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Research Article

Special transforms of the generalized bivariate Fibonacci and Lucas polynomials

Nazmiye Yılmaz, İbrahim Aktaş*

Karamanoğlu Mehmetbey University, Kamil Özdağ Science Faculty, Department of Mathematics, Karaman, Turkey

Abstract

This paper deals with the Catalan, Hankel, binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials. Also, some useful results such as generating functions, Binet formulas, summations of transforms defined by using recurrence relations of these special polynomials are presented. Furthermore, certain important relations among these transforms are deduced by using obtained new formulas. Finally, the Catalan, Cassini, Vajda and d'Ocagne formulas for these transforms are also derived.

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1. Introduction and preliminaries

As any very well-studied object in mathematics, the Fibonacci, Lucas, Pell, Chebyshev numbers possess many kinds of generalizations. One of the most important generalizations is the Fibonacci polynomial [10,15,25,26,28,29]. Due to common usage of this polynomial in the applied sciences its some generalizations have been defined in the literature. In [12] and its references, interested readers may find a short history and comprehensive informations about the Fibonacci polynomial. The Fibonacci numbers are defined as

$$F_n = F_{n-1} + F_{n-2}, \quad (F_0 = 0,\ F_1 = 1, n \geq 2)\,. \tag{1.1}$$

In [11], the authors gave new generalizations of the Fibonacci and Lucas polynomials which are called generalized bivariate Fibonacci and Lucas polynomials.

Let $p(\xi,\varsigma)$ and $q(\xi,\varsigma)$ are polynomials which coefficients are all real. The generalized bivariate Fibonacci and Lucas polynomials are, respectively,

$$H_n(\xi,\varsigma) = p(\xi,\varsigma)H_{n-1}(\xi,\varsigma) + q(\xi,\varsigma)H_{n-2}(\xi,\varsigma) \tag{1.2}$$

and

$$K_n(\xi,\varsigma) = p(\xi,\varsigma)K_{n-1}(\xi,\varsigma) + q(\xi,\varsigma)K_{n-2}(\xi,\varsigma), \tag{1.3}$$

*Corresponding Author.
Email addresses: yilmaznzmy@gmail.com (N. Yılmaz), aktasibrahim38@gmail.com (İ. Aktaş)
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where $n \geq 2$, $H_0(\xi,\varsigma) = 0$, $H_1(\xi,\varsigma) = 1$, $K_0(\xi,\varsigma) = 2$, $K_1(\xi,\varsigma) = p(\xi,\varsigma)$ and $p^2(\xi,\varsigma) + 4q(\xi,\varsigma) > 0$. The Binet formulas, relations of the generalized bivariate Fibonacci and Lucas polynomials are, respectively, (see[11])

$$H_n(\xi,\varsigma) = \frac{\alpha^n(\xi,\varsigma) - (-q(\xi,\varsigma))^n \alpha^{-n}(\xi,\varsigma)}{\alpha(\xi,\varsigma) + q(\xi,\varsigma)\alpha^{-1}(\xi,\varsigma)}, \tag{1.4}$$

$$K_n(\xi,\varsigma) = \alpha^n(\xi,\varsigma) + (-q(\xi,\varsigma))^n \alpha^{-n}(\xi,\varsigma) \tag{1.5}$$

and

$$K_n(\xi,\varsigma) = H_{n+1}(\xi,\varsigma) + q(\xi,\varsigma)H_{n-1}(\xi,\varsigma), \tag{1.6}$$

where $\alpha(\xi,\varsigma)$ and $\beta(\xi,\varsigma) = -q(\xi,\varsigma)\alpha^{-1}(\xi,\varsigma)$ are roots of characteristic equation (1.2) and (1.3). For the different $p(\xi,\varsigma)$ and $q(\xi,\varsigma)$, we obtain different polynomial sequences by using recursive relation. These polynomial sequences are given in the Table 1 below in [11]:

$p(\xi,\varsigma)$	$q(\xi,\varsigma)$	$H_n(\xi,\varsigma)$	$K_n(\xi,\varsigma)$
ξ	ς	Bivariate Fibonacci, $F_n(\xi,\varsigma)$	Bivariate Lucas, $L_n(\xi,\varsigma)$
ξ	1	Fibonacci, $F_n(\xi)$	Lucas, $L_n(\xi)$
2ξ	1	Pell, $P_n(\xi)$	Pell-Lucas, $Q_n(\xi)$
1	2ξ	Jacobsthal, $J_n(\xi)$	Jacobsthal-Lucas, $j_n(\xi)$
2ξ	-1	Chebyshev of the second kind, $U_{n-1}(\xi)$	Chebyshev of the first kind, $2T_n(\xi)$
3ξ	-2	Fermat, $\mathcal{F}_n(\xi)$	Fermat-Lucas, $\mathcal{F}_n(\xi)$

