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Computing Eigenvalues of Sturm–Liouville Operators with a Family of Trigonometric Polynomial Potentials

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Abstract
We provide estimates for the periodic and antiperiodic eigenvalues of non-self-adjoint Sturm–Liouville operators with a family of complex-valued trigonometric polynomial potentials. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we give a numerical example with error analysis.

Keywords: *Eigenvalue estimations; periodic and antiperiodic boundary conditions; trigonometric polynomial potentials*

AMS Subject Classification (2020): *34L05; 34L15; 65L15.*

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1. Introduction

In the present paper, we consider the operators $T_s(v)$, for $s = 0, 1$, generated in $L_2[0, \pi]$ by the differential expression

$$-y''(x) + v(x)y(x) \tag{1.1}$$

and the boundary conditions

$$y(\pi) = e^{i\pi s}y(0), \qquad y'(\pi) = e^{i\pi s}y'(0), \tag{1.2}$$

which are periodic and antiperiodic boundary conditions, where v is the complex-valued trigonometric polynomial potential of the form

$$v(x) = v_{-1}e^{-i2x} + v_2 e^{i4x}, \qquad v_{-1}, v_2 \in \mathbb{C}. \tag{1.3}$$

Note that, the trigonometric polynomial potential (1.3) is a PT-symmetric potential if $v_{-1}, v_2 \in \mathbb{R}$. For the properties of the general PT-symmetric potentials, see [1–6] and references therein. Here, we only note that, the investigations of PT-symmetric periodic potentials were begun by Bender et al. [7].

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It is well known that the spectra of the operators $T_0(v)$ and $T_1(v)$ are discrete and for large enough n, there are two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counting with multiplicities) in the neighborhood of n^2. See the basic and detailed classical results in [8–11] and references therein.

The eigenvalues of the operators $T_0(0)$ and $T_1(0)$ are $(2n)^2$ and $(2n+1)^2$, for $n \in \mathbb{Z}$, respectively and all eigenvalues of $T_0(0)$ and $T_1(0)$, except 0, are double. The eigenvalues of $T_0(v)$ and $T_1(v)$ are called the periodic and antiperiodic eigenvalues and they are denoted by $\mu_n(v)$, for $n \in \mathbb{Z}$ and $\lambda_n(v)$, for $n \in \mathbb{Z} - \{0\}$, respectively.

It is well known that (see [10–12]), if v is real-valued, then all eigenvalues of the operator $T_s(v)$ are real, for all $s \in (-1, 1]$, and the spectrum $\sigma(T(v))$ of the Hill operator $T(v)$, generated in $L_2(-\infty, \infty)$ by expression (1.1) with the real-valued potential (1.3), consists of the real intervals

$$\Gamma_1 := [\mu_0(v), \lambda_{-1}(v)], \quad \Gamma_2 := [\lambda_{+1}(v), \mu_{-1}(v)], \quad \Gamma_3 := [\mu_{+1}(v), \lambda_{-2}(v)], \quad \Gamma_4 := [\lambda_{+2}(v), \mu_{-2}(v)], \ldots,$$

where $\mu_0(v)$, $\mu_{-n}(v)$, $\mu_{+n}(v)$, for $n = 1, 2, \ldots$ are the eigenvalues of $T_0(v)$ and $\lambda_{-n}(v)$, $\lambda_{+n}(v)$, for $n = 1, 2 \ldots$ are the eigenvalues of $T_1(v)$ and the following inequalities hold:

$$\mu_0(v) < \lambda_{-1}(v) \le \lambda_{+1}(v) < \mu_{-1}(v) \le \mu_{+1}(v) < \lambda_{-2}(v) \le \lambda_{+2}(v) < \mu_{-2}(v) \le \mu_{+2}(v) < \cdots .$$

The bands $\Gamma_1, \Gamma_2, \ldots$ of the spectrum $\sigma(T(v))$ of $T(v)$ are separated by the gaps

$$\Delta_1 := (\lambda_{-1}(v), \lambda_{+1}(v)), \quad \Delta_2 := (\mu_{-1}(v), \mu_{+1}(v)), \quad \Delta_3 := (\lambda_{-2}(v), \lambda_{+2}(v)), \ldots$$

if and only if the eigenvalues at the endpoints of the intervals are simple. In other notation, $\Gamma_n = \{\gamma_n(s) : s \in [0, 1]\}$, where $\gamma_1(s), \gamma_2(s), \ldots$ are the eigenvalues of $T_s(v)$, called as Bloch eigenvalues corresponding to the quasimomentum s. The Bloch eigenvalue $\gamma_n(s)$, continuously depends on s and $\gamma_n(-s) = \gamma_n(s)$.

Obviously, $\mu_{-n}(v)$ and $\mu_{+n}(v)$, for $n = 1, 2, \ldots$ are the $(2n)$th and $(2n+1)$th periodic eigenvalues; $\lambda_{-n}(v)$ and $\lambda_{+n}(v)$, for $n = 1, 2, \ldots$ are the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues, respectively.

If one of the numbers v_{-1} and v_2 is zero and the other is real in (1.3), then all eigenvalues of the operator $T_0(v)$, except 0, are double and they are equal to $(2n)^2$. This fact was proved for the first time in [13]. This case was investigated also in [14–16].

In this paper, we provide estimates for the eigenvalues of $T_0(v)$, when $v_{-1}, v_2 \in \mathbb{C}$. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we give a numerical example with error analysis using Rouche's theorem.

It is well known that [17]

$$|\mu_{\pm n}(v) - \mu_{\pm n}(0)| \le \sup_{x \in [0,\pi]} |v(x)| = M,$$
$$|\lambda_{\pm n}(v) - \lambda_{\pm n}(0)| \le \sup_{x \in [0,\pi]} |v(x)| = M,$$

for $n = 1, , 2 \ldots$, where

$$\mu_{\pm n}(0) = (2n)^2, \qquad \lambda_{\pm n}(0) = (2n-1)^2$$

and $M \le |v_{-1}| + |v_2| \le 2c$, $c = \max\{|v_{-1}|, |v_2|\}$. Moreover, for $n = 0$, $|\mu_0(v)| \le M$ holds. Therefore, we have

$$(2n)^2 - M \le |\mu_n| \le (2n)^2 + M \tag{1.4}$$

and

$$|\mu_n - (2k)^2| \ge |(2n)^2 - (2k)^2| - M = 4|n-k||n+k| - M \ge 4|2n-1| - M,$$

for $n \in \mathbb{Z}$ and $k \ne \pm n$. In particular, if $n = 1$, we have $|\mu_{\pm 1}| \le 4 + M$ and

$$|\mu_{\pm 1} - (2k)^2| \ge ||\mu_{\pm 1}| - (2k)^2| \ge 16 - |\mu_{\pm 1}| \ge 12 - M, \tag{1.5}$$

for $k \ge 2$. Besides, if $|n| \ge 2$, we have $|\mu_n| \ge |\mu_{-2}| \ge 16 - M$ and

$$|\mu_n - (2k)^2| \ge ||\mu_{-2}| - (2k)^2| \ge |\mu_{-2}| - 4 \ge 12 - M, \tag{1.6}$$

for $k \ne \pm n$. The analogous inequalities can be written for the antiperiodic eigenvalues from

$$(2n-1)^2 - M \le |\lambda_{\pm n}| \le (2n-1)^2 + M,$$

for $n = 1, 2, \ldots$.

