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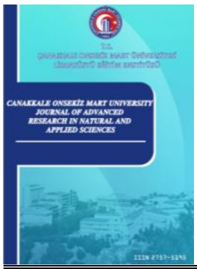

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Gaussian-bihyperbolic Numbers Containing Pell and Pell-Lucas Numbers

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Research Article

Abstract − In this study, we define a new type of Pell and Pell-Lucas numbers which are called Gaussian-bihyperbolic Pell and Pell-Lucas numbers. We also define negaGaussian-bihyperbolic Pell and Pell-Lucas numbers. Moreover, we obtain Binet's formulas, generating function formulas, d'Ocagne's identities, Catalan's identities, Cassini's identities and some sum formulas for these new type numbers and we investigate some algebraic proporties of these. Furthermore, we give the matrix representation of Gaussian-bihyperbolic Pell and Pell-Lucas numbers.

Keywords − *biHyperbolic, Gaussian-bihyperbolic, Gaussian-bihyperbolic Pell, Gaussian-bihyperbolic Pell-Lucas*

1. Introduction

Complex numbers, Dual numbers and Hyperbolic numbers arise in many areas such as coordinate transformation, dynamic analysis, displacement analysis, static analysis, velocity analysis, rigid body dynamics, matrix modeling, geometry, mechanics, mathematics, physics, kinematics and transformation. Horadam (Horadam, 1963) introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers $GF_n = F_n + iF_{n-1}$ where $F_n \in \mathbb{R}, i^2 = -1$ and F_n, nth Fibonacci numbers. Fjelstad and Gal (Fjelstad and Gal, 1998) defined the hyperbolic numbers $H = h + jh^*$ where $h, h^* \in \mathbb{R}, j^2 = 1$ and $j \neq \pm 1$. Clifford (Clifford, 1871) described the dual numbers $D = d + \varepsilon d^*$ where $d, d^* \in \mathbb{R}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. Messelmi (Messelmi, 2022) expressed the dual-complex numbers $Z = z + \varepsilon z^*$ where $z, z^* \in \mathbb{C}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. There are a number of studies in the literature concerned with these numbers (Catarino, 2019; Gül, 2020; Soykan, 2021; Vajda, 1989; Gürses, Şentürk and Yüce, 2021). Fjelstad and Gal (Fjelstad and Gal, 1998) inspected the extensions of the hyperbolic complex numbers to n-dimensions and they gave n-dimensional dual complex numbers in algebra and analysis. Matsuda (Matsuda, Kaji and Ochiai, 2014) et al. inspected the two-dimensional rigid transformation which is more concise and efficient than the standart matrix presentation, by modifying the ordinary dual number construction for the complex numbers. Majernik (Majernik, 1996) gave three types of the four-component number systems which are formed by using the complex, binary and dual two-component numbers. Akar (Akar, Yüce and Şahin, 2018) et al. introduced arithmetical operations on dual-hyperbolic numbers. They investigated dual hyperbolic number and hyperbolic complex number valued functions. Brod (Brod, Szynal-Liana, Wloch, 2020) et al. formulated any bihyperbolic number by $w = x_1 + jx_2 + j_1x_3 + j_2x_4$. Addition, substraction and multiplication of bihyperbolic numbers and was defined as

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$$w_1 \pm w_2 = (x_1 + y_1) \pm j(x_2 + y_2) \pm j_1(x_3 \pm y_3) + j_2(x_4 \pm y_4)$$
$$w_1 \times w_2 = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4) + j(x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3)$$
$$+j_1(x_1y_3 + x_2y_4 + x_3y_1 + x_4y_2) + j_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1).$$

Table 1

Multiplication scheme of bihyperbolic numbers.

$\times$	1	j	j_1	j_2
1	1	j	j_1	j_2
j	j	1	j_2	j_1
j_1	j_1	j_2	1	j
j_2	j_2	j_1	j	1

For $n \in \mathbb{N}_0$, Pell and Pell-Lucas numbers are defined by the recurrence relations, respectively, in the following way: $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 2$, $Q_1 = 2$. The nth Pell and Pell-Lucas number are also formulized as $P_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $Q_n = \alpha^n + \beta^n$, where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$. These formulas are called as their Binet's formulas (Koshy, 2014).

