J. Algebra Comb. Discrete Appl.
9(2) • 15–29

Received: 20 September 2021
Accepted: 16 February 2022

Journal of Algebra Combinatorics Discrete Structures and Applications

A note on $GDD(1, n, n, 4; \lambda_1, \lambda_2)$

Research Article

Dinesh G. Sarvate*, **Dinkayehu M. Woldemariam****

Abstract: The present note is motivated by two papers on group divisible designs (GDDs) with the same block size three but different number of groups: three and four where one group is of size 1 and the others are of the same size n. Here we present some interesting constructions of GDDs with block size 4 and three groups: one of size 1 and other two of the same size n. We also obtain necessary conditions for the existence of such GDDs and prove that they are sufficient in several cases. For example, we show that the necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.

2010 MSC: 05B05, 05B30

Keywords: t-designs, Group divisible designs with unequal group sizes, BIBDs

1. Introduction

Among all combinatorial designs, probably the most widely studied design is a Balanced Incomplete Block Design (BIBD). For definitions and and background please see Lindner and Rodger [6].

Definition 1.1. *A Balanced Incomplete Block Design, BIBD(v, k, λ), is an arrangement of v distinct points into b proper subsets (called blocks) of size k each, such that every point appears in exactly r blocks and every pair of distinct points occurs together in exactly λ blocks.*

The numbers v, b, r, k and λ are parameters of the BIBD and satisfy the necessary conditions $vr = bk$ and $\lambda(v-1) = r(k-1)$ for the existence of a BIBD(v, k, λ).

In 1961, Haim Hanani [4] proved that the necessary conditions are sufficient for the existence of BIBDs with block size three as well as four. Specifically he proved:

* *This author thanks Professor Jurisich for support through departmental research funds.*
Dinesh G. Sarvate (Corresponding Author); College of Charleston, SC 29424, USA (email: SarvateD@cofc.edu).
** *This author is supported by Adama Science and Technology University research grant number ASTU/SP-R/011/19.*
Dinkayehu M. Woldemariam; Adama Science and Technology University, Adama, Ethiopia (email: dinkumen@gmail.com).

Theorem 1.2. *A BIBD*$(v, 4, \lambda)$ *exists if and only if*
$\lambda \equiv 1, 5 \pmod 6$ *and* $v \equiv 1, 4 \pmod{12}$*;*
$\lambda \equiv 2, 4 \pmod 6$ *and* $v \equiv 1 \pmod 3$*;*
$\lambda \equiv 3 \pmod 6$ *and* $v \equiv 0, 1 \pmod 4$*;*
$\lambda \equiv 0 \pmod 6$ *and* $v \geq 4$*.*

Group divisible designs defined below play a role in the construction of BIBDs as well as other designs. For example, in the construction of t-designs where instead of each pair occurring in λ blocks each t-tuple occurs in λ blocks.

Definition 1.3. *A group divisible design, GDD*$(n_1, n_2, ..., n_m, k; \lambda_1, \lambda_2)$*, is a triple* $(X, \mathcal{G}, \mathcal{B})$*, where* X *is a* v*-set,* $\mathcal{G}$ *is a partition of* X *into* m *subsets (called* groups*) of size* $n_1, n_2, ..., n_m$ *respectively and* $\mathcal{B}$ *is a collection of* k*-subsets of* X *(called* blocks*) such that each pair of points within the same group appear together in* λ_1 *blocks, whereas each pair of points from different groups appear together in* λ_2 *blocks. The points in the same group are called* first associate *of each other and elements not in the same group are called* second associates *of each other.*

Fu, Rodger and Sarvate [2, 3] obtained complete results on group divisible designs with m groups of size n and block size 3, namely GDD$(n, n, ..., n, 3; \lambda_1, \lambda_2)$. In 1992, Colbourn, Hoffman and Rees [1] proved the sufficiency of the necessary conditions for the existence of a GDD$(n, n, ..., n, u, 3; 0, 1)$. In 2011, Pabhapote and Punnim [8] studied all triples of positive integers (n_1, n_2, λ) for which a GDD$(n_1, n_2, 3; \lambda, 1)$ exists. Later, Pabhapote [7] proved the existence of a GDD$(n_1, n_2, 3; \lambda_1, \lambda_2)$ for all $n_1 \neq 2$ and $n_2 \neq 2$ in which $\lambda_1 \geq \lambda_2$.

This note is specially motivated by the papers of Sakda and Uiyyasathian [9] and Lapchinda, Punnim and Pabhapote [5]. In 2014, Lapchinda, Punnim and Pabhapote [5] gave a complete solution for the existence of a group divisible design with block size 3 and 3 groups of sizes n, n and 1. In 2017, Sakda and Uiyyasathian, obtained complete result on group divisible designs with block size 3 and 4 groups of sizes n, n, n and 1, namely GDD$(1, n, n, n, 3; \lambda_1, \lambda_2)$. In this note we study the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ with three groups G_1, G_2, G_3 of sizes 1, n, and n respectively. In general, when the number of groups is less than the block size the work is more involved and possibly making them harder to construct. It is well known that GDDs are used as a building block for BIBDs, but the converse is also true, for example, an easy observation is the following result.

Theorem 1.4. *If a BIBD*$(n_1 + n_2 + ... + n_m, k, \lambda_2)$ *and a BIBD*(n_i, k, λ_1) *exist for* $i = 1, 2, ...m$*, then a GDD*$(n_1, n_2, ..., n_m, k; \lambda_1 + \lambda_2, \lambda_2)$ *exists.*

Corollary 1.5. *If a BIBD*$(mn + 1, 4, \lambda_2)$ *and a BIBD*$(n, 4, \lambda)$ *exist, then a GDD*$(1, n, n, ..., n, 4; \lambda_1 = \lambda_2 + \lambda, \lambda_2)$ *exists.*

The converse of the above corollary is not true, for example, we show in Section 5 that a GDD$(1, n, n, 4; \lambda_2 + \lambda, \lambda_2)$ exist for $n = 2$ or $n = 3$ but clearly a BIBD$(2, 4, \lambda)$ or BIBD$(3, 4, \lambda)$ does not exist. One can find such examples for larger values of n by using the construction given in the next section. For example, the construction gives a GDD$(1, 7, 7, 4; 9, 6)$ but a BIBD$(7, 4, 3)$ does not exist.

Another observation gives,

Theorem 1.6. *A GDD*$(n_1, n_2, ..., n_m, k; \lambda_1, 0)$ *exists if and only if a BIBD*$(n_i, k, \ \lambda_1)$ *exist for* $i = 1, 2, ...m$*.*

Corollary 1.7. *A GDD*$(1, n, n, ..., n, k; \lambda_1 = \lambda, 0)$ *exists if and only if a BIBD*(n, k, λ) *exists.*

One may notice that it is much easier to construct GDDs with $\lambda_1 \geq \lambda_2$, specially when a BIBD(n, k, λ) exists. In the next section we present an important construction technique which produces GDDs where λ_1 is less than λ_2.

2. A new construction of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$

A K_n on G_i means the vertices of the complete graph K_n are labeled with the elements of G_i for $i = 2, 3$. Let n be even. Then the complete graph K_n on G_2 (respectively on G_3) has a 1-factorization, say $\{E_1, E_2, ..., E_{n-1}\}$ (respectively $\{F_1, F_2, ..., F_{n-1}\}$). For $x = 1, 2, ..., n-1$, if $E_x = \{e_1, e_2, ..., e_{\frac{n}{2}}\}$ and $F_x = \{f_1, f_2, ..., f_{\frac{n}{2}}\}$, then we can form blocks $e_l \cup f_m$ of size 4, for $1 \le l, m \le \frac{n}{2}$.