2. Main results

We shall focus only on the operator $T_0(v)$ which is associated with the periodic boundary conditions. The investigation of $T_1(v)$ is similar. From now on, when we use the notation μ_n, we mean the $(2n)$th and $(2n+1)$th periodic eigenvalues $\mu_{-n}(v)$ and $\mu_{+n}(v)$, for $n = 1, 2, \ldots$. In order to obtain iteration formulas, we use the equations

$$(\mu_N - (2n)^2)(\Psi_N, e^{i2nx}) = (v\Psi_N, e^{i2nx}), \tag{2.1}$$

$$(\mu_N - (2n)^2)(\Psi_N, e^{-i2nx}) = (v\Psi_N, e^{-i2nx}), \tag{2.2}$$

which are obtained from

$$-\Psi_N''(x) + v(x)\Psi_N(x) = \mu_N \Psi_N(x),$$

by multiplying both sides of the equality by e^{i2nx} and e^{-i2nx}, respectively, where $\Psi_N(x)$ is the eigenfunction corresponding to the eigenvalue μ_N. Iterating equation (2.1) k times, the way it was done in the paper [18], we obtain

$$\Big(\mu_n - (2n)^2 - \sum_{j=1}^{k} a_j(\mu_n)\Big)(\Psi_n, e^{i2nx}) - \Big(v_{2n} + \sum_{j=1}^{k} b_j(\mu_n)\Big)(\Psi_n, e^{-i2nx}) = r_k(\mu_n), \tag{2.3}$$

where

$$a_j(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$

$$b_j(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{2n-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$

$$r_k(\mu_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{v_{n_1} v_{n_2} \cdots v_{n_k} v_{n_{k+1}} (v\Psi_n, e^{i2(n-n_1-\cdots-n_{k+1})x})}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-\cdots-n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = -1, 2$, $\sum_{i=1}^{l} n_i \neq 0, 2n$ for $l = 1, 2, ..., k+1$. Note that, for the trigonometric polynomial potential of the form (1.3), we have $v_i = 0$ for $i \neq -1, 2$.

Similarly, iterating equation (2.2) k times, we obtain

$$\Big(\mu_n - (2n)^2 - \sum_{j=1}^{k} a_j^*(\mu_n)\Big)(\Psi_n, e^{-i2nx}) - \Big(v_{-2n} + \sum_{j=1}^{k} b_j^*(\mu_n)\Big)(\Psi_n, e^{i2nx}) = r_k^*(\mu_n), \tag{2.4}$$

where

$$a_j^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$b_j^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-2n-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$r_k^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{v_{n_1} v_{n_2} \cdots v_{n_k} v_{n_{k+1}} (v\Psi_n, e^{-i2(n+n_1+\cdots+n_{k+1})x})}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = -1, 2$, $\sum_{i=1}^{l} n_i \neq 0, -2n$ for $l = 1, 2, ..., k+1$. Since the potential v is the trigonometric polynomial potential of the form (1.3), we have the followings, after some calculations:

$$a_{3j-1}^*(\mu_n) = a_{3j-1}(\mu_n), \qquad a_{3j-2}^*(\mu_n) = a_{3j-2}(\mu_n) = a_{3j}^*(\mu_n) = a_{3j}(\mu_n) = 0, \tag{2.5}$$

for $j = 1, 2, \ldots$. Now, in order to give the main results, we prove the following lemma. Without loss of generality, we assume that $\Psi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue μ_n.

Lemma 2.1. *The statements*

(a) $|p_n|^2 + |q_n|^2 > 0$, *where* $p_n = (\Psi_n, e^{i2nx})$ *and* $q_n = (\Psi_n, e^{-i2nx})$,

(b) $\lim_{k\to\infty} r_k(\mu_n) = 0, \qquad \lim_{k\to\infty} r_k^*(\mu_n) = 0$,

hold in the following cases:

(i) *if* $\max\{|v_{-1}|, |v_2|\} = c \leq 97/50$, *for* $n = 1$,

(ii) *if* $c < 2t - 1$, *for* $n \geq t$, $t = 2, 3, \ldots$.

Proof. **(a)** Assume the contrary, $p_n = 0$ and $q_n = 0$. Since the system of the root functions $\{e^{2ikx}/\sqrt{\pi} : k \in \mathbb{Z}\}$ of $T_0(0)$ forms an orthonormal basis for $L_2[0, \pi]$, we have the decomposition

$$\pi\Psi_n = p_n e^{i2nx} + q_n e^{-i2nx} + \sum_{k\in\mathbb{Z}, k\neq\pm n} \left(\Psi_n, e^{i2kx}\right) e^{i2kx}$$

for the normalized eigenfunction Ψ_n corresponding to the eigenvalue μ_n of $T_0(v)$. By Parseval's equality, we obtain

$$\sum_{k\in\mathbb{Z}, k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi.$$

On the other hand, by (1.4)-(1.6), we have

$$|\mu_1 - 16| \geq 12 - M, \qquad |\mu_1 - 36| \geq 32 - M, \qquad |\mu_1 - 64| \geq 60 - M, \tag{2.6}$$

$$16 - M \leq |\mu_2| \leq 16 + M, \qquad |\mu_2 - 4| \geq 12 - M, \qquad |\mu_2 - 36| \geq 20 - M. \tag{2.7}$$

First, we consider the case $n = 1$, namely, the case (i). Using the relations (2.1) and (2.6), the Bessel inequality, and taking

$$(v\Psi_1, 1) = v_{-1}(\Psi_1, e^{i2x}) + v_2(\Psi_1, e^{-i4x}) = v_2(\Psi_1, e^{-i4x}),$$
$$(v\Psi_1, e^{-i4x}) = v_{-1}(\Psi_1, e^{-i2x}) + v_2(\Psi_1, e^{-i8x}) = v_2(\Psi_1, e^{-i8x})$$

into account, we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}, k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 &= \frac{|(v\Psi_1, 1)|^2}{|\mu_1|^2} + \sum_{k\neq 0,\pm 1} \frac{|(v\Psi_1, e^{i2kx})|^2}{|\mu_1 - (2k)^2|^2} \\
&< \frac{|v_2|^4 |(v\Psi_1, e^{-i8x})|^2}{|\mu_1|^2 |\mu_1 - 16|^2 |\mu_1 - 64|^2} + \sum_{k\neq 0,\pm 1} \frac{|(v\Psi_1, e^{i2kx})|^2}{|\mu_1 - 16|^2} \\
&\leq \frac{c^4 \pi (2c)^2}{(4 - 2c)^2 (12 - 2c)^2 (60 - 2c)^2} + \frac{1}{(12 - 2c)^2} \sum_{k\neq 0,\pm 1} |(v\Psi_1, e^{i2kx})|^2 \\
&\leq \frac{4\pi (97/50)^6}{(3/25)^2 (203/25)^2 (1403/25)^2} + \frac{\pi (97/50)^2}{(203/25)^2} < \frac{13\pi}{100} < \pi,
\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z}, k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 = \pi$.