Table 1: Special conditions of the Generalized Bivariate Fibonacci and Lucas Polynomials

Catalan numbers are described by [2]

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.7}$$

and the generating function of Catalan number sequence is given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1.8}$$

In [6,13], the authors gave the properties for Hankel transform. Many authors have also applied these transformations to the number sequences in [1,7,16–18,20,23,24]. Morever, several transforms which matrix based may be established for a given sequence. Binomial transform is one of them and there exist such kind of other transforms known as rising and falling binomial transforms too (see [3,5,8,9,19,21,22,27,30,31]). Furthermore, there is an interesting study on the watermarking and the binomial transform. For example, a binomial transform based on the fragile image watermarking technique is proposed for the color image authentication in [14].

We would like to remind here some basic concepts related to the our study. Let an integer sequence $X = \{x_0, x_1, x_2, \dots\}$. Then, the binomial transform B of the given sequences X is shown by $B(X) = \{b_n\}$ and it is defined by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} x_i. \tag{1.9}$$

Also, in [4], the author studied the following properties of the binomial transform:

$$\sum_{i=0}^{n} \binom{n}{i} i x_i = n(b_n - b_{n-1}) \tag{1.10}$$

and

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x_i}{i} = \sum_{j=1}^{n} \frac{b_j}{j}. \tag{1.11}$$

The main purpose of this article is to apply the Catalan, Hankel, binomial transforms to the generalized bivariate Fibonacci and Lucas polynomials. Also, the generating functions of the transforms mentioned here are found with the help of recurrence relations. Finally, it is shown the Catalan, Cassini, Vajda and d'Ocagne formulas and the relations among of the binomial transforms by deriving new formulas.

2. Catalan and Hankel transforms of the generalized bivariate Fibonacci and Lucas polynomials

In this part, we are going to determine Catalan transforms of the generalized bivariate Fibonacci and Lucas polynomials. Firstly, we examine the generating functions of these special polynomials and then, we will get the Hankel transforms of these transforms.

Definition 2.1. Let $H_n(\xi,\varsigma)$ and $K_n(\xi,\varsigma)$ be the generalized bivariate Fibonacci and Lucas polynomials, respectively. The Catalan transforms of these polynomials may be expressed as follows:

$$CH_n(\xi,\varsigma) = \sum_{i=1}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} H_i(\xi,\varsigma)$$

and

$$CK_n(\xi,\varsigma) = \sum_{i=1}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} K_i(\xi,\varsigma).$$

It is given the first few members of Catalan transform of the generalized bivariate Fibonacci polynomial:

$$\begin{aligned}
CH_1(\xi,\varsigma) =& 1\\
CH_2(\xi,\varsigma) =& p(\xi,\varsigma)+1\\
CH_3(\xi,\varsigma) =& p^2(\xi,\varsigma)+2p(\xi,\varsigma)+q(\xi,\varsigma)+2\\
CH_4(\xi,\varsigma) =& p^3(\xi,\varsigma)+3p^2(\xi,\varsigma)+2p(\xi,\varsigma)q(\xi,\varsigma)+5p(\xi,\varsigma)+3q(\xi,\varsigma)+5
\end{aligned}$$

It is given the first few members of Catalan transform of the generalized bivariate Lucas polynomial:

$$\begin{aligned}
CK_1(\xi,\varsigma) =& p(\xi,\varsigma)\\
CK_2(\xi,\varsigma) =& p^2(\xi,\varsigma)+p(\xi,\varsigma)+2q(\xi,\varsigma)\\
CK_3(\xi,\varsigma) =& p^3(\xi,\varsigma)+2p^2(\xi,\varsigma)+3p(\xi,\varsigma)q(\xi,\varsigma)+2p(\xi,\varsigma)+4q(\xi,\varsigma)\\
CK_4(\xi,\varsigma) =& p^4(\xi,\varsigma)+4p^2(\xi,\varsigma)q(\xi,\varsigma)+2q^2(\xi,\varsigma)+3p^3(\xi,\varsigma)+5p^2(\xi,\varsigma)\\
&+9p(\xi,\varsigma)q(\xi,\varsigma)+5p(\xi,\varsigma)+10q(\xi,\varsigma)
\end{aligned}$$