On the other hand when n is odd, a K_n on G_2 (respectively on G_3) has a 2-factorization, say $\{E_1, E_2, ..., E_{\frac{n-1}{2}}\}$ (respectively $\{F_1, F_2, ..., F_{\frac{n-1}{2}}\}$). For $x = 1, 2, ..., \frac{n-1}{2}$, if $E_x = \{e_1, e_2, ..., e_n\}$ and $F_x = \{f_1, f_2, ..., f_n\}$, then we can form blocks $e_l \cup f_m$ of size 4, for $1 \le l, m \le n$. Now we define

$\mathcal{B}_4 = \{e_l \cup f_m : e_l \in E_x$ and $f_m \in F_x$ for $x = 1, 2, ..., n-1$ and $1 \le l, m \le \frac{n}{2}$ $\}$ if n is even, and

$\mathcal{B}_4 = \{e_l \cup f_m : e_l \in E_x$ and $f_m \in F_x$ for $x = 1, 2, ..., \frac{n-1}{2}$ and $1 \le l, m \le n$ $\}$ if n is odd.

Theorem 2.1. *Suppose a BIBD$(2n, 3, \lambda)$ and a BIBD$(n, 3, \mu)$ exist.*

(a) Suppose n is even and there are nonnegative integers i, j, u and v such that

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + v(n-1). \tag{1}$$

Then there exists a GDD$(1, n, n, 4; i\lambda + j\mu + un\mu + vn/2, \frac{i(2n-1)\lambda + j\mu(n-1)}{2})$.

(b) Suppose n is odd and there are nonnegative integers i, j, u and v such that

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + 2v(n-1). \tag{2}$$

Then there exists a GDD$(1, n, n, 4; i\lambda + j\mu + un\mu + vn, \frac{i(2n-1)\lambda + j\mu(n-1)}{2})$.

Proof. Let $G_1 = \{x\}$, $G_2 = \{a_1, a_2, ..., a_n\}$ and $G_3 = \{b_1, b_2, ..., b_n\}$. Then consider the following sets:

- $\mathcal{B}_1 = \{G_1 \cup B : B$ is a block of BIBD$(2n, 3, \lambda)$ on $G_2 \cup G_3\}$;
- $\mathcal{B}_2 = \{G_1 \cup B : B$ is a block of BIBD$(n, 3, \mu)$ on G_2 and $G_3\}$;
- $\mathcal{B}_3 = \{\{a\} \cup B : a \in G_i$ and B is a block of BIBD$(n, 3, \mu)$ on G_j for $i, j = 2, 3$ and $i \neq j\}$.

In $\mathcal{B}_1$, every element from $G_2 \cup G_3$ comes with the point of G_1 $\frac{\lambda(2n-1)}{2}$ times and every pair of elements from $G_2 \cup G_3$ comes λ times. In $\mathcal{B}_2$, every element from G_2 and G_3 comes with the point of G_1 $\frac{\mu(n-1)}{2}$ times and first associate pair from G_2 and G_3 comes μ times. In $\mathcal{B}_3$, first associate pair from G_2 and G_3 comes μn times and second associate pair from G_2 and G_3 comes $\mu(n-1)$ times. In $\mathcal{B}_4$, first and second associate pairs from G_2 and G_3 occur n and $2(n-1)$ times respectively if n is odd while first and second associate pairs from G_2 and G_3 occur $\frac{n}{2}$ and $n-1$ times respectively if n is even.

Suppose we have i copies of $\mathcal{B}_1$, j copies of $\mathcal{B}_2$, u copies of $\mathcal{B}_3$ and v copies of $\mathcal{B}_4$. Then the following matrix displays the replication number of each pair (a_1, x), (a_1, a_2) and (a_1, b_1) in $i\mathcal{B}_1$, $j\mathcal{B}_2$, $u\mathcal{B}_3$ and $v\mathcal{B}_4$, where i, j, u and v are any nonnegative integers.

For n even,

$$\begin{array}{c} \\ i\mathcal{B}_1 \\ j\mathcal{B}_2 \\ u\mathcal{B}_3 \\ v\mathcal{B}_4 \end{array} \begin{array}{c} \begin{array}{ccc} (a_1, x) & (a_1, a_2) & (a_1, b_1) \end{array} \\ \begin{pmatrix} \frac{i(2n-1)\lambda}{2} & i\lambda & i\lambda \\ \frac{j(n-1)\mu}{2} & j\mu & 0 \\ 0 & un\mu & u\mu(n-1) \\ 0 & vn/2 & v(n-1) \end{pmatrix} \end{array}.$$

So we have a GDD$(1, n, n, 4; i\lambda + j\mu + un\mu + vn/2, \frac{i(2n-1)\lambda + j\mu(n-1)}{2})$ when

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + v(n-1). \tag{3}$$

For n odd,

$$\begin{array}{c} \\ i\mathcal{B}_1 \\ j\mathcal{B}_2 \\ u\mathcal{B}_3 \\ v\mathcal{B}_4 \end{array} \begin{array}{c} \begin{array}{ccc} (a_1, x) & (a_1, a_2) & (a_1, b_1) \end{array} \\ \begin{pmatrix} \frac{i(2n-1)\lambda}{2} & i\lambda & i\lambda \\ \frac{j(n-1)\mu}{2} & j\mu & 0 \\ 0 & un\mu & u\mu(n-1) \\ 0 & vn & 2v(n-1) \end{pmatrix} \end{array}.$$

So we have a GDD$(1, n, n, 4; i\lambda + j\mu + un\mu + vn, \frac{i(2n-1)\lambda + j\mu(n-1)}{2})$ when

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + 2v(n-1). \tag{4}$$

□

Theorem 2.2. *(a) If a BIBD$(2n, b, r, 3, \lambda)$ exists for odd n and if $r - \lambda = 2(n-1)t$, then a GDD$(1, n, n, 4; \lambda + nt, r)$ exists.*

(b) If a BIBD$(2n, b, r, 3, \lambda)$ exists for even n and if $r - \lambda = (n-1)t$, then a GDD$(1, n, n, 4; \lambda + \frac{nt}{2}, r)$ exists.

Proof. Let $\mathcal{B} = \{G_1 \cup B : B$ is a block of BIBD$(2n, b, r, 3, \lambda)\}$. Then t copies of $\mathcal{B}_4$ along with $\mathcal{B}$ give the required GDD. □

Example 2.3. *As a BIBD$(6, 40, 20, 3, 8)$ exists, $r - \lambda = 12 = 4 \times 3$. We get a GDD$(1, 3, 3, 4; 8 + 3 \times 3 = 17, 20 = (8 + 3 \times 4))$ using 3 copies of $\mathcal{B}_4$.*

Essentially, $\mathcal{B}_4$ is a GDD on two groups of size n, where the indices depend on n odd or even.

For n even, $\mathcal{B}_4$= GDD$(n, n, 4; \lambda_1 = \frac{n}{2}, \lambda_2 = n-1)$ and for n odd, $\mathcal{B}_4$= GDD$(n, n, 4; \lambda_1 = n, \lambda_2 = 2(n-1))$. Now the replication number r for a BIBD$(2n, 3, \lambda)$ is $\frac{\lambda(2n-1)}{2}$. If we wish $r - \lambda$ to be a multiple of $(n-1)$, say $s(n-1)$ when n is even (respectively $2s(n-1)$ when n is odd), then $\lambda = \frac{2s(n-1)}{2n-3}$ (respectively $\lambda = \frac{4s(n-1)}{2n-3}$). For $s = 2n-3$, $\lambda = 2(n-1)$ for n even (respectively $\lambda = 4(n-1)$ for n odd).