Now, we consider the case (ii), namely the case $c < 2t - 1$, for $n \geq t$, $t = 2, 3, \ldots$. Using

$$(2n)^2 - 2c \leq |\mu_n| \leq (2n)^2 + 2c,$$

we obtain

$$\begin{aligned}
|\mu_n - (2k)^2| \geq |\mu_n - (2(n-1))^2| &\geq (2n)^2 - 2c - (2(n-1))^2 \\
&= 4(2n - 1) - 2c > 4(2t - 1) - 2(2t - 1) = 4t - 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}, k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 &= \sum_{k\in\mathbb{Z}, k\neq\pm n} \frac{|(v\Psi_n, e^{i2kx})|^2}{|\mu_n - (2k)^2|^2} \\
&< \frac{1}{(4t - 2)^2} \sum_{k\in\mathbb{Z}, k\neq\pm n} |(v\Psi_n, e^{i2kx})|^2 \leq \frac{\pi (2c)^2}{(4t - 2)^2} < \pi,
\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq\pm n}|(\Psi_n,e^{i2kx})|^2=\pi$ and completes the proof of (a).

(b) By the definition of $r_k(\mu_n)$ and the conditions imposed on the summations, the number of each of the greatest summands

$$\frac{(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_1,e^{-i4x})}{\mu_1(\mu_1-16)^{k+1}(\mu_1-36)^{k-1}(\mu_1-64)^{k-1}}$$

and

$$\frac{(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_2,e^{-i2x})}{\mu_2^k(\mu_2-4)^{2k}}$$

of $r_{3k-1}(\mu_1)$ and $r_{3k-1}(\mu_2)$ in absolute value, is not greater than 4^k. Therefore, using (2.6), (2.7) and $M\leq|v_{-1}|+|v_2|\leq 2c$ and considering the greatest summands of $r_{3k-1}(\mu_n)$ in absolute value, we obtain for case (i)

$$\begin{aligned}|r_{3k-1}(\mu_1)|&<\frac{4^k|v_{-1}|^{2k-1}|v_2|^{k+1}M\sqrt{\pi}}{|\mu_1||\mu_1-16|^{k+1}|\mu_1-36|^{k-1}|\mu_1-64|^{k-1}}\leq\frac{4^kc^{2k-1}c^{k+1}2c\sqrt{\pi}}{(4-2c)(12-2c)^{k+1}(32-2c)^{k-1}(60-2c)^{k-1}}\\&\leq\frac{2\sqrt{\pi}4^k(97/50)^{3k+1}}{(3/25)(203/25)^{k+1}(703/25)^{k-1}(1403/25)^{k-1}}<6284\sqrt{\pi}(\frac{1}{438})^k,\qquad k\geq 2\end{aligned}$$

and for case (ii)

$$\begin{aligned}|r_{3k-1}(\mu_n)|\leq|r_{3k-1}(\mu_2)|&<\frac{4^k|v_{-1}|^{2k-1}|v_2|^{k+1}M\sqrt{\pi}}{|\mu_2|^k|\mu_2-4|^{2k}}\leq\frac{4^kc^{3k+1}2\sqrt{\pi}}{(16-2c)^k(12-2c)^{2k}}\\&<\frac{4^k3^{3k+1}2\sqrt{\pi}}{10^k6^{2k}}=6\sqrt{\pi}\frac{4^k27^k}{10^k36^k}=6\sqrt{\pi}(\frac{3}{10})^k.\end{aligned}$$

Thus, in any case $|r_{3k-1}(\mu_n)|<\alpha a^k$, for some constant $\alpha>0$ and $0<a<1$, which implies $\lim_{k\to\infty}r_k(\mu_n)=0$. Similarly, we prove that $\lim_{k\to\infty}r_k^*(\mu_n)=0$. □

Now, we consider the statements of Lemma 2.1 for the case $n=0$:

Lemma 2.2. *If* $\max\{|v_{-1}|,|v_2|\}=c\leq 36/25$, *for* $n=0$, *then the statements* ***(a)*** $|(\Psi_0,1)|>0$ *and* ***(b)*** $\lim_{k\to\infty}r_k(\mu_0)=0$ *are valid.*

Proof. **(a)** Assume the contrary $(\Psi_0,1)=0$. Isolating the terms $|(\Psi_0,e^{-i2x})|^2$ and $|(\Psi_0,e^{i2x})|^2$ in Parseval's equality, we can write

$$|(\Psi_0,e^{-i2x})|^2+|(\Psi_0,e^{i2x})|^2+\sum_{k\neq 0,\pm 1}|(\Psi_0,e^{i2kx})|^2=\pi.$$

First, we estimate $|(v\Psi_0,e^{-i2x})|^2+|(v\Psi_0,e^{i2x})|^2$. Using (2.1), the relations

$$|\mu_0-4|\geq 4-M,\qquad|\mu_0-16|\geq 16-M,\qquad|\mu_0-36|\geq 36-M,\tag{2.8}$$

and

$$\begin{aligned}(v\Psi_0,e^{-i2x})&=v_{-1}(\Psi_0,1)+v_2(\Psi_0,e^{-i6x})=v_2(\Psi_0,e^{-i6x}),\\(v\Psi_0,e^{i2x})&=v_{-1}(\Psi_0,e^{i4x})+v_2(\Psi_0,e^{-i2x}),\\(v\Psi_0,e^{i4x})&=v_{-1}(\Psi_0,e^{i6x})+v_2(\Psi_0,1)=v_{-1}(\Psi_0,e^{i6x}),\end{aligned}$$

we obtain

$$|(v\Psi_0,e^{-i2x})|\leq\frac{|v_2(v\Psi_0,e^{-i6x})|}{|\mu_0-36|}\leq\frac{2c^2\sqrt{\pi}}{(36-2c)}\leq\frac{2(36/25)^2\sqrt{\pi}}{(828/25)}<\frac{13\sqrt{\pi}}{100}$$

and

$$\begin{aligned}|(v\Psi_0,e^{i2x})|&\leq\frac{|(v_{-1})^2(q\Psi_0,e^{i6x})|}{|\mu_0-16||\mu_0-36|}+\frac{|(v_2)^2(q\Psi_0,e^{-i6x})|}{|\mu_0-4||\mu_0-36|}\\&\leq\frac{2c^3\sqrt{\pi}}{(16-2c)(36-2c)}+\frac{2c^3\sqrt{\pi}}{(4-2c)(36-2c)}\\&\leq\frac{2(36/25)^3\sqrt{\pi}}{(328/25)(828/25)}+\frac{2(36/25)^3\sqrt{\pi}}{(28/25)(828/25)}<\frac{3\sqrt{\pi}}{10},\end{aligned}$$

and hence,

$$|(v\Psi_0, e^{-i2x})|^2 + |(v\Psi_0, e^{i2x})|^2 < \pi/9. \tag{2.9}$$

Using (2.1), (2.8), (2.9) and the Bessel inequality, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 &= \frac{|(v\Psi_0, e^{-i2x})|^2}{|\mu_0 - 4|^2} + \frac{|(v\Psi_0, e^{i2x})|^2}{|\mu_0 - 4|^2} + \sum_{k\neq 0,\pm 1} \frac{|(v\Psi_0, e^{i2kx})|^2}{|\mu_0 - (2k)^2|^2} \\ &< \frac{\pi}{9(4-2c)^2} + \frac{1}{(16-2c)^2} \sum_{k\neq 0,\pm 1} |(v\Psi_0, e^{i2kx})|^2 \\ &\le \frac{\pi}{9(28/25)^2} + \frac{4(36/25)^2\pi}{(328/25)^2} < \frac{\pi}{11} + \frac{\pi}{20} = \frac{31\pi}{220} < \pi,\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 = \pi$ and completes the proof of (a).