We know that the generating functions of the generalized bivariate Fibonacci and Lucas polynomials and Catalan numbers are

$$h(t) = \frac{t}{1-p(\xi,\varsigma)t-q(\xi,\varsigma)t^2},$$

$$k(t) = \frac{2-p(\xi,\varsigma)t}{1-p(\xi,\varsigma)t-q(\xi,\varsigma)t^2},$$

$$c(t) = \frac{1-\sqrt{1-4t}}{2t},$$

respectively, in [2,11]. Let $c(t)$ and $A(t)$ be the generating functions of the sequence of C_n and any sequence a_n. In [2], the author proved that $A(tc(t))$ is the generating function of the Catalan transform of this sequence. As a result, by making use of the composition of functions, it can be deduced the generating functions of the Catalan transforms of the generalized bivariate Fibonacci and Lucas polynomials as follow:

$$Ch(t) = \frac{1-\sqrt{1-4t}}{2-p(\xi,\varsigma)-q(\xi,\varsigma)+(p(\xi,\varsigma)+q(\xi,\varsigma))\sqrt{1-4t}+2q(\xi,\varsigma)t},$$

$$Ck(t) = \frac{4-p(\xi,\varsigma)+p(\xi,\varsigma)\sqrt{1-4t}}{2-p(\xi,\varsigma)-q(\xi,\varsigma)+(p(\xi,\varsigma)+q(\xi,\varsigma))\sqrt{1-4t}+2q(\xi,\varsigma)t},$$

respectively. Also, we write matrix CH_n as the product of the lower triangular matrix C and H_n matrix for generalized bivariate Fibonacci polynomials

$$\begin{aligned} CH_n &= C \cdot H_n \\ \begin{bmatrix} CH_1(\xi,\varsigma) \\ CH_2(\xi,\varsigma) \\ CH_3(\xi,\varsigma) \\ CH_4(\xi,\varsigma) \\ \vdots \end{bmatrix} &= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 5 & 5 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} H_1(\xi,\varsigma) \\ H_2(\xi,\varsigma) \\ H_3(\xi,\varsigma) \\ H_4(\xi,\varsigma) \\ \vdots \end{bmatrix}. \end{aligned}$$

Similarly, we write matrix CK_n as the product of the lower triangular matrix C and K_n matrix for generalized bivariate Lucas polynomials

$$\begin{aligned} CK_n &= C \cdot K_n \\ \begin{bmatrix} CK_1(\xi,\varsigma) \\ CK_2(\xi,\varsigma) \\ CK_3(\xi,\varsigma) \\ CK_4(\xi,\varsigma) \\ \vdots \end{bmatrix} &= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 5 & 5 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} K_1(\xi,\varsigma) \\ K_2(\xi,\varsigma) \\ K_3(\xi,\varsigma) \\ K_4(\xi,\varsigma) \\ \vdots \end{bmatrix}. \end{aligned}$$

By motivation [6, 13, 23], the Hankel transforms of Catalan transforms for these polynomials are given

$$\begin{aligned} HCH_1(\xi,\varsigma) =& 1 \\ HCH_2(\xi,\varsigma) =& q(\xi,\varsigma)+1 \\ HCH_3(\xi,\varsigma) =& q^2(\xi,\varsigma)+3q(\xi,\varsigma)+1 \end{aligned}$$

and

$$\begin{aligned} HCK_1(\xi,\varsigma) =& p(\xi,\varsigma) \\ HCK_2(\xi,\varsigma) =& -p^2(\xi,\varsigma)q(\xi,\varsigma)+p^2(\xi,\varsigma)-4q^2(\xi,\varsigma) \\ HCK_3(\xi,\varsigma) =& -48p^2(\xi,\varsigma)q(\xi,\varsigma)-71p^3(\xi,\varsigma)-88p^4(\xi,\varsigma)-16p^6(\xi,\varsigma)-44p^5(\xi,\varsigma) \\ & -64q^3(\xi,\varsigma)+24p^4(\xi,\varsigma)q(\xi,\varsigma)+196p^2(\xi,\varsigma)q^2(\xi,\varsigma)-7p^3(\xi,\varsigma)q(\xi,\varsigma) \\ & +156p(\xi,\varsigma)q^2(\xi,\varsigma)+8p^6(\xi,\varsigma)q(\xi,\varsigma)+16p^4(\xi,\varsigma)q^2(\xi,\varsigma) \\ & +16p^5(\xi,\varsigma)q(\xi,\varsigma)+73p^3(\xi,\varsigma)q^2(\xi,\varsigma)-48p^2(\xi,\varsigma)q^3(\xi,\varsigma) \\ & -104p(\xi,\varsigma)q^3(\xi,\varsigma)-4p^7(\xi,\varsigma)-16p^3(\xi,\varsigma)q^3(\xi,\varsigma)-4p(\xi,\varsigma)q^4(\xi,\varsigma) \\ & -4p^5(\xi,\varsigma)q^2(\xi,\varsigma), \end{aligned}$$

respectively.

3. Binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials

In this section, we center on binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials in order to obtain several important results. Firstly, we present the recurrence relations, generating functions, Binet formulas and summations. Eventually, we will arrive at some relations among of these transforms. We would like to mention here that, we write $\alpha = \alpha(\xi, \varsigma), \beta = \beta(\xi, \varsigma)$ and $p = p(\xi, \varsigma), q = q(\xi, \varsigma)$ for the sake of brevity.

Definition 3.1. Let $H_n(\xi, \varsigma)$ and $K_n(\xi, \varsigma)$ be the generalized bivariate Fibonacci and Lucas polynomials, respectively. The binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials are, respectively,

$$b_n(\xi, \varsigma) = \sum_{i=0}^{n} \binom{n}{i} H_i(\xi, \varsigma) \text{ and } c_n(\xi, \varsigma) = \sum_{i=0}^{n} \binom{n}{i} K_i(\xi, \varsigma).$$

The following lemma is a key tool for the proof of the following theorem.

Lemma 3.2. *The following identities hold true for $n \geq 0$:*

i) $b_{n+1}(\xi, \varsigma) - b_n(\xi, \varsigma) = \sum_{i=0}^{n} \binom{n}{i} H_{i+1}(\xi, \varsigma)$,
ii) $c_{n+1}(\xi, \varsigma) - c_n(\xi, \varsigma) = \sum_{i=0}^{n} \binom{n}{i} K_{i+1}(\xi, \varsigma)$.

Proof. Here, we just prove the identity *i*), since the identity *ii*) may be obtained with the same manner.

i) From the Definition 3.1 and the binomial identity

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1} \text{ for } 1 \leqslant i \leqslant n, \tag{3.1}$$

we deduce

$$\begin{aligned} b_{n+1}(\xi, \varsigma) &= \sum_{i=1}^{n+1} \binom{n+1}{i} H_i(\xi, \varsigma) + H_0(\xi, \varsigma) \\ &= \sum_{i=0}^{n} \binom{n}{i} H_i(\xi, \varsigma) + \sum_{i=1}^{n+1} \binom{n}{i-1} H_i(\xi, \varsigma) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(H_i(\xi, \varsigma) + H_{i+1}(\xi, \varsigma) \right), \end{aligned}$$

which is desired result.

□

Theorem 3.3. *i) Suppose that $b_0(\xi, \varsigma) = 0$ and $b_1(\xi, \varsigma) = 1$. For $n > 0$, the sequences $\{b_n(\xi, \varsigma)\}$ has a recurrence relation as follow:*

$$b_n(\xi, \varsigma) = (p+2)\, b_{n-1}(\xi, \varsigma) + (q - p - 1)\, b_{n-2}(\xi, \varsigma). \tag{3.2}$$

ii) Let $c_0(\xi, \varsigma) = 2$ and $c_1(\xi, \varsigma) = p + 2$. For $n > 0$, the sequences $\{c_n(\xi, \varsigma)\}$ has a recurrence relation as follow:

$$c_n(\xi, \varsigma) = (p+2)\, c_{n-1}(\xi, \varsigma) + (q - p - 1)\, c_{n-2}(\xi, \varsigma). \tag{3.3}$$

Proof. Just as the proof of previous lemma, we prove the identity *i*) only.

i) By considering Lemma 3.2, we obtain

$$\begin{aligned} b_{n+1}(\xi,\varsigma) &= \sum_{i=0}^{n}\binom{n}{i}(H_i(\xi,\varsigma)+H_{i+1}(\xi,\varsigma)) \\ &= \sum_{i=1}^{n}\binom{n}{i}(H_i+H_{i+1})(\xi,\varsigma)+H_0(\xi,\varsigma)+H_1(\xi,\varsigma) \\ &= (p+1)b_n(\xi,\varsigma)+q\sum_{i=1}^{n}\binom{n}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma). \end{aligned} \tag{3.4}$$

By taking $n \to n-1$, we have

$$b_n(\xi,\varsigma) = (p+1)b_{n-1}(\xi,\varsigma)+q\sum_{i=1}^{n-1}\binom{n-1}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma).$$

