.094
	500	*ML*	0.003	0.999	0.998	0.001	0.000	0.001	0.002
		LS	0.017	1.000	0.986	0.051	0.000	0.032	0.085
5	25	*ML*	0.073	1.002	0.934	0.035	0.005	0.037	0.077
		LS	0.087	0.998	0.930	0.143	0.008	0.117	0.268
	50	*ML*	0.038	0.999	0.966	0.015	0.003	0.016	0.034
		LS	0.044	0.998	0.964	0.118	0.004	0.092	0.214
	100	*ML*	0.020	0.999	0.983	0.007	0.000	0.007	0.014
		LS	0.032	1.001	0.974	0.061	0.001	0.047	0.110
	300	*ML*	0.007	1.000	0.994	0.002	0.000	0.002	0.004
		LS	0.009	1.000	0.992	0.028	0.000	0.020	0.048
	500	*ML*	0.005	1.000	0.995	0.001	0.000	0.001	0.002
		LS	0.007	1.000	0.994	0.019	0.000	0.014	0.033
6	25	*ML*	0.071	0.998	0.941	0.032	0.002	0.033	0.067
		LS	0.056	0.999	0.953	0.122	0.004	0.105	0.231
	50	*ML*	0.037	0.999	0.966	0.013	0.001	0.015	0.029
		LS	0.038	0.999	0.967	0.071	0.002	0.060	0.133
	100	*ML*	0.020	0.999	0.982	0.007	0.000	0.007	0.014
		LS	0.018	1.000	0.984	0.041	0.000	0.034	0.075
	300	*ML*	0.006	1.000	0.994	0.002	0.000	0.002	0.004
		LS	0.007	1.000	0.993	0.019	0.000	0.015	0.034
	500	*ML*	0.004	1.000	0.997	0.001	0.000	0.001	0.002
		LS	0.005	1.000	0.996	0.010	0.000	0.008	0.018

The value of the shape parameter β is taken to be greater than or equal to 3. The reason of this is that the moments of the DIW distribution do not exist unless $\beta \geq 3$. Simulation results given in Table 2 show that the ML estimators of $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\sigma}$ performed better than the LS estimator for all values of the shape parameter β and sample size *n* according to the bias, the MSE and the DEF values.

4. Discussion and Conclusion

In this paper, estimation of the parameters of the DIW distribution and its implementation to simple linear regression are considered. Two different Monte Carlo simulation studies are performed. First is to compare the efficiencies of the estimators of the unknown parameters of the DIW distribution using five different estimation methods. Simulation results show that the ML and AD estimators are more preferable according to the MSE and DEF criteria in most of the cases. Second is to compare the estimators of the unknown parameters in a simple linear regression model by using two different estimation methods when the random error terms have the DIW distribution. Simulation study shows that the ML method performs better than the LS method for all sample sizes in terms of MSE and DEF criteria.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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