(b) The number of the greatest summand

$$\frac{2(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_0, e^{-i6x})}{(\mu_0 - 4)^{k+1}(\mu_0 - 16)^{k-1}(\mu_0 - 36)^k}$$

of $r_{3k-1}(\mu_0)$ in absolute value, is not greater than 4^k. Hence, using (2.8) and $M \le |v_{-1}| + |v_2| \le 2c$, we obtain

$$\begin{aligned}|r_{3k-1}(\mu_0)| &< \frac{4^k 2|v_{-1}|^{2k-1}|v_2|^{k+1} M\sqrt{\pi}}{|\mu_0 - 4|^{k+1}|\mu_0 - 16|^{k-1}|\mu_0 - 36|^k} \le \frac{4^{k+1}c^{3k+1}\sqrt{\pi}}{(4-2c)^{k+1}(16-2c)^{k-1}(36-2c)^k} \\ &\le \frac{\sqrt{\pi}4^{k+1}(36/25)^{3k+1}}{(28/25)^{k+1}(328/25)^{k-1}(828/25)^k} < 68\sqrt{\pi}(\frac{1}{40})^k, \qquad k \ge 2,\end{aligned}$$

which implies $\lim_{k\to\infty} r_k(\mu_0) = 0$. □

Now, letting k tend to infinity in the equations (2.3) and (2.4), we obtain the following results. First, we consider the case $n \ge 2$.

Theorem 2.1. *If $\max\{|v_{-1}|, |v_2|\} = c < 2t - 1$, for $n \ge t$, $t = 2, 3, \ldots$, then μ is an eigenvalue of $T_0(v)$ if and only if it is a root of the equation*

$$\big(\mu - (2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu)\big)^2 - \sum_{j=1}^{\infty} b_j(\mu) \sum_{j=1}^{\infty} b_j^*(\mu) = 0 \tag{2.10}$$

lying inside the circle $C_n := \{\mu \in \mathbb{C} : |\mu - (2n)^2| = 2c\}$ and each of the series in equation (2.10) converges uniformly to an analytic function on the disk $D_n := \{\mu \in \mathbb{C} : |\mu - (2n)^2| \le 2c\}$.

Proof. **(a)** By Lemma 2.1, letting k tend to infinity in the equations (2.3) and (2.4), we obtain

$$\big(\mu_n - (2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu_n)\big)p_n = \big(v_{2n} + \sum_{j=1}^{\infty} b_j(\mu_n)\big)q_n, \tag{2.11}$$

$$\big(\mu_n - (2n)^2 - \sum_{j=1}^{\infty} a^*_{3j-1}(\mu_n)\big)q_n = \big(v_{-2n} \sum_{j=1}^{\infty} b^*_j(\mu_n)\big)p_n, \tag{2.12}$$

where $p_n = (\Psi_n, e^{i2nx})$ and $q_n = (\Psi_n, e^{-i2nx})$. If one of the numbers p_n and q_n is zero, then the proof is obvious. If they are both different from zero, multiplying these equations side by side and then cancelling the term $p_n q_n$, by (2.5), we have

$$\big(\mu_n - (2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu_n)\big)^2 - \big(v_{2n} + \sum_{j=1}^{\infty} b_j(\mu_n)\big)\big(v_{-2n} \sum_{j=1}^{\infty} b^*_j(\mu_n)\big) = 0. \tag{2.13}$$

Since $v_{2n} = v_{-2n} = 0$ for $n \ge 2$, the eigenvalue μ of $T_0(v)$ is a root of (2.10).

Now, we prove that the roots of (2.10) lying in the disk D_n are the eigenvalues of T_0. The equation $f(\mu) := (\mu-(2n)^2)^2 = 0$, has two roots in the disk D_n and

$$|f(\mu_n)| = |\mu_n-(2n)^2|^2 = 4c^2,$$

for all $\mu_n \in C_n$. Define the function

$$g(\mu) := \big(\mu-(2n)^2-\sum_{j=1}^{\infty} a_{3j-1}(\mu)\big)^2-\sum_{j=1}^{\infty} b_j(\mu)\sum_{j=1}^{\infty} b_j^*(\mu) = 0.$$

Estimating the summands of $|a_{3j-1}(\mu_n)|$, $|b_j(\mu_n)|$ and $|b_j^*(\mu_n)|$ for $n=2$, we obtain

$$|a_{3j-1}(\mu_2)| < \frac{2^j|v_{-1}|^{2j}|v_2|^j}{|\mu_2|^j|\mu_2-4|^{2j-1}}, \quad |b_{3j-2}(\mu_2)| < \frac{2^{j-1}|v_{-1}|^{2j-2}|v_2|^{j+1}}{|\mu_2|^j|\mu_2-4|^{2j-2}}, \quad |b_{3j}^*(\mu_2)| < \frac{|v_{-1}|^{2j+2}|v_2|^{j-1}}{|\mu_2|^j|\mu_2-4|^{2j}}, \tag{2.14}$$

for $j \geq 1$. Using the relations $|\mu_2| \geq 16-2c$ and $|\mu_2-4| \geq 12-2c$, it follows by the geometric series formula that

$$\begin{aligned}\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)| \leq \sum_{j=1}^{\infty}|a_{3j-1}(\mu_2)| &< \frac{2c^3}{(16-2c)(12-2c)}+\frac{2^2c^6}{(16-2c)^2(12-2c)^3}+\frac{2^3c^9}{(16-2c)^3(12-2c)^5}+\cdots\\ &= \frac{2c^3}{(16-2c)(12-2c)}(1+\frac{2c^3}{(16-2c)(12-2c)^2}+\frac{2^2c^6}{(16-2c)^2(12-2c)^4}+\cdots)\\ &= \frac{2c^3}{(16-2c)(12-2c)}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{2c^3(12-2c)}{(16-2c)(12-2c)^2-2c^3} < \frac{12.3^3}{360-54} = \frac{18}{17},\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{\infty}|b_j(\mu_n)| \leq \sum_{j=1}^{\infty}|b_{3j-2}(\mu_2)| &< \frac{c^2}{(16-2c)}+\frac{2c^5}{(16-2c)^2(12-2c)^2}+\frac{2^2c^8}{(16-2c)^3(12-2c)^4}+\cdots\\ &= \frac{c^2}{(16-2c)}(1+\frac{2c^3}{(16-2c)(12-2c)^2}+\frac{2^2c^6}{(16-2c)^2(12-2c)^4}+\cdots)\\ &= \frac{c^2}{(16-2c)}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{c^2(12-2c)^2}{(16-2c)(12-2c)^2-2c^3} < \frac{18}{17},\end{aligned}$$

and that

$$\begin{aligned}\sum_{j=1}^{\infty}|b_j^*(\mu_n)| \leq \sum_{j=1}^{\infty}|b_{3j}^*(\mu_2)| &< \frac{c^4}{(16-2c)(12-2c)^2}+\frac{c^7}{(16-2c)^2(12-2c)^4}+\frac{c^{10}}{(16-2c)^3(12-2c)^6}+\cdots\\ &= \frac{c^4}{(16-2c)(12-2c)^2}(1+\frac{c^3}{(16-2c)(12-2c)^2}+\frac{c^6}{(16-2c)^2(12-2c)^4}+\cdots)\\ &= \frac{c^4}{(16-2c)(12-2c)^2}\frac{1}{1-\frac{c^3}{(16-2c)(12-2c)^2}} = \frac{c^4}{(16-2c)(12-2c)^2-c^3} < \frac{3^4}{360-27} = \frac{9}{37}.\end{aligned}$$