By considering recurrence relations of the generalized bivariate Fibonacci polynomials and the equation (3.1), we obtain

$$\begin{aligned} b_n(\xi,\varsigma) &= (p+1)b_{n-1}(\xi,\varsigma)+q\sum_{i=1}^{n-1}\binom{n-1}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma) \\ &= pb_{n-1}(\xi,\varsigma)+\sum_{i=1}^{n-1}\binom{n-1}{i-1}H_{i-1}(\xi,\varsigma)+q\sum_{i=1}^{n}\binom{n-1}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma) \\ &= pb_{n-1}(\xi,\varsigma)+(1-q)\sum_{i=0}^{n-1}\binom{n-1}{i}H_i(\xi,\varsigma)+q\sum_{i=1}^{n}\binom{n}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma) \\ &= (p+1-q)b_{n-1}(\xi,\varsigma)+q\sum_{i=1}^{n}\binom{n}{i}H_{i-1}(\xi,\varsigma)+H_1(\xi,\varsigma). \end{aligned}$$

From (3.4), we have

$$b_{n+1}(\xi,\varsigma) = (p+1)b_n(\xi,\varsigma)+b_n(\xi,\varsigma)-(p+1-q)b_{n-1}(\xi,\varsigma),$$

which completes the proof.

□

The generating functions for the generalized bivariate Fibonacci and Lucas polynomials play a vital role to find out many important identities for these polynomials. In the next theorem, we improve the generating functions for the binomial transforms of these polynomials.

Theorem 3.4. *The generating functions of the binomial transforms for* $\{b_n(\xi,\varsigma)\}$ *and* $\{c_n(\xi,\varsigma)\}$ *are*

i) $\sum\limits_{n=0}^{\infty} b_n(\xi,\varsigma)t^n = \frac{t}{1-(p+2)t-(q-p-1)t^2}$,

ii) $\sum\limits_{n=0}^{\infty} c_n(\xi,\varsigma)t^n = \frac{2-(p+2)t}{1-(p+2)t-(q-p-1)t^2}$,

where α *and* β *denote the zeros of characteristic equation* $\lambda^2-(p+2)\lambda-(q-p-1)=0$.

Proof. Let suppose that $b(\xi,\varsigma,t)$ and $c(\xi,\varsigma,t)$ are the generating functions of $\{b_n(\xi,\varsigma)\}$ and $\{c_n(\xi,\varsigma)\}$, respectively. Then, we want to emphasize here that, $b(\xi,\varsigma,t)$ and $c(\xi,\varsigma,t)$ may be obtained by using the generating functions of the generalized bivariate Fibonacci and Lucas polynomials in [11],

$$f(t) = \frac{t}{1-pt-qt^2}$$

and

$$g\left(t\right)=\frac{2-pt}{1-pt-qt^{2}}.$$

By using the result proven by Prodinger in [19] we can easily see the followings:

$$b(\xi,\varsigma,t)=\frac{1}{1-t}f\left(\frac{t}{1-t}\right)$$

and

$$c\left(\xi,\varsigma,t\right)=\frac{1}{1-t}g\left(\frac{t}{1-t}\right).$$

So, the proof is completed. □

To derive new identities, we now present an explicit formula of $\{b_n(\xi,\varsigma)\}$ and $\{c_n(\xi,\varsigma)\}$ for $n\geq 0$.

Theorem 3.5. *The Binet's formulas of sequences* $\{b_n(\xi,\varsigma)\}$ *and* $\{c_n(\xi,\varsigma)\}$ *are*

i) $b_n(\xi,\varsigma)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$,

ii) $c_n(\xi,\varsigma)=\alpha^n+\beta^n$,

where α *and* β *denote the zeros of charecteristic equation* $\lambda^2-(p+2)\lambda-(q-p-1)=0$.

Proof. *i)* From Theorem 3.4, we have

$$\sum_{n=0}^{\infty}b_n(\xi,\varsigma)t^n=\frac{t}{1-(p+2)\,t-(q-p-1)\,t^{2}}.$$

It is easily seen that

$$\begin{aligned} b\left(\xi,\varsigma,t\right)&=\frac{\frac{1}{\sqrt{p^2+4q}}}{1-\frac{p+2+\sqrt{p^2+4q}}{2}t}-\frac{\frac{1}{\sqrt{p^2+4q}}}{1-\frac{p+2-\sqrt{p^2+4q}}{2}t}\\ &=\frac{1}{\alpha-\beta}\left[\sum_{n=0}^{\infty}\alpha^n t^n-\sum_{n=0}^{\infty}\beta^n t^n\right].\end{aligned}$$

Thus, by the equality of generating function, we get $b_n(\xi,\varsigma)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$.

ii) Since this part can be proven similarly to the part *i)* we omitted the details. □

Theorem 3.6. *The exponential generating functions of the binomial transforms for* $\{b_n(\xi,\varsigma)\}$ *and* $\{c_n(\xi,\varsigma)\}$ *are*

i) $\sum\limits_{n=0}^{\infty}b_n(\xi,\varsigma)\frac{t^n}{n!}=\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$,

ii) $\sum\limits_{n=0}^{\infty}c_n(\xi,\varsigma)\frac{t^n}{n!}=e^{\alpha t}+e^{\beta t}$,

where α *and* β *denote the zeros of charecteristic equation* $\lambda^2-(p+2)\lambda-(q-p-1)=0$.