Many researchers have studied several areas of these number sequences. Brod (Brod, Szynal-Liana, Wloch, 2021) et al. examined a new one-parameter of bihyperbolic Pell numbers. They also gave some important features of these newly defined numbers. Soykan and Göcen (Soykan and Göcen, 2020) introduced the generalized hyperbolic Pell numbers over the bidimensional Clifford algebra of hyperbolic numbers. Azak and Güngör (Azak and Güngör, 2017) defined the dual complex Fibonacci and Lucas numbers and gave the well-known properties for these numbers. Aydın (Aydın, 2019) investigated the hyperbolic Fibonacci sequence and the hyperbolic Fibonacci numbers and also gave some algebraic properties of them. Dikmen (Dikmen, 2019) introduced the hyperbolic Jacobsthal numbers and presented recurrence relations, Binet's formula, generating function and the summation formulas for these numbers. Petoukhov (Petoukhov, 2019) desribed the applications of two-dimensional hyperbolic numbers and their algebraic two-dimensional extensions in modeling some genetic phenomena. He also gave the properties of hyperbolic numbers and their matrix representations. Bilgin and Ersoy (Bilgin and Ersoy, 2020) studied the four-dimensional real algebra of bihyperbolic numbers. They also showed conjugates, three hyperbolic valued moduli, real moduli and multiplicative inverse of this numbers.

2. Gaussian-bihyperbolic Pell and Pell-Lucas Numbers

In this sections, the Gaussian-bihyperbolic Pell and Pell-Lucas numbers will be defined. A variety of algebraic properties of Gaussian-bihyperbolic Pell and Pell-Lucas numbers are presented in a unified manner. Some identities will be given for Gaussian-bihyperbolic Pell and Pell-Lucas numbers such as Binet's formulas, generating function formulas, d'Ocagne's identities, Catalan's identities, Cassini's identities and some sum formulas. The properties of Gaussian-bihyperbolic Pell and the Pell-Lucas numbers will also be obtained using their matrix representation.

Horadam (Horadam, 1963) introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers $GF_n = F_n + iF_{n-1}$ where $F_n \in \mathbb{R}, i^2 = -1$ and F_n, the nth Fibonacci numbers. In view of this definition, we will call Gaussian numbers as the numbers whose components are formed by ordering the consecutive terms of a number sequence from largest to smallest. After these explanations, we can give the following definition.

Definition 2.1. For $n \in \mathbb{N}_0$, the Gaussian-bihyperbolic Pell and Pell-Lucas numbers are defined by

$$GP_{n+3} = P_{n+3} + jP_{n+2} + j_1P_{n+1} + j_2P_n$$

$$GQ_{n+3} = Q_{n+3} + jQ_{n+2} + j_1Q_{n+1} + j_2Q_n$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers. j, j_1 and j_2 denotes ($j^2 = j_1^2 = j_2^2 = 1; j, j_1, j_2 \neq \mp 1$).

$$GP_0 = j - 2j_1 + 5j_2,\ GP_1 = 1 + j_1 - 2j_2,\ GP_2 = 2 + j + j_2, \dots$$

$$GQ_0 = 2 - 2j + 6j_1 - 14j_2,\ GQ_1 = 2 + 2j - 2j_1 + 6j_2,\ GQ_2 = 6 + 2j + 2j_1 - 2j_2, \dots$$

$GP_n = 2GP_{n-1} + GP_{n-2}$ and $GQ_n = 2GQ_{n-1} + GQ_{n-2}$ are a recurrence relationship between Gaussian-bihyperbolic Pell and Gaussian-bihyperbolic Pell-Lucas numbers.

Let GP_{a+3} and GP_{b+3} be two Gaussian-bihyperbolic Pell numbers. The addition, substraction and multiplication of the Gaussian-bihyperbolic Pell numbers are given by

$$GP_{a+3} \pm GP_{b+3} = (P_{a+3} \pm P_{b+3}) + j(P_{a+2} \pm P_{b+2}) + j_1(P_{a+1} \pm P_{b+1}) + j_2(P_a \pm P_b)$$

$$\begin{aligned} GP_{a+3} \times GP_{b+3} = {} & (P_{a+3}P_{b+3} + P_{a+2}P_{b+2} + P_{a+1}P_{b+1} + P_aP_b) \\ & + j(P_{a+3}P_{b+2} + P_{a+2}P_{b+3} + P_{a+1}P_b + P_aP_{b+1}) + j_1(P_{a+3}P_{b+1} + P_{a+2}P_b + P_{a+1}P_{b+3} + P_aP_{b+2}) \\ & + j_2(P_{a+3}P_b + P_{a+2}P_{b+1} + P_{a+1}P_{b+2} + P_aP_{b+3}). \end{aligned}$$