Example 2.4. *For $n = 4$, we have GDD$(1, 4, 4, 4; 16, 21)$ by using blocks of a BIBD$(8, b, 21, 3, 6)$ and $s = 5$ copies of $\mathcal{B}_4$ =GDD$(4, 4, 4; 2, 3)$.*

Example 2.5. *For $n = 5$, we have GDD$(1, 5, 5, 4; 51, 72)$ by using blocks of a BIBD$(10, b, 72, 3, 16)$ and $s = 7$ copies of $\mathcal{B}_4$ =GDD$(5, 5, 4; 5, 8)$.*

In general :

Theorem 2.6. *For n even, using a BIBD$(2n, 3, 2(n-1))$ and $2n-3$ copies of a GDD$(n, n, 4; \frac{n}{2}, n-1)$, a GDD$(1, n, n, 4; \frac{2n^2+n-4}{2}, (n-1)(2n-1))$ and for n odd, using a BIBD$(2n, 3, 4(n-1))$ and $2n-3$ copies of a GDD$(n, n, 4; n, 2(n-1))$, a GDD$(1, n, n, 4; 2n^2 + n - 4, 2(n-1)(2n-1))$ exists.*

In the next section, we obtain some necessary conditions for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$. Towards this aim, assuming a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ exists, we count the number of blocks, r_i, containing a given element x of G_i for $i = 1, 2, 3$, and the required number of blocks, say b, for the GDD.

3. Necessary conditions

Suppose a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ exists with groups G_1, G_2, G_3 of size 1, n, n respectively. Let r_i be the replication number of each element of G_i for $i = 1, 2, 3$. As the size of G_2 is equal to the size of G_3, $r_2 = r_3$. Then by counting argument, $r_1 = \frac{2n\lambda_2}{3}$ and $r_2 = r_3 = \frac{\lambda_1(n-1)+\lambda_2(n+1)}{3}$. Let b be the required number of blocks for a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ if it exists. Since $4 \times b = r_1 \times 1 + r_2 \times (n + n)$, we have $b = \frac{\lambda_1(n^2-n)+\lambda_2(n^2+2n)}{6}$.

As r_1 and r_2 must be integers, we have the following.

- If $n \equiv 0 \pmod 3$, then $\lambda_1 \equiv \lambda_2 \pmod 3$.
- If $n \equiv 1 \pmod 3$, then $\lambda_2 \equiv 0 \pmod 3$.
- If $n \equiv 2 \pmod 3$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

Since b must be an integer, we have the following.

- If $n \equiv 0, 4 \pmod 6$, then no restriction on λ_1 and λ_2.
- If $n \equiv 1, 3 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 2$.
- If $n \equiv 2 \pmod 6$, then $\lambda_1 + \lambda_2 \equiv 0 \pmod 3$.
- If $n \equiv 5 \pmod 6$, then $2\lambda_1 + 5\lambda_2 \equiv 0 \pmod 6$.

Hence some basic necessary conditions for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ are

- If $n \equiv 0 \pmod 6$, then $\lambda_1 \equiv \lambda_2 \pmod 3$,
- If $n \equiv 1 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 6$,
- If $n \equiv 2 \pmod 6$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$,
- If $n \equiv 3 \pmod 6$, then $\lambda_1 \equiv \lambda_2 \pmod 3$, $\lambda_2 \equiv 0 \pmod 2$,
- If $n \equiv 4 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 3$, and
- If $n \equiv 5 \pmod 6$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$.

Above necessary conditions are summarized in Table 1, where "None" means the design does not exist for any n. λ_1 is given in modulo 3 and λ_2 is given in modulo 6.

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	all n	None	None	n even	None	None
1	$n \equiv 1 \pmod 3$	$n \equiv 0 \pmod 6$	None	$n \equiv 4 \pmod 6$	$n \equiv 0 \pmod 3$	None
2	$n \equiv 1 \pmod 3$	None	$n \equiv 0 \pmod 3$	$n \equiv 4 \pmod 6$	None	$n \equiv 0 \pmod 6$

Table 1. The necessary conditions for GDD$(1, n, n, 4; \lambda_1, \lambda_2)$

A side application of the table is that for $n \equiv 3 \pmod 6$ instead of constructing three families: GDD$(1, 6m+3, 6m+3, 4; 3t, 6s)$, GDD$(1, 6m+3, 6m+3, 4; 3t+1, 6s+4)$ and GDD$(1, 6m+3, 6m+3, 4; 3t+2, 6s+2)$, one needs to construct just one family GDD$(1, 6m+3, 6m+3, 4; 3t, 6s)$. Then the family GDD$(1, 6m+3, 6m+3, 4; 3t+2, 6s+2)$ can be obtained by taking the blocks of a GDD$(1, 6m+$

$3, 6m+3, 4; 3t, 6s)$ and the blocks of a BIBD$(12m+7, 4, 2)$.
Similarly, a GDD$(1, 6m+3, 6m+3, 4; 3t+1, 6s+4)$ can be obtained by taking the blocks of a GDD$(1, 6m+3, 6m+3, 4; 3(t-1), 6s)$ and the blocks of a BIBD$(12m+7, 4, 4)$ where m is any nonnegative integer.

As a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ has 3 groups and blocks of size 4, each block contains at least one associate pair. Then $b \leq [\binom{n}{2} + \binom{n}{2}]\lambda_1 = n(n-1)\lambda_1$. Now substituting the value of b, we have the following theorem:

Theorem 3.1. *A necessary condition for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ is*

$$\lambda_2 \leq \frac{5(n-1)}{n+2}\lambda_1.$$

Corollary 3.2. *For the existence of a GDD$(1, n, n, 4, \lambda_1, \lambda_2)$, $\lambda_2 \leq 5\lambda_1$.*

The blocks of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$, if exists, have $(n^2+2n)\lambda_2$ second associate pairs. There can be at most r_1 blocks of type $(1, 1, 2)$ which account for $5r_1$ second associate pairs, we have $b - r_1 \geq \frac{(n^2+2n)\lambda_2 - 5r_1}{4}$ as all other blocks can have at the most 4 second associate pairs. Thus, we have the following:

Theorem 3.3. *A necessary condition for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ is $b \geq \frac{(n^2+2n)\lambda_2 - r_1}{4}$.*

Corollary 3.4. *A necessary condition for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ is $\lambda_2 \leq \frac{2(n-1)}{n}\lambda_1 < 2\lambda_1$.*

Proof. Substituting the values of b and r_1 in $b \geq \frac{(n^2+2n)\lambda_2 - r_1}{4}$, we have $\lambda_2 \leq \frac{2(n-1)}{n}\lambda_1 < 2\lambda_1$. □

As a consequence of the above corollary, we have

Corollary 3.5. *For any nonnegative integer t and any positive integers x and s, where $x \leq s$ following GDDs do not exist.*

1. *A GDD$(1, 6t+4, 6t+4, 4; 3x+1, 6s+3)$.*
2. *A GDD$(1, 3t, 3t, 4; 3x+1, 6s+4)$.*
3. *A GDD$(1, 6t, 6t, 4; 3x+2, 6s+5)$.*
4. *A GDD$(1, n, n, 4; 3x, 6s)$.*

4. Existence of families of GDD$(1, n, n, 4; \lambda_1, \lambda_2)$

Unless otherwise stated in this section we are assuming $\lambda_1 \geq \lambda_2$.

Remark 4.1. *A GDD$(1, n, n, 4; 0, \lambda_2)$ does not exist as the number of groups is less than the block size.*

As a consequence of Theorem 1.4 we have following theorem where unless otherwise stated λ, s and t are nonnegative integers and $n > 1$.