Hence,

$$\begin{aligned}|g(\mu_n)-f(\mu_n)| &\leq 2|\mu_n-(2n)^2|\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)|+\big(\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)|\big)^2+\sum_{j=1}^{\infty}|b_j(\mu_n)|\sum_{j=1}^{\infty}|b_j^*(\mu_n)|\\ &< \frac{8c^4(12-2c)}{(16-2c)(12-2c)^2-2c^3}+\big(\frac{2c^3(12-2c)}{(16-2c)(12-2c)^2-2c^3}\big)^2\\ &+\frac{c^2(12-2c)^2}{(16-2c)(12-2c)^2-2c^3}\frac{c^4}{(16-2c)(12-2c)^2-c^3} < 4c^2.\end{aligned}$$

Therefore, $|g(\mu)-f(\mu)| < |f(\mu)|$ holds for all $\mu \in C_n$. By Rouche's theorem, $g(\mu)$ has two roots inside the circle C_n. Hence, T_0 has two eigenvalues (counting with multiplicities) lying inside C_n, which are the roots of (2.10). On the other hand, equation (2.10) has exactly two roots (counting with multiplicities) inside C_n. Thus, $\mu \in C_n$ is an eigenvalue of T_0 if and only if, it is a root of (2.10) and the roots of (2.10) coincide with the eigenvalues μ_{-n} and μ_{+n} of T_0.

Now, in order to estimate $\sum\limits_{j=1}^{\infty}|a'_{3j-1}(\mu_n)|$, $\sum\limits_{j=1}^{\infty}|b'_j(\mu_n)|$ and $\sum\limits_{j=1}^{\infty}|b_j^{*\prime}(\mu_n)|$, first we estimate the summands $|a'_{3j-1}(\mu_2)|$, $|b'_j(\mu_2)|$ and $|b_j^{*\prime}(\mu_2)|$, by differentiating $a_{3j-1}(\mu_2)$, $b_j(\mu_2)$ and $b_j^*(\mu_2)$ with respect to μ_2:

$$|\frac{d(a_{3j-1}(\mu_2))}{d\mu_2}|< \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{|\mu_2|^j|\mu_2-4|^{2j}}, \qquad |\frac{d(b_{3j}^*(\mu_2))}{d\mu_2}|< \frac{3^{j+1}|v_{-1}|^{2j+2}|v_2|^{j-1}}{2^j|\mu_2|^j|\mu_2-4|^{2j+1}},, \qquad j\geq 1$$
$$|\frac{d(b_1(\mu_2))}{d\mu_2}|< \frac{|v_2|^2}{|\mu_2|^2}, \qquad |\frac{d(b_{3j-2}(\mu_2))}{d\mu_2}|< \frac{2^{j+1}|v_{-1}|^{2j-2}|v_2|^{j+1}}{|\mu_2|^j|\mu_2-4|^{2j-1}}, \qquad j\geq 2,$$

and hence, we have

$$\begin{aligned}\sum_{j=1}^{\infty}|a'_{3j-1}(\mu_n)| &\leq \sum_{j=1}^{\infty}|a'_{3j-1}(\mu_2)| < \frac{2^2c^3}{(16-2c)(12-2c)^2}+\frac{2^3c^6}{(16-2c)^2(12-2c)^4}+\frac{2^4c^9}{(16-2c)^3(12-2c)^6}+\cdots\\ &= \frac{2^2c^3}{(16-2c)(12-2c)^2}(1+\frac{2c^3}{(16-2c)(12-2c)^2}+\frac{2^2c^6}{(16-2c)^2(12-2c)^4}+\cdots)\\ &= \frac{4c^3}{(16-2c)(12-2c)^2}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{4c^3}{(16-2c)(12-2c)^2-2c^3} < \frac{4.3^3}{360-54}=\frac{6}{17},\end{aligned}$$

and

$$\sum_{j=1}^{\infty}|b'_j(\mu_n)| \leq \sum_{j=1}^{\infty}|b'_{3j-2}(\mu_2)| < \frac{c^2}{(16-2c)^2}+\frac{8c^5}{(16-2c)(12-2c)[(16-2c)(12-2c)^2-2c^3]} < \frac{7}{37},$$
$$\sum_{j=1}^{\infty}|b_j^{*\prime}(\mu_n)| \leq \sum_{j=1}^{\infty}|b_{3j}^{*\prime}(\mu_2)| < \frac{9c^4}{(12-2c)[2(16-2c)(12-2c)^2-3c^3]} < \frac{27}{142}.$$

Therefore, each of the series $\sum\limits_{j=1}^{\infty}a_{3j-1}(\mu_n)$, $\sum\limits_{j=1}^{\infty}b_j(\mu_n)$ and $\sum\limits_{j=1}^{\infty}b_j^*(\mu_n)$, converges uniformly to an analytic function on the disk D_n. □

Now, to estimate the periodic eigenvalues μ_{-1} and μ_1, we consider the case $n=1$. In this case, substituting $b_{3j-1}(\mu_1)=0$, $b_{3j-2}(\mu_1)=0$, for $j\geq 1$, and $\sum\limits_{j=1}^{\infty}b_j^*(\mu_1)=b_1^*(\mu_1)=(v_{-1})^2/\mu_1$, in (2.3) and (2.4) as $k\to\infty$, by Lemma 2.1, we obtain

$$\big(\mu_1-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu_1)\big)^2-\frac{(v_{-1})^2}{\mu_1}\big(v_2-\sum_{j=1}^{\infty}b_{3j}(\mu_1)\big)=0. \tag{2.15}$$

Therefore, we have the following results.

Theorem 2.2. *If* $\max\{|v_{-1}|,|v_2|\}=c\leq 97/50$, *for* $n=1$, *then* μ *is an eigenvalue of* $T_0(v)$ *if and only if it is a root of the equation*

$$\big(\mu-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu)\big)^2-\frac{(v_{-1})^2v_2}{\mu}-\frac{(v_{-1})^2}{\mu}\sum_{j=1}^{\infty}b_{3j}(\mu)=0 \tag{2.16}$$

lying inside the circle $C_1:=\{\mu\in\mathbb{C}:|\mu|=4+2c\}$ *and each of the series in equation* (2.16) *converges uniformly to an analytic function on the disk* $D_1:=\{\mu\in\mathbb{C}:|\mu|\leq 4+2c\}$.