Lemma 2.2. Let P_n and Q_n be the Pell and the Pell-Lucas numbers, respectively. The following relations are satisfied (Koshy, 2014)

i. $P_mP_{n+r} - P_{m+r}P_n = (-1)^n P_{m-n}P_r$
ii. $P_mP_{n+r} + P_{m+r}P_n = \frac{2Q_{m+n+r} - (-1)^n Q_{m-n}Q_r}{8}$
iii. $\sum_{k=1}^{n} P_k = \frac{Q_{n+1} - 1}{2}$
iv. $\sum_{k=1}^{n} P_{2k-1} = \frac{P_{2n}}{2}$
v. $\sum_{k=1}^{n} P_{2k} = \frac{P_{2n+1} - 1}{2}$.

Similar equations for Pell-Lucas numbers is obtained.

Definition 2.3. The negaGaussian-bihyperbolic Pell and the negaGaussian-bihyperbolic Pell-Lucas numbers are defined by

$$GP_{-n} = (-1)^{n+1}[P_n - jP_{n+1} + j_1P_{n+2} - j_2P_{n+3}]$$

$$GQ_{-n} = (-1)^{n-3}[-Q_n + jQ_{n+1} - j_1Q_{n+2} + j_2Q_{n+3}]$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers.

Corollary 2.4. Let GP_n and GQ_n be the Gaussian-bihyperbolic Pell and the Gaussian-bihyperbolic Pell-Lucas numbers, respectively. The following relations are satisfied

i. $2(GP_{n+1} + GP_n) = GQ_{n+1}$
ii. $2(GP_{n+1} - GP_n) = GQ_n$
iii. $GP_{n+1} + GP_{n-1} = GQ_n$
iv. $GP_{n+1} - GP_{n-1} = 2GP_n$
v. $GP_{n+2} + GP_{n-2} = 6GP_n$
vi. $GP_{n+2} - GP_{n-2} = 2GQ_n$
vii. $GQ_{n+1} + GQ_n = 4GP_{n+1}$
viii. $GQ_{n+1} - GQ_n = 4GP_n$
ix. $GQ_{n+1} + GQ_{n-1} = 4GP_{n+1}$
x. $GQ_{n+1} - GQ_{n-1} = 2GQ_n$
xi. $GQ_{n+2} + GQ_{n-2} = 6GQ_n$
xii. $GQ_{n+2} - GQ_{n-2} = 16GP_n$

Proof.

i. $$\begin{aligned} 2(GP_{n+1} + GP_n) &= 2(P_{n+1} + jP_n + j_1P_{n-1} + j_2P_{n-2} + P_n + jP_{n-1} + j_1P_{n-2} + j_2P_{n-3}) \\ &= 2(P_{n+1} + P_n) + 2j(P_n + P_{n-1}) + 2j_1(P_{n-1} + P_{n-2}) + 2j_2(P_{n-2} + P_{n-3}) \\ &= Q_{n+1} + jQ_n + j_1Q_{n-1} + j_2Q_{n-2} = GQ_{n+1}. \end{aligned}$$

The remaining equations can be proven by the same method.

Theorem 2.5. (Generating Function Formula) Let GP_n be the Gaussian-bihyperbolic Pell numbers. Generating function formula for these numbers is as follows

$$h(t)=\frac{(j-2j_1+5j_2)+t(1-2j+5j_1-12j_2)}{1-2t-t^2}.$$

Proof.

Let $h(t)$ be the generating function for Gaussian-bihyperbolic Pell numbers as $h(t)=\sum_{n=0}^{\infty} GP_n t^n$. Using $h(t)$, $2th(t)$ and $t^2h(t)$, we get the following equations, $th(t)=\sum_{n=0}^{\infty} GP_n t^{n+1}$, $t^2h(t)=\sum_{n=0}^{\infty} GP_n t^{n+2}$. After the needed calculations, the generating function for Gaussian-bihyperbolic Pell numbers is obtained as

$$h(t)=\frac{GP_0+GP_1t-2GP_0t}{1-2t-t^2}$$

$$h(t)=\frac{(j-2j_1+5j_2)+t(1-2j+5j_1-12j_2)}{1-2t-t^2}.$$

Similarly, the generating function formula for Gaussian-bihyperbolic Pell-Lucas numbers is obtained.