Theorem 4.2. *If a BIBD$(2n+1, 4, \lambda_2)$ and a BIBD$(n, 4, \lambda)$ exist, then a GDD$(1, n, n, 4; \lambda_1 = \lambda_2 + \lambda, \lambda_2)$ exists. In particular, we have*

1. *A GDD$(1, n, n, 4; 6t, 6s)$ exists for all $n \geq 4$, where $t \geq s$.*
2. *A GDD$(1, n, n, 4; 6s+3t, 6s)$ exists for $n \equiv 0, 1 \pmod 4$.*

3. *A* $GDD(1, n, n, 4; 6t + 3s, 3s)$ *exists for* $n \equiv 0 \pmod 2$.
4. *A* $GDD(1, n, n, 4; 6t + 2s, 2s)$ *exists when* $n \equiv 0 \pmod 3$.
5. *A* $GDD(1, n, n, 4; 3t + s, s)$ *exists for* $n \equiv 0 \pmod{12}$.
6. *A* $GDD(1, n, n, 4; 3t + 2s, 2s)$ *exists for* $n \equiv 9 \pmod{12}$.
7. *A* $GDD(1, n, n, 4; 6t + \lambda, \lambda)$ *exists for* $n \equiv 0 \pmod 6$.
8. *A* $GDD(1, n, n, 4; 3t, 3s)$ *exists for* $n \equiv 0 \pmod 4$, *where* $t \geq s$.
9. *A* $GDD(1, n, n, 4; 2t + 6s, 6s)$ *exists for* $n \equiv 1 \pmod 6$.
10. *A* $GDD(1, n, n, 4; 2t + 3s, 3s)$ *exists for* $n \equiv 4 \pmod 6$.
11. *A* $GDD(1, n, n, 4; 6s + \lambda, 6s)$ *exists for* $n \equiv 1 \pmod{12}$.
12. *A* $GDD(1, n, n, 4; 3s + \lambda, 3s)$ *exists for* $n \equiv 4 \pmod{12}$.

Case 1: $\lambda_2 \equiv 0 \pmod 6$

From Theorem 4.2(1) we have

Corollary 4.3. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n, 4; \lambda_1, \lambda_2)$ *for* $\lambda_1 \equiv 0 \pmod 6$ *and* $\lambda_2 \equiv 0 \pmod 6$.

From Theorem 4.2(2) we have

Corollary 4.4. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n, 4; \lambda_1, \lambda_2)$ *for* $n \equiv 5 \pmod{12}$.

In the above family $\lambda_1 - \lambda_2 \equiv 3 \pmod 6$.

From the necessary conditions, when $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 0 \pmod 3$, n has to be even. From Theorem 4.2 (3), a GDD$(1, n, n, 4; 6t + 3, 6s + 3)$ exists for any even n and any nonnegative integers s and t, where $t \geq s$. From Theorem 4.2(8), a GDD$(1, n, n, 4; 6t, 6s + 3)$ exists when $n \equiv 0 \pmod 4$ for any nonnegative integers s and t, where $t > s$. Hence we have

Corollary 4.5. *Necessary conditions are sufficient for the existence of a* $GDD(1, n, n, 4; \lambda_1, \lambda_2)$ *for* $n \equiv 0 \pmod 4$, $\lambda_1 \equiv 0 \pmod 3$ *and* $\lambda_2 \equiv 0 \pmod 3$.

When $\lambda_1 \equiv 2 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$, from the necessary conditions, we have $n \equiv 1 \pmod 3$. For $n \equiv 1 \pmod 3$, a BIBD$(n, 4, 2)$ and a BIBD$(2n + 1, 4, 6)$ exist for $n \geq 4$. From Theorem 4.2(9) and (10), we have

Lemma 4.6. *A* $GDD(1, n, n, 4; 6t + 2, 6s)$ *exists when* $n \equiv 1 \pmod 3$ *for any nonnegative integers* s *and* t, *where* $t \geq s$.

For $n \equiv 1, 4 \pmod{12}$, a BIBD$(n, 4, 5)$ and a BIBD$(2n + 1, 4, 6)$ exist. From Theorem 4.2(11) and (12), we have

Lemma 4.7. *A* $GDD(1, n, n, 4; 6t + 5, 6s)$ *exists when* $n \equiv 1, 4 \pmod{12}$ *for any nonnegative integers* s *and* t, *where* $t \geq s$.

For $\lambda_2 \equiv 0 \pmod 6$, and $\lambda_1 \equiv 1, 2 \pmod 3$ from the necessary conditions, we have $n \equiv 1 \pmod 3$. From Theorem 4.2 (9) and (10), we have

Lemma 4.8. *A GDD$(1, n, n, 4; 6t+4, 6s)$ exists when $n \equiv 1 \pmod 3$ for any nonnegative integers s and t, where $t \geq s$.*

From Theorem 4.2 (11) and (12), we have the following:

Lemma 4.9. *A GDD$(1, n, n, 4; 6t+1, 6s)$ exists for $n \equiv 1, 4 \pmod{12}$, where s and t are nonnegative integers such that $t \geq s$.*

For $n \equiv 9 \pmod{12}$, a BIBD$(n, 4, 3)$ and BIBD$(2n+1, 4, 6)$ exist. So we have

Lemma 4.10. *A GDD$(1, n, n, 4; \lambda_1 = 3t, \lambda_2 = 6s)$ exists when $n \equiv 9 \pmod{12}$ for any nonnegative integers s and t, where $\lambda_1 \geq \lambda_2$.*

Hence, we have :

Corollary 4.11. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $\lambda_1 \geq \lambda_2$ for $n \equiv 1, 4 \pmod{12}$, $\lambda_1 \equiv 1 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$.*

Case 2: $\lambda_2 \equiv 1 \pmod 6$

In this case from the necessary conditions, we have $\lambda_1 \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 6$. From Theorem 4.2 (7):

Lemma 4.12. *A GDD$(1, n, n, 4; 6t+1, 6s+1)$ exists when $n \equiv 0 \pmod 6$ for any nonnegative integers s and t, where $t \geq s$.*

From Theorem 4.2 (5):

Lemma 4.13. *A GDD$(1, n, n, 4; 6t+4, 6s+1)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integers s and t, where $t \geq s$.*

Corollary 4.14. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $n \equiv 0 \pmod{12}$, and $\lambda_2 \equiv 1 \pmod 6$.*

Case 3: $\lambda_2 \equiv 2 \pmod 6$

In this case, $\lambda_1 \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 3$. From Theorem 4.2(4) we have the following lemma.