Proof. **(a)** Equation (2.16) follows from (2.15). Let $F(\mu):=(\mu-4)^2=0$ and

$$G(\mu):=\big(\mu-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu)\big)^2-\frac{(v_{-1})^2v_2}{\mu}-\frac{(v_{-1})^2}{\mu}\sum_{j=1}^{\infty}b_{3j}(\mu)=0.$$

Then, $|F(\mu_1)| = |\mu_1 - 4|^2 \geq (|\mu_1| - 4)^2 = 4c^2$, for all $\mu_1 \in C_1$. Using the estimations

$$\sum_{j=1}^{\infty} |a_{3j-1}(\mu_1)| < \sum_{j=1}^{\infty} \frac{(3/2)^j |v_{-1}|^{2j} |v_2|^j}{|\mu_1||\mu_1 - 16|^j |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}} \tag{2.17}$$

$$< \frac{3c^3(32-2c)(60-2c)}{(4+2c)[2(12-2c)(32-2c)(60-2c) - 3c^3]} < \frac{9}{50},$$

$$\sum_{j=1}^{\infty} |b_{3j}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j-1} |v_{-1}|^{2j} |v_2|^{j+1}}{|\mu_1||\mu_1 - 16|^{j+1} |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}} \tag{2.18}$$

$$< \frac{c^4(32-2c)(60-2c)}{(4+2c)(12-2c)[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{7}{250},$$

and

$$\sum_{j=1}^{\infty} |a'_{3j-1}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j+1} |v_{-1}|^{2j} |v_2|^j}{|\mu_1|^2 |\mu_1 - 16|^j |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}}$$

$$< \frac{4c^3(32-2c)(60-2c)}{(4+2c)^2[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{3}{50},$$

$$\sum_{j=1}^{\infty} |b'_{3j}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j+1} |v_{-1}|^{2j} |v_2|^{j+1}}{|\mu_1|^2 |\mu_1 - 16|^{j+1} |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}}$$

$$< \frac{4c^4(32-2c)(60-2c)}{(4+2c)^2(12-2c)[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{7}{500},$$

for all $\mu_1 \in C_1$, and arguing as in the proof of Theorem 2.1, by Rouche's theorem, we complete the proof. □

Finally, in order to estimate the first periodic eigenvalue μ_0, we consider the case $n = 0$. By Lemma 2.2, we have:

Theorem 2.3. *If* $\max\{|v_{-1}|, |v_2|\} = c \leq 36/25$, *for* $n = 0$, *then* μ *is an eigenvalue of* $T_0(v)$ *if and only if it is the root of the equation*

$$\mu - \frac{(v_{-1})^2 v_2}{(\mu - 4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu - 4)(\mu - 16)} - \sum_{j=2}^{\infty} a_{3j-1}(\mu) = 0 \tag{2.19}$$

lying inside the circle $C_0 := \{\mu \in \mathbb{C} : |\mu| = 2c\}$ *and the series in equation (2.19) converges uniformly to an analytic function on the disk* $D_0 := \{\mu \in \mathbb{C} : |\mu| \leq 2c\}$.

Proof. **(a)**Iterating $\mu_N(\Psi_N, 1) = (v\Psi_N, 1)$, for $N = 0$, k times, by isolating the terms containing $(\Psi_0, 1)$ gives

$$\Big(\mu_0 - \sum_{j=1}^{k} a_j(\mu_0)\Big)(\Psi_0, 1) = r_k(\mu_0). \tag{2.20}$$

Letting k tend to infinity in (2.20), by Lemma 2.2 and (2.5), we obtain (2.19). Let

$$H(\mu) := \mu - \frac{(v_{-1})^2 v_2}{(\mu_0 - 4)^2} = 0$$

and

$$K(\mu) := \mu - \sum_{j=1}^{\infty} a_{3j-1}(\mu) = \mu - \frac{(v_{-1})^2 v_2}{(\mu - 4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu - 4)(\mu - 16)} - \sum_{j=2}^{\infty} a_{3j-1}(\mu) = 0.$$

Then,

$$|H(\mu_0)| \geq \Big||\mu_0| - \frac{|v_{-1}|^2 |v_2|}{|\mu_0 - 4|^2}\Big| \geq 2c - \frac{c^3}{(4-2c)^2},$$

for all $\mu_0 \in C_0$. Using the estimations

$$\begin{aligned}
|K(\mu_0) - H(\mu_0)| &\le \frac{2|v_{-1}|^2|v_2|}{|\mu_0-4||\mu_0-16|} + \sum_{j=2}^{\infty} |a_{3j-1}(\mu_0)| \\
&< \frac{2|v_{-1}|^2|v_2|}{|\mu_0-4||\mu_0-16|} + \sum_{j=2}^{\infty} \frac{2^j |v_{-1}|^{2j}|v_2|^j}{|\mu_0-4|^{j+1}|\mu_0-16|^{j-1}|\mu_0-36|^{j-1}} \\
&< \frac{2|v_{-1}|^2|v_2|}{(4-2c)(16-2c)} + \sum_{j=2}^{\infty} \frac{2^j |v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+1}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{2c^3}{(4-2c)(16-2c)} + \frac{4c^6}{(4-2c)^2[(4-2c)(16-2c)(36-2c)-2c^3]} < \frac{47}{100}
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
\sum_{j=2}^{\infty} |a'_{3j-1}(\mu_0)| &< \sum_{j=2}^{\infty} \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{|\mu_0-4|^{j+2}|\mu_0-16|^{j-1}|\mu_0-36|^{j-1}} \\
&< \sum_{j=2}^{\infty} \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+2}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{8c^6}{(4-2c)^3[(4-2c)(16-2c)(36-2c)-2c^3]} < \frac{11}{100}
\end{aligned}$$

and arguing as in the proof of Theorem 2.1, by Rouche's theorem, we complete the proof. □

In order to estimate eigenvalues numerically, we take finite summations instead of the infinite series in the equations (2.10), (2.16) and (2.19). When we say the $(3k)$th approximations, we mean the equations containing $\sum_{j=1}^{k} a_{3j-1}(\mu)$, $\sum_{j=1}^{3k} b_j(\mu)$ and $\sum_{j=1}^{3k} b_j^*(\mu)$ instead of $\sum_{j=1}^{\infty} a_{3j-1}(\mu)$, $\sum_{j=1}^{\infty} b_j(\mu)$ and $\sum_{j=1}^{\infty} b_j^*(\mu)$. For instance, in the cases $n=0$, $n=1$ and $n=2$, the $(3k)$th approximations of (2.19), (2.16) and (2.10) are

$$\mu - \frac{(v_{-1})^2 v_2}{(\mu-4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu-4)(\mu-16)} - \sum_{j=2}^{k} a_{3j-1}(\mu) = 0, \tag{2.22}$$

$$\big(\mu - 4 - \sum_{j=1}^{k} a_{3j-1}(\mu)\big)^2 - \frac{(v_{-1})^2 v_2}{\mu} - \frac{(v_{-1})^2}{\mu}\sum_{j=1}^{k} b_{3j}(\mu) = 0, \tag{2.23}$$