Proof. These assertions can be easily proven by using Theorem 3.5. □

Now, we give the summations of binomial transforms for the generalized bivariate Fibonacci and Lucas polynomials.

Theorem 3.7. *Sums of sequences* $\{b_n(\xi,\varsigma)\}$ *and* $\{c_n(\xi,\varsigma)\}$ *are*

i) $\sum\limits_{i=0}^{n-1}b_i(\xi,\varsigma)=\frac{b_n(\xi,\varsigma)-(p+1-q)b_{n-1}(\xi,\varsigma)-1}{q}$,

ii) $\sum\limits_{i=0}^{n-1}c_i(\xi,\varsigma)=\frac{c_n(\xi,\varsigma)-(p+1-q)c_{n-1}(\xi,\varsigma)+p}{q}$.

Proof. We omit the proof of part $i)$ since it is quite similar to the part $ii)$. Now, considering Theorem 3.5, we deduce

$$\begin{aligned}\sum_{i=0}^{n-1} c_i(\xi,\varsigma) &= \sum_{i=0}^{n-1}\left(\alpha^i+\beta^i\right)\\ &= \frac{\alpha^n-1}{\alpha-1}+\frac{\beta^n-1}{\beta-1}\\ &= \frac{\alpha^n\beta-\alpha^n-\beta+\alpha\beta^n-\beta^n-\alpha+2}{\alpha\beta-\alpha-\beta+1}.\end{aligned}$$

Simplifying the last fraction the proof of part $ii)$ can be obtained easily. □

The combinatorial equalities of the binomial transforms for the generalized bivariate Fibonacci and Lucas polynomials may be given as the followings:

Theorem 3.8. *We have*

$i)$ $\sum_{i=0}^{n}\binom{n}{i}\left(\frac{p+2}{q-p-1}\right)^i b_i(\xi,\varsigma)=\frac{1}{(q-p-1)^n}b_{2n}(\xi,\varsigma)$,

$ii)$ $\sum_{i=0}^{n}\binom{n}{i}\left(\frac{p+2}{q-p-1}\right)^i c_i(\xi,\varsigma)=\frac{1}{(q-p-1)^n}c_{2n}(\xi,\varsigma)$,

$iii)$ $\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}(p+2)^{n-2i-1}(p^2+4q)^i=2^{n-1}b_n(\xi,\varsigma)$,

$iv)$ $\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}(p+2)^{n-2i}(p^2+4q)^i=2^{n-1}c_n(\xi,\varsigma)$,

where $n\geqslant 1$.

Proof. $i)$ From Theorem 3.5, we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(p+2)^i(q-p-1)^{n-i}b_i(\xi,\varsigma) &= \sum_{i=0}^{n}\binom{n}{i}(p+2)^i(q-p-1)^{n-i}\left(\frac{\alpha^i-\beta^i}{\alpha-\beta}\right)\\ &= \frac{((p+2)\alpha+q-p-1)^n-((p+2)\beta+q-p-1)^n}{\alpha-\beta}\\ &= \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\\ &= b_{2n}(\xi,\varsigma).\end{aligned}$$

The part $ii)$ can be proven similarly as in the proof of $i)$. Also, the parts $iii)$ and $iv)$ can be seen by method of induction for the binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials.

□

The relationships among these binomial transforms are in the following theorem.

Theorem 3.9. *If* $n\geq 0$*, then the followings are true:*

$i)$ $b_{n+1}(\xi,\varsigma)+(q-p-1)b_{n-1}(\xi,\varsigma)=c_n(\xi,\varsigma)$,

$ii)$ $c_{n+1}(\xi,\varsigma)+(q-p-1)c_{n-1}(\xi,\varsigma)=(p^2+4q)b_n(\xi,\varsigma)$,

$iii)$ $b_n(\xi,\varsigma)c_n(\xi,\varsigma)=b_{2n}(\xi,\varsigma)$,

$iv)$ $b_n(\xi,\varsigma)b_m(\xi,\varsigma)=\frac{1}{p^2+4q}\left[c_{n+m}(\xi,\varsigma)-(-q+p+1)^m c_{n-m}(\xi,\varsigma)\right], n\geq m$.

$v)$ $c_n(\xi,\varsigma)c_m(\xi,\varsigma)=c_{n+m}(\xi,\varsigma)+(-q+p+1)^m c_{n-m}(\xi,\varsigma), n\geq m$.