Lemma 4.15. *A GDD$(1, n, n, 4; 6t+2, 6s+2)$ exists when $n \equiv 0 \pmod 3$ for any nonnegative integers s and t, where $t \geq s$.*

From Theorem 4.2(5) and (6), we have

Lemma 4.16. *A GDD$(1, n, n, 4; 6t+5, 6s+2)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integer s and t, where $t \geq s$.*

Hence we have:

Corollary 4.17. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $n \equiv 0, 9 \pmod{12}$, and $\lambda_2 \equiv 2 \pmod 6$.*

Case 4: $\lambda_2 \equiv 3 \pmod 6$

In this case from the necessary conditions, when $\lambda_1 \equiv 1, 2 \pmod 3$, $n \equiv 4 \pmod 6$. For $n \equiv 4 \pmod 6$, a BIBD$(n, 4, 2)$, a BIBD$(n, 4, 4)$ and a BIBD$(2n+1, 4, 3)$ exist. From Theorem 4.2(10), we have the following lemma:

Lemma 4.18. *When $n \equiv 4 \pmod 6$, a GDD$(1, n, n, 4; 6t+1, 6s+3)$ exists for any nonnegative integers s and t, where $t > s$ and a GDD$(1, n, n, 4; 6t+5, 6s+3)$ exists where $t \geq s$.*

For $n \equiv 4 \pmod{12}$, a BIBD$(n, 4, 1)$ and a BIBD$(2n+1, 4, 3)$ exist. Hence we have a GDD$(1, n, n, 4; 3, 3)$, a GDD$(1, n, n, 4; 4, 3)$ and a GDD$(1, n, n, 4; 5, 3)$. From Theorem 4.2 (12), we have

Corollary 4.19. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, 6s+3)$ when $n \equiv 4 \pmod{12}$ for any nonnegative integers s and λ_1, where $\lambda_1 \geq 6s+3$.*

When $\lambda_1 \equiv 0 \pmod 3$, $n \equiv 0 \pmod 2$, from Theorem 4.2 (8), we have

Corollary 4.20. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, 6s+3)$ when $n \equiv 0 \pmod 4$ for any nonnegative integers s and λ_1, where $\lambda_1 \geq 6s+3$.*

Case 5: $\lambda_2 \equiv 4 \pmod 6$

In this case from the necessary conditions, we have $\lambda_1 \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$. For $n \equiv 0 \pmod{12}$, a BIBD$(2n+1, 4, 1)$, and a BIBD$(n, 4, 3)$ exist. Hence, a GDD$(1, n, n, 4; 3x+1, 4)$ exists for $x > 1$. Hence

Corollary 4.21. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1 = 3t+1, \lambda_2 = 6s+4)$ for $n \equiv 0 \pmod{12}$.*

Similarly from Theorem 4.2 (6), we have a GDD$(1, n, n, 4; \lambda_1 = 3x+4, \lambda_2 = 4)$ for a nonnegative integer x, hence

Corollary 4.22. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1 = 3t+1, \lambda_2 = 6s+4)$ for $n \equiv 9 \pmod{12}$ where $\lambda_1 > \lambda_2$.*

For $n \equiv 0 \pmod 3$, a BIBD$(2n+1, 4, 2)$, a BIBD$(2n+1, 4, 4)$ and a BIBD$(n, 4, 6)$ exist for $n \geq 4$. From Theorem 4.2(4), we have

Lemma 4.23. *A GDD$(1, n, n, 4; 6t+4, 6s+4)$ exists when $n \equiv 0 \pmod 3$ for any nonnegative integers s and t, where $t \geq s$.*

Case 6: $\lambda_2 \equiv 5 \pmod 6$

Here, from the necessary conditions, we have $\lambda_1 \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 6$. But for $n \equiv 0 \pmod{12}$, a BIBD$(2n+1, 4, 5)$ and a BIBD$(n, 4, 3)$ exist. From Theorem 4.2(5) and (6), we have

Lemma 4.24. *A GDD$(1, n, n, 4; 6t+2, 6s+5)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integers s and t, where $t > s$.*

From Theorem 4.2(7), we have

Lemma 4.25. *A GDD$(1, n, n, 4; 6t+5, 6s+5)$ exist when $n \equiv 0 \pmod 6$ for any nonnegative integers s and t, where $t \geq s$.*

Hence we have

Corollary 4.26. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1 = 3t+2, \lambda_2 = 6s+5)$ for $n \equiv 0 \pmod{12}$ where $\lambda_1 > \lambda_2$.*

We have summarized main results from this section in Table 2.

From Table 2, we have

Theorem 4.27. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.*

$n \equiv$		The existence is not known for
0 (mod 12)	Corollary 4.3, Corrolary 4.5, Corollary 4.14, Corollary 4.17, Corollary 4.20, Corollary 4.21, Corollary 4.26, Lemma 4.12, Lemma 4.13, Lemma 4.15, Lemma 4.16, Lemma , Lemma 4.23, Lemma 4.24	
1 (mod 12)	Corollary 4.3, Corollary 4.11, Lemma 4.6, Lemma 4.7, Lemma 4.8, Lemma 4.9, Theorem 4.2 (11)	
2 (mod 12)	Corollary 4.3	GDD$(1,n,n,4;3t,6s+3)$ GDD$(1,n,n,4;6t+3,6s)$
3 (mod 12)	Corollary 4.3, Lemma 4.15, Lemma 4.23	GDD$(1,n,n,4;6t+3,6s)$ GDD$(1,n,n,4;6t+1,6s+4)$ GDD$(1,n,n,4;6t+5,6s+2)$
4 (mod 12)	Corollary 4.3, Corollary 4.5, Corollary 4.11, Corollary 4.19, Corollary 4.20, Lemma 4.6, Lemma 4.7, Lemma 4.8, Lemma 4.9, Lemma 4.18	
5 (mod 12)	Corollary 4.3, Corollary 4.4	
6 (mod 12)	Corollary 4.3, Lemma 4.12, Lemma 4.15, Lemma 4.25	GDD$(1,n,n,4;6t+3,6s)$ GDD$(1,n,n,4;3t,6s+3)$ GDD$(1,n,n,4;6t+4,6s+1)$ GDD$(1,n,n,4;6t+5,6s+2)$ GDD$(1,n,n,4;6t+1,6s+4)$ GDD$(1,n,n,4;6t+2,6s+5)$
7 (mod 12)	Corollary 4.3, Lemma 4.6, Lemma 4.8	GDD$(1,n,n,4;6t+1,6s)$ GDD$(1,n,n,4;6t+3,6s)$ GDD$(1,n,n,4;6t+5,6s)$
8 (mod 12)	Corollary 4.3, Corollary 4.5, Corollary 4.20,	
9 (mod 12)	Corollary 4.3, Corollary 4.17, Corollary 4.22, Lemma 4.10, Lemma 4.15, Lemma 4.23	
10 (mod 12)	Corollary 4.3, Lemma 4.6, Lemma 4.8, Lemma 4.18	GDD$(1,n,n,4;6t+3,6s)$ GDD$(1,n,n,4;6t+1,6s)$ GDD$(1,n,n,4;6t+5,6s)$ GDD$(1,n,n,4;6t+2,6s+3)$
11 (mod 12)	Corollary 4.3	GDD$(1,n,n,4;6t+3,6s)$

Table 2. For the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$, $\lambda_1 \geq \lambda_2$

5. Specific GDDs

In this section, we study the existence of GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for specific values of the parameters.

5.1. $\lambda_1 = 1$

Theorem 5.1. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;1,\lambda_2)$. Specifically, a GDD$(1,n,n,4;1,\lambda_2)$ exists when $\lambda_2 = 1$ and $n \equiv 0 \pmod 6$ and when $\lambda_2 = 0$ and $n \equiv 1,4 \pmod{12}$.*

Proof. For $\lambda_1 = 1$, by Corollary 3.4, $\lambda_2 < 2$, hence λ_2 can only be 0 or 1. A GDD$(1,n,n,4;1,1)$ exists for $n \equiv 0 \pmod 6$ as a BIBD$(2n+1,4,1)$ on $G_1 \cup G_2 \cup G_3$ exists. A GDD$(1,n,n,4;1,0)$ exists for $n \equiv 1,4 \pmod{12}$ as a BIBD$(n,4,1)$ on G_i for $i = 2,3$, where G_1, G_2, G_3 are groups of size 1, n, n

respectively. □

5.2. $\lambda_1 = 2$

Theorem 5.2. *A GDD$(1, n, n, 4; 2, \lambda_2)$ exists for $\lambda_2 \leq 2$, specifically when $\lambda_2 = 2$ and $n \equiv 0 \pmod 3$ and when $\lambda_2 = 0$ and $n \equiv 1, 4 \pmod 6$.*