and

$$\big(\mu - 16 - \sum_{j=1}^{k} a_{3j-1}(\mu)\big)^2 - \sum_{j=1}^{3k} b_j(\mu) \sum_{j=1}^{3k} b_j^*(\mu) = 0,$$

respectively. Then, by (2.14), (2.17), (2.18) and (2.21), we have the following estimations for the remaining terms of the series in these equations:

$$\begin{aligned}
\Big|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_0)\Big| &\le \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_0)| < \sum_{j=k+1}^{\infty} \frac{2^j |v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+1}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{2^{k+1}c^{3k+3}}{(4-2c)^{k+1}(16-2c)^{k-1}(36-2c)^{k-1}[(4-2c)(16-2c)(36-2c)-2c^3]} < 2.41(\frac{1}{162})^k,
\end{aligned}$$

for $n = 0$;

$$\Big|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_1)\Big| \leq \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_1)| < \sum_{j=k+1}^{\infty} \frac{(3/2)^j |v_{-1}|^{2j} |v_2|^j}{(4-2c)(12-2c)^j(32-2c)^{j-1}(60-2c)^{j-1}}$$
$$< \frac{2(3/2)^{k+1} c^{3k+3}}{(4-2c)(12-2c)^k(32-2c)^{k-1}(60-2c)^{k-1}[2(12-2c)(32-2c)(60-2c)-3c^3]} < 11.25(\frac{1}{1170})^k,$$
$$\Big|\sum_{j=k+1}^{\infty} b_{3j}(\mu_1)\Big| \leq \sum_{j=k+1}^{\infty} |b_{3j}(\mu_1)| < \sum_{j=k+1}^{\infty} \frac{2^{j-1} |v_{-1}|^{2j} |v_2|^{j+1}}{(4-2c)(12-2c)^{j+1}(32-2c)^{j-1}(60-2c)^{j-1}}$$
$$< \frac{2^k c^{3k+4}}{(4-2c)(12-2c)^{k+1}(32-2c)^{k-1}(60-2c)^{k-1}[(12-2c)(32-2c)(60-2c)-2c^3]} < 1.8(\frac{1}{877})^k,$$

for $n = 1$; and

$$\Big|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_2)\Big| \leq \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{2^j |v_{-1}|^{2j} |v_2|^j}{(16-2c)^j(12-2c)^{2j-1}}$$
$$< \frac{2^{k+1} c^{3k+3}}{(16-2c)^k(12-2c)^{2k-1}[(16-2c)(12-2c)^2-2c^3]} < \frac{18}{17}(\frac{3}{20})^k,$$
$$\Big|\sum_{j=k+1}^{\infty} b_{3j-2}(\mu_2)\Big| \leq \sum_{j=k+1}^{\infty} |b_{3j-2}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{2^{j-1} |v_{-1}|^{2j-2} |v_2|^{j+1}}{(16-2c)^j(12-2c)^{2j-2}}$$
$$< \frac{2^k c^{3k+2}}{(16-2c)^k(12-2c)^{2k-2}[(16-2c)(12-2c)^2-2c^3]} < \frac{18}{17}(\frac{3}{20})^k,$$
$$\Big|\sum_{j=k+1}^{\infty} b^*_{3j}(\mu_2)\Big| \leq \sum_{j=k+1}^{\infty} |b^*_{3j}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{|v_{-1}|^{2j+2} |v_2|^{j-1}}{(16-2c)^j(12-2c)^{2j}}$$
$$< \frac{c^{3k+4}}{(16-2c)^k(12-2c)^{2k}[(16-2c)(12-2c)^2-c^3]} < \frac{9}{37}(\frac{3}{40})^k,$$

for $n = 2$. Obviously, we have better approximations as k grows.

Now, we approach the periodic eigenvalues by the roots of the polynomials derived from the $(3k)$th approximations (2.22) and (2.23), the way it was done in [19]. For example, for $n = 0$ and $n = 1$, the sixth approximations are

$$Q_0(\mu) := \mu - \frac{(v_{-1})^2 v_2}{(\mu-4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu-4)(\mu-16)} - \frac{2(v_{-1})^4 (v_2)^2}{(\mu-4)^3(\mu-16)(\mu-36)}$$
$$- \frac{2(v_{-1})^4 (v_2)^2}{(\mu-4)^2(\mu-16)^2(\mu-36)} - \frac{2(v_{-1})^4 (v_2)^2}{(\mu-4)(\mu-16)^2(\mu-36)(\mu-64)} = 0,$$

and

$$Q_1(\mu) := \Big(\mu - 4 - \frac{(v_{-1})^2 v_2}{\mu(\mu-16)} - \frac{(v_{-1})^2 v_2}{(\mu-16)(\mu-36)} - \frac{(v_{-1})^4 (v_2)^2}{\mu(\mu-16)^2(\mu-36)(\mu-64)}$$
$$- \frac{(v_{-1})^4 (v_2)^2}{(\mu-16)^2(\mu-36)^2(\mu-64)} - \frac{(v_{-1})^4 (v_2)^2}{(\mu-16)(\mu-36)^2(\mu-64)(\mu-100)}\Big)^2$$
$$- \frac{(v_{-1})^2 v_2}{\mu} - \frac{(v_{-1})^4 (v_2)^2}{\mu^2(\mu-16)^2} - \frac{2(v_{-1})^6 (v_2)^3}{\mu^2(\mu-16)^3(\mu-36)(\mu-64)} = 0,$$

respectively. Then, the corresponding polynomials are

$$P_0(\mu) := (\mu-4)^3(\mu-16)^2(\mu-36)(\mu-64) Q_0(\mu), \tag{2.24}$$

and

$$P_1(\mu) := \mu^2(\mu-16)^4(\mu-36)^4(\mu-64)^2(\mu-100)^2 Q_1(\mu), \tag{2.25}$$

respectively. By the same token, we can derive polynomials to approximate the other periodic eigenvalues, as well. Now, we present a numerical example.

Example 2.1. Consider the potential $v(x) = e^{i4x} + e^{-i2x}$. In this case, $v_{-1} = v_2 = 1$, and we have the following approximations for the first periodic eigenvalues μ_0, μ_{-1} and μ_{-1}:

First, we show that μ_0 is the eigenvalue lying inside the circle

$$c_0 := \{\mu \in \mathbb{C} : |\mu - 0.0978293068037| = 8.8 \times 10^{-8}\}.$$

The root of the polynomial $P_0(\mu)$ defined by (2.24), lying in the disk $D_0 = \{\mu \in \mathbb{C} : |\mu| \le 2\}$, is

$$z_1 = 0.0978293068037.$$

The other roots of $P_0(\mu)$ are

$$z_2 = 3.43962569257, \quad z_3 = 3.99479646224, \quad z_4 = 4.45733974252,$$
$$z_5 = (16.0052043612 - 0.00233054592651i), \quad z_6 = (16.0052043612 + 0.00233054592651i),$$
$$z_7 = 36.0000000654 and z_8 = 64.0000000081.$$

Using the decomposition

$$Q_0(\mu) = \frac{(\mu - z_1)(\mu - z_2)\cdots(\mu - z_8)}{(\mu - 4)^3(\mu - 16)^2(\mu - 36)(\mu - 64)},$$

we obtain by direct calculation $|Q_0(\mu)| > 7.0297 \times 10^{-8}$, for all $\mu \in c_0$. On the other hand, again by direct calculations, we have

$$|K(\mu) - Q_0(\mu)| \le \sum_{j=3}^{\infty} |a_{3j-1}(\mu)| < 6.8948 \times 10^{-8},$$

for all $\mu \in c_0$. Therefore, by Rouche's theorem, equation (2.19) has only one root inside the circle c_0. Thus, using Theorem 2.3, we conclude that μ_0 is the eigenvalue lying inside the circle c_0.