Proof. *i)* It follows from Theorem 3.5 that

$$\begin{aligned} b_{n+1}(\xi,\varsigma)+(q-p-1)b_{n-1}(\xi,\varsigma) &= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}+(q-p-1)\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} \\ &= \frac{\alpha^{n-1}}{\alpha-\beta}\left(\alpha^2-\alpha\beta\right)-\frac{\beta^{n-1}}{\alpha-\beta}\left(\beta^2-\alpha\beta\right) \\ &= c_n(\xi,\varsigma), \end{aligned}$$

which is desired.

The rest assertions of the theorem may be proven similarly as in the proof of *i)*. □

We extend the definitions of $b_n(\xi,\varsigma)$ and $c_n(\xi,\varsigma)$ to negative subscripts in the following proposition.

Proposition 3.10. *For any integer $n \geq 1$, the followings are valid:*

i) $b_{-n}(\xi,\varsigma)=-\frac{1}{(-q+p+1)^n}b_n(\xi,\varsigma)$,

ii) $c_{-n}(\xi,\varsigma)=\frac{1}{(-q+p+1)^n}c_n(\xi,\varsigma)$.

Using the Binet formulas we can give the Catalan formulas of these transforms, as the next proposition shows.

Proposition 3.11. *For integers n, r with $r < n$, we have*

i) $b_{n-r}(\xi,\varsigma)b_{n+r}(\xi,\varsigma)-b_n^2(\xi,\varsigma)=-(-q+p+1)^{n-r}b_r^2(\xi,\varsigma)$,

ii) $c_{n-r}(\xi,\varsigma)c_{n+r}(\xi,\varsigma)-c_n^2(\xi,\varsigma)=(-q+p+1)^{n-r}(p^2+4q)b_r^2(\xi,\varsigma)$.

Remark 3.12. For $r=1$ in Proposition 3.11,

$$b_{n-1}(\xi,\varsigma)b_{n+1}(\xi,\varsigma)-b_n^2(\xi,\varsigma)=-(-q+p+1)^{n-1}$$

and

$$c_{n-1}(\xi,\varsigma)c_{n+1}(\xi,\varsigma)-c_n^2(\xi,\varsigma)=(-q+p+1)^{n-1}(p^2+4q),$$

which we call as Cassini formulas for the binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials.

Using the Binet formulas we can give the Vajda formulas of these transforms, as the next proposition shows.

Proposition 3.13. *For integers n, i, j, we have*

i) $b_{n+i}(\xi,\varsigma)b_{n+j}(\xi,\varsigma)-b_n(\xi,\varsigma)b_{n+i+j}(\xi,\varsigma)=(-q+p+1)^n b_i(\xi,\varsigma)b_j(\xi,\varsigma)$,

ii) $c_{n+i}(\xi,\varsigma)c_{n+j}(\xi,\varsigma)-c_n(\xi,\varsigma)c_{n+i+j}(\xi,\varsigma)=-(-q+p+1)^n(p^2+4q)b_i(\xi,\varsigma)b_j(\xi,\varsigma)$.

Remark 3.14. In the above Proposition 3.13, by giving $i=1, j=m-n$, we have the following d'Ocagne identities:

i) $b_{n+1}(\xi,\varsigma)b_m(\xi,\varsigma)-b_n(\xi,\varsigma)b_{m+1}(\xi,\varsigma)=(-q+p+1)^n b_{m-n}(\xi,\varsigma)$,

ii) $c_{n+1}(\xi,\varsigma)c_m(\xi,\varsigma)-c_n(\xi,\varsigma)c_{m+1}(\xi,\varsigma)=-(-q+p+1)^n(p^2+4q)b_{m-n}(\xi,\varsigma)$.

Proposition 3.15. *For any integer $n > 0$, we have*

i) $n\left(b_n(\xi,\varsigma)-b_{n-1}(\xi,\varsigma)\right)=\sum_{i=0}^{n}\binom{n}{i}iH_i(\xi,\varsigma)$,

ii) $n\left(c_n(\xi,\varsigma)-c_{n-1}(\xi,\varsigma)\right)=\sum_{i=0}^{n}\binom{n}{i}iK_i(\xi,\varsigma)$,

iii) $\sum_{j=1}^{n}\frac{b_j(\xi,\varsigma)}{j}=\sum_{i=1}^{n}\binom{n}{i}\frac{H_i(\xi,\varsigma)}{i}$,

iv) $\sum_{j=1}^{n}\frac{c_j(\xi,\varsigma)}{j}=\sum_{i=1}^{n}\binom{n}{i}\frac{K_i(\xi,\varsigma)}{i}+\sum_{j=1}^{n}\frac{2}{j}$.