Proof. For $\lambda_1 = 2$, by Corollary 3.4, $\lambda_2 < 4$, hence λ_2 can be 0, 1, 2 and 3. A GDD$(1, n, n, 4; 2, 0)$ exists for $n \equiv 1 \pmod 3$ as a BIBD$(n, 4, 2)$ on G_i for $i = 2, 3$, where G_1, G_2, and G_3 are groups of size 1, n, and n respectively. A GDD$(1, n, n, 4; 2, 1)$ does not exists for any n from the necessary conditions. A GDD$(1, n, n, 4; 2, 2)$ exists for $n \equiv 0 \pmod 3$ as a BIBD$(2n + 1, 4, 2)$ on $G_1 \cup G_2 \cup G_3$ exists. □

As a GDD$(1, n, n, 4; 2, 3)$ exists for $n = 4$ (see Example 5.13) and for $n = 10$ (see Example 6.8), we have

Theorem 5.3. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; 2, \lambda_2)$ except possibly for $n \equiv 4 \pmod 6$, $n \neq 4$, 10 and $\lambda_2 = 3$.*

5.3. $\lambda_1 = 3$

Theorem 5.4. *Necessary conditions are sufficient for the existence of a GDD$(1, n, n, 4; 3, \lambda_2)$. Specifically, a GDD$(1, n, n, 4; 3, \lambda_2)$ exists when $\lambda_2 = 3$ and $n \equiv 0 \pmod 2$ and when $\lambda_2 = 0$ and $n \equiv 0, 1 \pmod 4$.*

Proof. For $\lambda_1 = 3$, by Corollary 3.4, $\lambda_2 < 6$. Hence λ_2 can be 0 and 3. A GDD$(1, n, n, 4; 3, 0)$ exists for $n \equiv 0, 1 \pmod 4$ as a BIBD$(n, 4, 3)$ on G_i for $i = 2, 3$ exists. But a GDD$(1, n, n, 4; 3, 0)$ does not exist for $n \equiv 2, 3 \pmod 4$ by Corollary 1.7. A GDD$(1, n, n, 4; 3, 3)$ exists for $n \equiv 0 \pmod 2$ as a BIBD$(2n+1, 4, 3)$ on $G_1 \cup G_2 \cup G_3$ exists. □

5.4. n=2

When $n = 2$, both λ_1 and λ_2 are 0 modulo 3. In a GDD$(1, 2, 2, 4; \lambda_1, \lambda_2)$, there are no blocks of type $(0, 4)$ and $(1, 3)$. Hence, a GDD$(1, 2, 2, 4; \lambda_1, 0)$ does not exist. Let b_1 and b_2 be the number of blocks of type $(1, 1, 2)$ and $(2, 2)$ for a GDD$(1, 2, 2, 4; \lambda_1, \lambda_2)$ if it exists. Then $b_1 + 2b_2 = 2\lambda_1$ and $5b_1 + 4b_2 = 8\lambda_2$. Hence

$$3b_1 = 8\lambda_2 - 4\lambda_1$$

As $b_1 = r_1 = \frac{4}{3}\lambda_2$, we have

Lemma 5.5. *A necessary condition for the existence of a GDD$(1, 2, 2, 4; \lambda_1, \lambda_2)$ is $\lambda_1 = \lambda_2$.*

As a GDD$(1, n, n, k; \lambda, \lambda)$ is a BIBD$(2n + 1, k, \lambda)$ and as a BIBD$(5, 4, 3)$ exists by Theorem 1.2, we have

Theorem 5.6. *Necessary conditions are sufficient for the existence of a GDD$(1, 2, 2, 4; \lambda_1, \lambda_2)$.*

5.5. n=3

Let $G_1 = \{x\}$, $G_2 = \{a, b, c\}$ and $G_3 = \{1, 2, 3\}$. In a GDD$(1, 3, 3, 4; \lambda_1, \lambda_2)$, there is no block of type $(0, 4)$. Let b_1, b_2 and b_3 be the number of blocks of type $(1, 1, 2)$, $(1, 3)$ and $(2, 2)$ respectively. Then $b_1 + 3b_2 + 2b_3 = 6\lambda_1$ and $5b_1 + 3b_2 + 4b_3 = 15\lambda_2$. As $6\lambda_1 = b_1 + 3b_2 + 2b_3 \leq 5b_1 + 3b_2 + 4b_3 = 15\lambda_2$, we have $\lambda_1 \leq \frac{5}{2}\lambda_2$. Also, by Corollary 3.4, $\lambda_2 \leq \frac{4}{3}\lambda_1$. Hence,

Lemma 5.7. *A necessary condition for the existence of a GDD*$(1,3,3,4;\lambda_1,\lambda_2)$ *is* $\frac{3}{4}\lambda_2 \le \lambda_1 \le \frac{5}{2}\lambda_2$.

In other words, $\frac{2}{5}\lambda_1 \le \lambda_2 \le \frac{4}{3}\lambda_1$.

Remark 5.8. *A GDD*$(1,3,3,4;\lambda_1,\ 0)$ *does not exist as there are no blocks of type* $(0,4)$.

Case 1.

$\lambda_1 < \lambda_2$. For $n = 3$, $\lambda_1 \equiv \lambda_2 \pmod 3$. Let $\lambda_2 - \lambda_1 = 3s$ for some positive integer s. Let $\lambda_1 = 3t + i$, for $i = 0, 1, 2$. Then from $\lambda_2 \le \frac{4\lambda_1}{3}$, $\lambda_2 \le 4t + i$. Therefore the difference $\lambda_2 - \lambda_1 = 3s$ is less than or equal to t. Using $\lambda_2 = 3s + \lambda_1$ and $\lambda_2 \le \frac{4\lambda_1}{3}$, the smallest λ_1 will be $9s$ and λ_2 will be $12s$. Hence the smallest GDD where $\lambda_2 - \lambda_1 = 3s$ is a GDD$(1,3,3,4;9s,12s)$. For $s = 1$, we construct a GDD$(1,3,3,4;9,12)$ as follows.

A relabeling construction

Let $X = \{0,1,2,3,4,5,6\}$, $G_1 = \{0\}$, $G_2 = \{1,2,3\}$ and $G_3 = \{4,5,6\}$. Then $\mathcal{B} = \{\{2,4,5,6\},\{3,5,6,0\},\{4,6,0,1\},\{5,0,1,2\},\{6,1,2,3\},\{0,2,3,4\},\ \{1,3,4,5\}\}$ is a collection of blocks of a BIBD$(7,4,\ 2)$ on $G_1 \cup G_2 \cup G_3 = X$.
We relabel elements of these blocks using different permutations (say α) on X to get following six isomorphic BIBDs.