Theorem 2.6. (Binet's Formula) Let GP_n be the Gaussian-bihyperbolic Pell numbers. Binet's formula for this numbers is as follows

$$GP_n=\frac{\hat{\alpha}\alpha^{n-3}+\hat{\beta}\beta^{n-3}}{\alpha-\beta}$$

where $\hat{\alpha}=\alpha^3+j\alpha^2+j_1\alpha^1+j_2$, $\alpha=1+\sqrt{2}$ and $\hat{\beta}=\beta^3+j\beta^2+j_1\beta^1+j_2$, $\beta=1-\sqrt{2}$.

Proof.

$$\begin{aligned}GP_n&=P_n+jP_{n-1}+j_1P_{n-2}+j_2P_{n-3}\\&=\left(\frac{\alpha^n+\beta^n}{\alpha-\beta}\right)+j\left(\frac{\alpha^{n-1}+\beta^{n-1}}{\alpha-\beta}\right)+j_1\left(\frac{\alpha^{n-2}+\beta^{n-2}}{\alpha-\beta}\right)+j_2\left(\frac{\alpha^{n-3}+\beta^{n-3}}{\alpha-\beta}\right)\\&=\frac{\alpha^{n-3}(\alpha^3+j\alpha^2+j_1\alpha^1+j_2)}{\alpha-\beta}+\frac{\beta^{n-3}(\beta^3+j\beta^2+j_1\beta^1+j_2)}{\alpha-\beta}\end{aligned}$$

$$GP_n=\frac{\hat{\alpha}\alpha^{n-3}+\hat{\beta}\beta^{n-3}}{\alpha-\beta}.$$

Similarly, Binet's formula for Gaussian-bihyperbolic Pell-Lucas numbers is obtained.

Theorem 2.7. (d'Ocagne's Identity) Let GP_n be the Gaussian-bihyperbolic Pell numbers. d'Ocagne's identity for this numbers is as follows

$$\begin{aligned}&GP_mGP_{n+1}-GP_{m+1}GP_n\\&\quad=j[(-1)^n(P_{m-n-1}-P_{m-n+1})]+j_2(-1)^n[P_{m-n-3}-P_{m-n-1}+P_{m-n+1}-P_{m-n+3}].\end{aligned}$$

Proof.

$$\begin{aligned}GP_mGP_{n+1}-GP_{m+1}GP_n&=(P_m+jP_{m-1}+j_1P_{m-2}+j_2P_{m-3})(P_{n+1}+jP_n+j_1P_{n-1}+j_2P_{n-2})\\&\quad-(P_{m+1}+jP_m+j_1P_{m-1}+j_2P_{m-2})(P_n+jP_{n-1}+j_1P_{n-2}+j_2P_{n-3})\\&=j[(-1)^n(P_{m-n-1}-P_{m-n+1})]+j_2(-1)^n[P_{m-n-3}-P_{m-n-1}+P_{m-n+1}-P_{m-n+3}].\end{aligned}$$

Similarly, d'Ocagne's identity for Gaussian-bihyperbolic Pell-Lucas numbers is obtained.

Theorem 2.8. (Catalan's Identity) Let GP_n be the Gaussian-bihyperbolic Pell numbers. Catalan's identity for this numbers is as follows

$$GP_n^2-GP_{n+r}GP_{n-r}=2j(-1)^{n-r}P_r[P_{r-1}-P_{r+1}]+j_2(-1)^{n-r}P_r[P_{r-3}-P_{r-1}+P_{r+1}-P_{r+3}].$$

Proof.

$$\begin{aligned}GP_n^2-GP_{n+r}GP_{n-r}&=(P_n+jP_{n-1}+j_1P_{n-2}+j_2P_{n-3})(P_n+jP_{n-1}+j_1P_{n-2}+j_2P_{n-3})\\&\quad-(P+jP_{n+r-1}+j_1P_{n+r-2}+j_2P_{n+r-3})(P_{n-r}+jP_{n-r-1}+j_1P_{n-r-2}+j_2P_{n-r-3})\\&=2j(-1)^{n-r}P_r[P_{r-1}-P_{r+1}]+j_2(-1)^{n-r}P_r[P_{r-3}-P_{r-1}+P_{r+1}-P_{r+3}].\end{aligned}$$

Similarly, Catalan's identity for Gaussian-bihyperbolic Pell-Lucas numbers is obtained.