Now, we show that μ_{-1} and μ_1 are the complex eigenvalues lying inside the circles

$$c_{-1} := \{\mu \in \mathbb{C} : |\mu - (3.9817022865 - 0.00000193582494331i)| = 2.4 \times 10^{-11}\}.$$

and

$$c_1 := \{\mu \in \mathbb{C} : |\mu - (3.9817022865 + 0.00000193582494331i)| = 2.4 \times 10^{-11}\}.$$

respectively. The roots of the polynomial $P_1(\mu)$ defined by (2.25), lying in the disk $D_1 = \{\mu \in \mathbb{C} : |\mu| \le 6\}$ are $x_1 = (3.9817022865 - 0.00000193582494331i)$, $x_2 = (3.9817022865 + 0.00000193582494331i)$ and $x_{3,4} = (0.0156946762466 \pm 0.0000000103890273399i)$. The other roots of $P_1(\mu)$ are

$$x_{5,6} = (15.6169913038 \pm 0.443784473665i), \quad x_{7,8} = (16.1045835081 \pm 0.60324291928i),$$
$$x_9 = 15.4238418891, \quad x_{10,11} = (16.5675433687 \pm 0.291363414949i), \quad x_{12} = 35.6520094578,$$
$$x_{13,14} = (35.8640598326 \pm 0.342968435657i), \quad x_{15,16} = (36.3114986519 \pm 0.241842187319i),$$
$$x_{17,18} = (63.9746729986 \pm 0.044045369207i), \quad x_{19} = 64.0506554041,$$
$$x_{20} = 99.9999172995, \quad x_{21} = 100.000082697.$$

Using the decomposition

$$Q_1(\mu) = \frac{(\mu - x_1)(\mu - x_2)\cdots(\mu - x_{21})}{\mu^2(\mu - 16)^4(\mu - 36)^4(\mu - 64)^2(\mu - 100)^2},$$

by direct calculations, we obtain $|Q_1(\mu)| > 1.0992 \times 10^{-11}$, for all $\mu \in c_{-1}$ and $|Q_1(\mu)| > 1.0992 \times 10^{-11}$, for all $\mu \in c_1$. On the other hand, one can easily calculate that

$$|G(\mu) - Q_1(\mu)| \le 2(|\mu - 4| + |a_2(\mu)| + |a_5(\mu)|)\sum_{j=3}^{\infty} |a_{3j-1}(\mu)| + \Big(\sum_{j=3}^{\infty} |a_{3j-1}(\mu)|\Big)^2 + \sum_{j=3}^{\infty} \frac{|b_{3j}(\mu)|}{|\mu|} < 4.7184 \times 10^{-12},$$

for all $\mu \in c_{-1} \cup c_1$. The proof follows from Rouche's theorem and Theorem 2.2; equation (2.16) has one root inside each of the circles c_{-1} and c_1 and μ_{-1} and μ_{+1} are the complex eigenvalues lying inside c_{-1} and c_1, respectively.

3. Conclusion

In this paper, we have given estimates for the periodic eigenvalues, when $v_{-1}, v_2 \in \mathbb{C}$. We have even approximated complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we have given a numerical example with error analysis using Rouche's theorem. In this paper, we have given a practical way to calculate the eigenvalues of the operator $T_0(v)$. The method used in this paper can be extended to compute the periodic eigenvalues of the Hill operator for different classes of potentials.

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References

[1] Bagarello, F., Gazeau, J.-P., Szafraniec, F., Znojil, M. (Eds.): Non-selfadjoint operators in quantum physics: Mathematical aspects. John Wiley & Sons (2015).

[2] Bender, C. M.: *PT-symmetric potentials having continuous spectra.* Journal of Physics A-Mathematical and Theoretical. **53** (37), 375302 (2020).

[3] Mostafazadeh, A.: *Psevdo-hermitian representation of quantum mechanics.* International Journal of Geometric Methods in Modern Physics. **11**, 1191-1306 (2010).

[4] Veliev, O. A.: *On the spectral properties of the Schrodinger operator with a periodic PT-symmetric potential.* International Journal of Geometric Methods in Modern Physics. **14**, 1750065 (2017).

[5] Veliev, O. A.: *On the finite-zone periodic PT-symmetric potentials.* Moscow Mathematical Journal. **19** (4), 807-816 (2019).

[6] Veliev, O. A.: Non-self-adjoint Schrödinger operator with a periodic potential. Springer, Cham (2021).

[7] Bender, C. M., Dunne, G. V., Meisinger, P. N.: *Complex periodic potentials with real band spectra.* Physics Letters A. **252**, 272-276 (1999).

[8] Brown, B.M., Eastham, M. S. P., Schmidt, K. M.: Periodic differential operators, Operator Theory: Advances and Applications, 230, Birkhuser/Springer: Basel AG, Basel (2013).

[9] Levy, M., Keller, B.: *Instability intervals of Hill's equation.* Communications on Pure and Applied Mathematics. **16**, 469-476 (1963).

[10] Magnus, W., Winkler, S.: Hill's equation. Interscience Publishers, New York (1966).

[11] Marchenko, V.: Sturm-Liouville operators and applications. Birkhauser Verlag, Basel (1986).

[12] Eastham, M. S. P.: The spectral theory of periodic differential operators. Hafner. New York (1974).

[13] Gasymov, M. G.: *Spectral analysis of a class of second-order nonself-adjoint differential operators.* Fankts. Anal. Prilozhen. **14**, 14-19 (1980).

[14] Kerimov, N. B.: *On a boundary value problem of N. I. Ionkin type.* Differential Equations. **49**, 1233-1245 (2013).

[15] Nur, C.: *On the estimates of periodic eigenvalues of Sturm-Liouville operators with trigonometric polynomial potentials.* Mathematical Notes. **109** (5), 794-807 (2021).

[16] Veliev, O. A.: *Spectral problems of a class of non-self-adjoint one-dimensional Schrodinger operators.* Journal of Mathematical Analysis and Applications. **422**, 1390-1401 (2015).

[17] Pöschel, J., Trubowitz, E.: Inverse Spectral Theory. Academic Press, Boston, Mass, USA (1987).

[18] Dernek, N., Veliev, O. A.: *On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operators.* Israel Journal of Mathematics. **145**, 113-123 (2005).

[19] Veliev, O. A.: *The spectrum of the Hamiltonian with a PT-symmetric periodic optical potential.* International Journal of Geometric Methods in Modern Physics. **15**, 1850008 (2018).

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