- $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha(2) = 3$, $\alpha(3) = 2$, $\alpha(4) = 4$, $\alpha(5) = 6$ and $\alpha(6) = 5$. Then, when we relabel the points of X using α, the blocks of $\mathcal{B}$ become $\mathcal{B}_1 = \{\{3,4,6,5\},\{2,6,5,0\},\{4,5,0,1\},\{6,0,1,3\},\{5,1,3,2\},\{0,3,2,4\},\{1,2,4,6\}\}$.
- $\alpha(0) = 0$, $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 3$, $\alpha(4) = 6$, $\alpha(5) = 5$ and $\alpha(6) = 4$. Then the blocks of $\mathcal{B}$ become $\mathcal{B}_2 = \{\{1,6,5,4\},\{3,5,4,0\},\ \{6,4,0,2\},\{5,0,2,1\},\{4,2,1,3\},\{0,1,3,6\},\{2,3,6,5\}\}$.
- $\alpha(0) = 2$, $\alpha(1) = 1$, $\alpha(2) = 0$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 6$. Then the blocks of $\mathcal{B}$ become $\mathcal{B}_3 = \{\{0,4,5,6\},\{3,5,6,2\},\ \{4,6,2,1\},\{5,2,1,0\},\{6,1,0,3\},\{2,0,3,4\},\{1,3,4,5\}\}$.
- $\alpha(0) = 2$, $\alpha(1) = 1$, $\alpha(2) = 0$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 6$. Then the blocks of $\mathcal{B}$ become $\mathcal{B}_4 = \{\{0,4,5,6\},\{3,5,6,2\},\ \{4,6,2,1\},\{5,2,1,0\},\{6,1,0,3\},\{2,0,3,4\},\{1,3,4,5\}\}$.
- $\alpha(0) = 6$, $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 0$. Then the blocks of $\mathcal{B}$ become $\mathcal{B}_5 = \{\{2,4,5,0\},\{3,5,0,6\},\ \{4,0,6,1\},\{5,6,1,2\},\{0,1,2,3\},\{6,2,3,4\},\{1,3,4,5\}\}$.
- $\alpha(0) = 6$, $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 0$. Then the blocks of $\mathcal{B}$ become $\mathcal{B}_6 = \{\{2,4,5,0\},\{3,5,0,6\},\ \{4,0,6,1\},\{5,6,1,2\},\{0,1,2,3\},\{6,2,3,4\},\{1,3,4,5\}\}$.

So $\mathcal{B} \cup \mathcal{B}_1 \cup ... \cup \mathcal{B}_6$ gives a BIBD$(7,4,14)$ which contains 49 blocks. Removing the ten blocks containing all three points $1,2,3$ and all three points $4,5,6$, we have a GDD$(1,3,3,4;14-5=9,14-2=12)$.

In general, as a BIBD$(7,4,14t)$ exists for any positive integer t, we have the following result.

Lemma 5.9. *A GDD*$(1,3,3,4;14t-5s,14t-2s)$ *for* $s = 0,1,\cdots,t$ *exists for any positive integer* t *where* $s = 0$ *gives a BIBD.*

For example, when $t = 1$, we have a GDD$(1,3,3,4;14-5=9,14-2=12)$, and hence by using BIBD(7,4,2) repeatedly, we have GDD$(1,3,3,4;9+2m,12+2m)$, specifically we are interested in GDD$(1,3,3,4;13,16)$, GDD$(1,3,3,4;15,18)$, GDD$(1,3,3,4;17,20)$, GDD$(1,3,3,4;19,22)$, GDD$(1,3,3,4;21,24)$.

Lemma 5.10. *Necessary conditions are sufficient for the existence of a GDD*$(1,3,3,4;\lambda_1,\lambda_2)$ *for* $\lambda_1 \le \lambda_2$.

Proof. A GDD$(1,3,3,4;9,12)$ and a BIBD$(7,4,2)$ exist. The smallest GDD$(1,3,3,4;\lambda_1,\lambda_2)$ when the difference $\lambda_2-\lambda_1=3s$ is GDD$(1,3,3,4;9s,12s)$. s copies of GDD$(1,3,3,4;9,12)$ and m copies of BIBD$(7,4,2)$ together give all required GDD$(1,3,3,4;9s+2m,12s+2m)$ where $\lambda_2-\lambda_1=3s$. Recall that for $n=3$, λ_2 is always even. When $\lambda_1=\lambda_2$, necessary conditions are the same as the conditions for the existence of a BIBD$(7,4,\lambda_1)$. □

Case 2

$\lambda_1>\lambda_2$. Since a BIBD$(6,3,2)$ and a BIBD$(3,3,1)$ exist, we have the following from Theorem 2:

$$\begin{array}{c} \\ i\mathcal{B}_1 \\ j\mathcal{B}_2 \\ u\mathcal{B}_3 \\ v\mathcal{B}_4 \end{array}\begin{array}{c} \begin{array}{ccc}(1,x) & (1,2) & (1,a)\end{array} \\ \begin{pmatrix} 5i & 2i & 2i \\ j & j & 0 \\ 0 & 3u & 2u \\ 0 & 3v & 4v \end{pmatrix}\end{array}$$

So when $5i+j=2i+2u+4v$, then we have a

$$GDD(1,3,3,4;2i+j+3u+3v,5i+j). \tag{5}$$

Let $\lambda_1-\lambda_2=3s$ for some nonnegative integer s. As $\lambda_1\le\frac{5\lambda_2}{2}$, $3s\ \le 1.5\lambda_2$. Hence when the difference $\lambda_1-\lambda_2=3s$, smallest value of λ_2 is $2s$ and corresponding smallest parameter GDD will be GDD$(1,3,3,4;5s,2s)$. For $s=1$, the required GDD is GDD$(1,3,3,4;5,2)$ which can be constructed from 5 by letting $i=0$, $j=2$, $u=1$, $v=0$.

Lemma 5.11. *Necessary conditions are sufficient for the existence of a GDD$(1,3,3,4;\lambda_1,\lambda_2)$ for $\lambda_1\ge\lambda_2$.*

Proof. A GDD$(1,3,3,4;5,2)$ and a BIBD$(7,4,2)$ exist. The smallest GDD$(1,3,3,4;\lambda_1,\lambda_2)$ when the difference $\lambda_1-\lambda_2=3s$ is GDD$(1,3,3,4;5s,2s)$. Note that s copies of GDD$(1,3,3,4;5,2)$ and m copies of BIBD$(7,4,2)$ together give all required GDD$(1,3,3,4;5s+2m,2s+2m)$ where the $\lambda_1-\lambda_2=3s$. Recall that for $n=3$, λ_2 is always even. When $\lambda_1=\lambda_2$, necessary conditions are the same as the conditions for the existence of a BIBD$(7,4,\lambda_1)$. □

Lemma 5.10 and Lemma 5.11 together complete the case for $n=3$ and we have

Theorem 5.12. *Necessary conditions are sufficient for the existence of a GDD$(1,3,3,4;\lambda_1,\lambda_2)$.*

5.6. $n=4$

Example 5.13. *A GDD$(1,4,4,4;2,3)$ with $G_1=\{x\}$, $G_2=\{a,b,c,d\}$ and $G_3=\{1,2,3,4\}$. The blocks are given below in columns.*

x	x	x	x	x	x	x	x	a	c	a	c	a	b	a	b
a	d	b	c	b	a	a	b	b	d	b	d	c	d	c	d
1	1	2	2	c	d	d	c	1	1	3	3	1	1	2	2
4	4	3	3	1	2	3	4	2	2	4	4	3	3	4	4

For $n=4$, from Theorem 3.3, $\lambda_2\le\frac{3}{2}\lambda_1$. Hence, we have the following corollary:

Corollary 5.14. *A necessary condition for the existence of a GDD$(1,4,4,4;\lambda_1,\lambda_2)$ is $\lambda_1\ge\frac{2}{3}\lambda_2$.*

From Table 1, we need to construct two families: GDD$(1,4,4,4;\lambda_1,6s)$ and GDD$(1,4,4,4;\lambda_1,6s+3)$ where s and λ_1 are nonnegative integers. For the first family, by Corollary 5.14, $\lambda_1\ge 4s$. Using $2s$ copies

of the GDD$(1,4,4,4;$
$2,3)$ and λ copies of BIBD$(4,4,1)$, we have a GDD$(1,4,4,4;4s+\lambda,6s)$, for any λ.