Theorem 2.9. (Cassini's Identity) Let GP_n be the Gaussian-bihyperbolic Pell numbers. Cassini's identity for this numbers is as follows

$$GP_n^2 - GP_{n+1}GP_{n-1} = -4j(-1)^{n-1} - 12j_2(-1)^{n-1}.$$

Proof. If $r = 1$ is taken in the Catalan's identity, Cassini's identity is obtained. Similarly, Cassini's identity for Gaussian-bihyperbolic Pell-Lucas numbers is obtained.

Theorem 2.10. Let GP_n be the Gaussian-bihyperbolic Pell numbers. In this case

i. $\sum_{k=1}^{n} GP_k = \left(\frac{Q_{n+1}-1}{2}\right) + j\left(\frac{Q_n-1}{2}\right) + j_1\left(\frac{Q_{n-1}+1}{2}\right) + j_2\left(\frac{Q_{n-2}-3}{2}\right)$

ii. $\sum_{k=1}^{n} GP_{2k-1} = \left(\frac{P_{2n}}{2}\right) + j\left(\frac{P_{2n-1}-1}{2}\right) + j_1\left(\frac{P_{2n-2}+2}{2}\right) + j_2\left(\frac{P_{2n-3}-5}{2}\right)$

iii. $\sum_{k=1}^{n} GP_{2k} = \left(\frac{P_{2n+1}-1}{2}\right) + j\left(\frac{P_{2n}}{2}\right) + j_1\left(\frac{P_{2n-1}-1}{2}\right) + j_2\left(\frac{P_{2n-2}+2}{2}\right).$

Proof.

i. $\sum_{k=1}^{n} GP_k = \sum_{k=1}^{n}(P_k + jP_{k-1} + j_1P_{k-2} + j_2P_{k-3})$

$\sum_{k=1}^{n} GP_k = \sum_{k=1}^{n} P_k + j\sum_{k=0}^{n-1} P_k + j_1\sum_{k=-1}^{n-2} P_k + j_2\sum_{k=-2}^{n-3} P_k$

$\sum_{k=1}^{n} GP_k = \left(\frac{Q_{n+1}-1}{2}\right) + j\left(\frac{Q_n-1}{2}\right) + j_1\left(\frac{Q_{n-1}+1}{2}\right) + j_2\left(\frac{Q_{n-2}-3}{2}\right).$

The remaining sums can be proven by the same method.

Theorem 2.11. Let GP_n be the Gaussian-bihyperbolic Pell numbers. For any positive integer n, we have the matrix representations of these sequences with both negative and positive indices are as follows

i. $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} GP_2 & GP_1 \\ GP_1 & GP_0 \end{bmatrix} = \begin{bmatrix} GP_{n+2} & GP_{n+1} \\ GP_{n+1} & GP_n \end{bmatrix}$

ii. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} GP_0 \\ GP_1 \end{bmatrix} = \begin{bmatrix} GP_n \\ GP_{n+1} \end{bmatrix}$

iii. $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} GP_2 & GP_1 \\ GP_1 & GP_0 \end{bmatrix} = \begin{bmatrix} GP_{-n+2} & GP_{-n+1} \\ GP_{-n+1} & GP_{-n} \end{bmatrix}$

iv. $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} GP_0 \\ GP_1 \end{bmatrix} = \begin{bmatrix} GP_{-n} \\ GP_{-n+1} \end{bmatrix}$

Proof.

i. For the prove, we utilize induction principle on n. The equality hold for $n = 1$. Now assume that the equality is true for $n > 1$. Then, we can verify for $n + 1$ as follows

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} GP_2 & GP_1 \\ GP_1 & GP_0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} GP_2 & GP_1 \\ GP_1 & GP_0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GP_{n+2} & GP_{n+1} \\ GP_{n+1} & GP_n \end{bmatrix} = \begin{bmatrix} GP_{n+3} & GP_{n+2} \\ GP_{n+2} & GP_{n+1} \end{bmatrix}.$$

Thus, the first step of the theorem can be proved easily. Similarly, the other steps of the proof are concluded by induction on n. Matrix representations for Gaussian-bihyperbolic Pell-Lucas numbers are obtained analogously.

3. Conclusion

This study presents the Gaussian-bihyperbolic Pell and Pell-Lucas numbers. We hope that these newly defined number sequences may be used in many areas such as quantum physics, applied mathematics, kinematic, differential equations and cryptology. Since this study includes some new results, it contributes to literature by providing essential information concerning these new numbers. Therefore, we hope that this new number system and its properties that we have found will offer a new perspective to the researchers.

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Conflicts of Interest

As the author of this study, we declare that we do not have any conflict of interest statement.

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