Proof. We will give only the proofs of *ii)* and *iii)*, since the proof of others are similar to them.

ii) From Definition 3.1, we obtain

$$\begin{aligned} n\left(c_n(\xi,\varsigma)-c_{n-1}(\xi,\varsigma)\right) =& n\left(\sum_{i=0}^{n}\binom{n}{i}K_i(\xi,\varsigma)-\sum_{i=0}^{n-1}\binom{n-1}{i}K_i(\xi,\varsigma)\right)\\ =& n\sum_{i=0}^{n}\binom{n}{i}K_i(\xi,\varsigma)-n\sum_{i=0}^{n}\frac{n-i}{n}\binom{n}{i}K_i(\xi,\varsigma)\\ =& \sum_{i=0}^{n}\binom{n}{i}iK_i(\xi,\varsigma). \end{aligned}$$

iii) From Definition 3.1, we get

$$\begin{aligned} \sum_{j=1}^{n}\frac{b_j(\xi,\varsigma)}{j} =& \sum_{j=1}^{n}\frac{1}{j}\left(\sum_{i=0}^{j}\binom{j}{i}H_i(\xi,\varsigma)\right)\\ =& \sum_{j=1}^{n}\frac{1}{j}\left(\binom{j}{0}H_0(\xi,\varsigma)+\binom{j}{1}H_1(\xi,\varsigma)+\binom{j}{2}H_2(\xi,\varsigma)+\ldots+\binom{j}{j}H_j(\xi,\varsigma)\right)\\ =& \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)H_0(\xi,\varsigma)+\left(1+\frac{2}{2}+\frac{3}{3}+\ldots+\frac{n}{n}\right)H_1(\xi,\varsigma)\\ &+\left(\frac{1}{2}+1+\frac{3}{2}+\ldots+\frac{n-1}{2}\right)H_2(\xi,\varsigma)+\ldots+\frac{1}{n}H_n(\xi,\varsigma). \end{aligned}$$

By considering $H_0(\xi,\varsigma)=0$ and the properties of summation, we can write

$$\begin{aligned} \sum_{j=1}^{n}\frac{b_j(\xi,\varsigma)}{j} =& \binom{n}{1}\frac{H_1(\xi,\varsigma)}{1}+\binom{n}{2}\frac{H_2(\xi,\varsigma)}{2}+\cdots+\binom{n}{n}\frac{H_n(\xi,\varsigma)}{n}\\ =& \sum_{i=1}^{n}\binom{n}{i}\frac{H_i(\xi,\varsigma)}{i}. \end{aligned}$$

□

Conclusion

In the presented paper, we introduce the Catalan, Hankel and binomial transforms of the generalized bivariate Fibonacci and Lucas polynomials, and give some interesting properties of these transforms. As a result, we obtained a new generalization for the Catalan, Hankel and binomial transforms that have the similar recurrence relation in the literature. Taking into account these transforms and their properties, it also can be obtained properties of the Catalan, Hankel, binomial transforms of the Fibonacci, Lucas, Pell, Jacobsthal, balancing numbers and polynomials. That is,

- If we replace $p=1, q=2k$ in $CK_n(\xi,\varsigma), HCK_n(\xi,\varsigma)$, we obtain the Catalan, Hankel transforms for k-Jacobsthal-Lucas numbers in [1].
- If we replace $p=k, q=1$ in $CH_n(\xi,\varsigma), HCH_n(\xi,\varsigma), b_n(\xi,\varsigma)$, we obtain the Catalan, Hankel, binomial transforms for k-Fibonacci numbers in [7, 8].
- If we replace $p=k, q=1$ in $CK_n(\xi,\varsigma), HCK_n(\xi,\varsigma), c_n(\xi,\varsigma)$, we obtain the Catalan, Hankel, binomial transforms for k-Lucas numbers in [3, 16].
- If we replace $p=1, q=k$ in $CH_n(\xi,\varsigma), HCH_n(\xi,\varsigma)$ we obtain the Catalan, Hankel transforms for k-Jacobsthal numbers in [23].
- If we replace $p=2, q=k$ in $CH_n(\xi,\varsigma), HCH_n(\xi,\varsigma), CK_n(\xi,\varsigma), HCK_n(\xi,\varsigma)$, we obtain the Catalan, Hankel transforms for k-Pell and k-Pell-Lucas numbers in [24].
- If we replace $p=6\xi, q=-1$ in $b_n(\xi,\varsigma)$, we obtain the binomial transforms for balancing polynomials in [30].

Eventually, we obtain the generating and exponential generating functions, Binet formulas, summations and relationships for the binomial transforms of the well-known sequences in the literature.

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