To construct a GDD$(1,4,4,4;\lambda_1,6s+3)$, we observe that by Corollary 5.14, $\lambda_1 \geq 4s+2$. Hence using $2s+1$ copies of the GDD$(1,4,4,4;2,3)$ and λ copies of a BIBD$(4,4,1)$, we have GDD$(1,4,4,4;4s+2+\lambda,6s+3)$, for any nonnegative integer λ. Hence we have:

Theorem 5.15. *Necessary conditions are sufficient for the existence of a GDD*$(1,4,4,4;\lambda_1,\lambda_2)$.

6. Difference families constructions

The aim of this section is to construct some examples of GDDs with a difference family. In the process we make some comments to show sufficiency in certain cases.

6.1. $n = 5$

Recall for $n = 5$, from the necessary conditions, $\lambda_2 \equiv 0 \pmod 6$. So let $\lambda_2 = 6t$ where t is a nonnegative integer. Now from the necessary condition, we have $\lambda_1 < \lambda_2 < 2\lambda_1$. Let $\lambda_2 - \lambda_1 = 3s$. Then using $\lambda_2 < 2\lambda_1$, $3s < \lambda_1$. If smallest possible value of λ_1 is $3s+3$, then $\lambda_2 = 6s+3$ not 0 mod 6. Hence the smallest possible value of λ_1 has to be $3s+6$ and $\lambda_2 = 6s+6$. For example, when $s = 1$ the smallest possible GDD will be GDD$(1,5,5,4;9,12)$.

We present a difference family construction for GDD$(1,5,5,4;9,12)$.

Example 6.1. *Let the groups be* $G_1 = \{\infty\}$, $G_2 = \{1,3,5,7,9\}$ *and* $G_3 = \{0,2,4,6,8\}$.
Difference family is $\{\{\infty,0,1,3\},\{\infty,0,1,4\},\{\infty,0,3,4\},\{\infty,0,3,6\},\{0,5,1,2\},\{0,5,1,3\},\{0,5,1,4\},\{0,5,2,3\},\{0,5,2,4\},\{0,5,3,4\}\}$.

Hence, we also have GDD$(1,5,5,4;6m+9t,6m+12t)$ *for nonnegative integers* m *and* t.

Theorem 6.2. *The necessary conditions for the existence of a GDD*$(1,5,5,4;6t,6s)$ *are sufficient for* $t \geq s$ *and the necessary conditions for the existence of a GDD*$(1,5,5,4;\ 6t+3,6s)$ *are sufficient for* $t \geq s-1$

On the other hand, let $\lambda_1 - \lambda_2 = 3s$, as $\lambda_2 \equiv 0 \pmod 6$, a GDD$(1,5,5,4;6t,6t)$ and a BIBD$(5,4,3)$ exist, hence a GDD$(1,5,5,4;6t+3s,6t)$ exists.

Theorem 6.3. *The necessary conditions for the existence of a GDD*$(1,5,5,4;\lambda_1,\lambda_2)$ *are sufficient for* $\lambda_1 \geq \lambda_2$.

6.2. $n = 6$

Let $G_1 = \{\infty\}$, $G_2 = \{1,3,5,7,9,11\}$ and $G_3 = \{0,2,4,6,8,10\}$ be groups.

Example 6.4. *The following multiset* $\{\{\infty,0,2,4\},\{0,1,4,5\},\{0,2,7,8\},\{0,2,4,6\},\{0,3,6,9\},\{0,3,6,9\}\}$ *is a difference family for a GDD*$(1,6,6,4;6,3)$. *Note that* $\{0,3,6,9\}$ *is a short difference set and gives only three blocks. These bocks cover difference* 3 *and* 6 *pairs only once. Hence, we also have a GDD*$(1,6,6,4;m+6t,m+3t)$ *for any nonnegative integers* m *and* t.

Example 6.5. *The following multiset* $\{\{\infty,0,1,5\},\{\infty,0,2,3\},\{\infty,0,3,4\},\{0,1,2,5\},\{0,1,4,5\},\{0,1,7,10\},\{0,2,7,8\},\{0,3,5,10\},\{0,3,6,9\},\{0,3,6,9\}\}$ *is a difference family for a GDD*$(1,6,6,4;\ 6,9)$. *Hence, we also have a GDD*$(1,6,6,4;m+6t,m+9t)$ *for any nonnegative integers* m *and* t.

Example 6.6. *A GDD*$(1,6,6,4;4,1)$ *can be constructed by difference family:* $\{\{\infty,0,\ 4,8\},\{0,1,4,6\},\{0,2,4,6\}\}$.

Example 6.7. *The difference family for a GDD*$(1,6,6,4;5,2)$ *is* $\{\{\infty,0,4,8\},\{\infty,0,4,8\},\{0,1,6,7\},$ $\{0,2,4,5\},\{0,2,4,10\},\{0,3,6,9\}$ *where* $G_1=\{\infty\}$, $G_2=\{1,3,5,7,9,11\}$ *and* $G_3=\{0,2,4,6,8,10\}$.

So we have a GDD$(1,6,6,4;3t+1,3s+1)$ for $t>s+1$, GDD$(1,6,6,4;3t+2,3s+2)$ for $t>s+1$, GDD$(1,6,6,4;6t+1,6s+1)$, GDD$(1,6,6,4;6t+1,6s+1)$ and GDD$(1,6,6,4;6t+3,6s+3)$ for $t\geq s$.

6.3. $n = 10$

Example 6.8. $G_1=\{\infty\}$, $G_2=\{1,3,5,...,19\}$ *and* $G_3=\{0,2,4,...,18\}$.

The difference family $\{\{\infty,0,3,6\},\{0,1,3,7\},\{0,1,8,9\},\{0,5,7,16\},\{0,5,10,15\},\ \{0,5,10,15\}\}$ *provides a GDD*$(1,10,10,4;2,3)$.

7. Summary

We used interesting construction techniques to construct specific examples for GDDs and obtained an important general construction for GDDs with three groups of sizes $1,n,n$ with block size 4. We obtained necessary conditions for the existence of these GDDs and proved that they are sufficient for specific values of n, specific values of λ_1 and for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for $n\equiv 0,1,4,5,8,9 \pmod{12}$ when $\lambda_1\geq\lambda_2$. The work leads to several open problems including questions on the existence of unknown families of GDDs.

Acknowledgment: The first author thanks Professor Jurisich for support through departmental research funds. The second author was supported by Adama Science and Technology University research grant number ASTU/SP-R/011/19.

References

[1] C. J. Colbourn, D. G. Hoffman, R. Rees, A new class of group divisible designs with block size three, J. Combin. Theory Ser. A 59(1) (1992) 73–89.
[2] H. L. Fu, C. A. Rodger, Group divisible designs with two associate cases: $n=2$ or $m=2$, J. Combin. Theory Ser. A 83(1) (1998) 94–117.
[3] H. L. Fu, C. A. Rodger, D. G. Sarvate, The existence of group divisible designs with first and second associates having block size three, Ars Combin. 54 (2000) 33–50.
[4] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 225–369.
[5] W. Lapchinda, N. Punnim, N. Pabhapote, GDDs with two associate classes with three groups of sizes 1, n and n, Australas. J. Combin. 58(2) (2014) 292–303.
[6] C. C. Lindner, C. A. Rodger, Design theory, 2nd Edition, Chapman & Hall/CRC, New York (2008).
[7] N. Pabhapote, Group divisible designs with two associate classes and with two unequal groups, Int. J. Pure Appl. Math. 81(1) (2012) 191–198.
[8] N. Pabhapote, N. Punim, Group divisible designs with two associate classes and $\lambda_2=1$, Int. J. Math. Sci. (2011) 1–10.
[9] A. Sakda, C. Uiyyasathian, Group divisible designs GDD$(n,n,n,1;\lambda_1,\lambda_2)$, Australas. J. Comb. 69(1) (2017) 18–28.