The bicyclic semigroup as the quotient inverse semigroup by any gauge inverse submonoid

Emil Daniel Schwab

Abstract: Every gauge inverse submonoid (including Jones-Lawson’s gauge inverse submonoid of the polycyclic monoid \(P_n\)) is a normal submonoid. In 2018, Alyamani and Gilbert introduced an equivalence relation on an inverse semigroup associated to a normal inverse subsemigroup. The corresponding quotient set leads to an ordered groupoid. In this note we shall show that this ordered groupoid is inductive if the normal inverse subsemigroup is a gauge inverse submonoid and the corresponding quotient inverse semigroup by any gauge inverse submonoid is isomorphic either to the bicyclic semigroup or to the bicyclic semigroup with adjoined zero.

2010 MSC: 20M18, 20L05

Keywords: Inverse semigroup, Ordered groupoid, Gauge inverse submonoid, Bicyclic semigroup

1. Introduction

An equivalence relation \(\simeq_N\) on an inverse semigroup \(S\) associated to a normal inverse subsemigroup \(N\) is introduced in [1]. Usually, it is not a congruence on \(S\). Following [1] the quotient set \(S/\simeq_N\) (also denoted by \(S/N\)) leads to an ordered groupoid [1, Theorem 3.6]. If this ordered groupoid is inductive then the set of all morphisms, that is \(S/N\), equipped with the "pseudoproduct" \(\otimes\) ([3, page 112]) forms an inverse semigroup (see [3, Proposition 4.1.7 (1)]), and we say, by abuse of language (since \(\simeq_N\) is not necessary a congruence), that this inverse semigroup \((S/N, \otimes)\) is the quotient inverse semigroup of \(S\) by the normal inverse subsemigroup \(N\).

The gauge inverse monoid \(G_M\) is a special submonoid of such a combinatorial bisimple (0-bisimple) inverse monoid \(S(M)\) for which the submonoid \(M\) of right units is an \(\ell\)-RILL monoid (see [5]). Any gauge inverse submonoid is normal ([5, Proposition 5.6]). Jones-Lawson’s gauge inverse monoid is the gauge inverse submonoid (denoted by \(G_n\)) of the polycyclic monoid \(P_n\) ([2, Section 3]).

Emil Daniel Schwab; Department of Mathematical Sciences, University of Texas at El Paso, 500 W. University Ave, El Paso, Texas 79968-0514, USA (email: eschwab@utep.edu).
The case of the polycyclic monoid $P_n$ is examined in Example 3.11 from [1]. The conclusion of this examination is that $P_n//G_n$ is isomorphic to the Brandt semigroup on the set of non-negative integers. In fact the product $\langle [(u, v)]_{G_n}[(s, t)]_{G_n} = [(u, t)]_{G_n}\rangle$ considered at the end of Section 3 in [1] is the composition of two morphisms (if it is defined) in the corresponding ordered groupoid and it is not the pseudoproduct $\otimes$ which defines the quotient inverse semigroup $P_n//G_n$.

The aim of this note is to show that for any gauge inverse submonoid $G_M$, the quotient inverse semigroup $(S(M)//G_M, \otimes)$ is isomorphic either to the bicyclic semigroup $B$ or to the bicyclic semigroup with adjoined zero $B^0$.

In the next section, we will survey the background results, particularly from [3] (Subsection 2.1), [1] (Subsection 2.2) and [5] (Subsection 2.3), needed to understand this paper. The symbol $\circ$ is used only for composition (from right to left) of two morphisms.

2. Background. Ordered groupoids, normal inverse subsemigroups and gauge inverse submonoids

2.1. Ordered groupoids

A groupoid $\mathcal{G}$ is a small category in which every morphism is an isomorphism, meaning that for any morphism $f : X \to Y$ there is a morphism $f^{-1} : Y \to X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$, where $1_X$ and $1_Y$ are the identity morphisms of $X$ and $Y$, respectively. A groupoid $\mathcal{G}_X$ is said to be connected simple system on the set $X$ (or simplicial groupoid on $X$) if the set of objects $\text{Ob}\mathcal{G}_X = X$ and there is exactly one morphism between any two objects. We call the groupoid $\mathcal{G}_X$ obtained from $\mathcal{G}_X$ by adjoining an extra object 0 such that the set of morphisms from $X$ to $Y$ is empty if either $X = 0, Y \neq 0$ or $X \neq 0, Y = 0$ and it is a singleton if $X = Y = 0$, the connected simple system with adjoined 0.

A groupoid $\mathcal{G}$ is said to be ordered if the set of all morphisms $\text{Mor}(\mathcal{G})$ of $\mathcal{G}$ is equipped with a partial order $\leq$ such that:

- $(O_1)$ $f \leq g$ implies $f^{-1} \leq g^{-1}$;
- $(O_2)$ If $f \leq g$, $f' \leq g'$ and $f \circ f'$ and $g \circ g'$ are defined then $f \circ f' \leq g \circ g'$;
- $(O_3)$ If $1_Z \leq 1_X$ and $f : X \to Y$ then there exists a unique morphism $f|_Z : Z \to \bullet$ called the restriction of $f$ to $Z$ such that $f|_Z \leq f$;
- $(O_4)$ If $1_Z \leq 1_Y$ and $f : X \to Y$ then there exists a unique morphism $f|Z : \bullet \to Z$ called the corestriction of $f$ to $Z$ such that $f|Z \leq f$.

The axiom $(O_4)$ is a consequence of axioms $(O_1) - (O_3)$.

An inverse semigroup $S$ (i.e. a semigroup $S$ in which every element $s \in S$ has a unique inverse $s^{-1} \in S$ in the sense that $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$) can be considered as an ordered groupoid $\mathcal{G}(S)$ in which the set of objects is the set of idempotents $E(S)$ of $S$, the set of morphisms from $e$ to $f$ is the set $\{s \in S|s^{-1}s = e$ and $ss^{-1} = f\}$ and the composition $s \circ t$ of two morphisms $s$ and $t$

$$t^{-1}t \xrightarrow{=} tt^{-1} = s^{-1}s \xrightarrow{=} ss^{-1}$$

is the usual product $st$ in $S$ (i.e., the composition is just the restriction of the multiplication of $S$ to composable pairs). The partial order on the set of all morphisms of $\mathcal{G}(S)$ is the natural partial order $\leq$ on the inverse semigroup $S$, i.e. $s \leq t \iff s = ss^{-1}t$ (or equivalently $s = ts^{-1}s$). In the ordered groupoid $\mathcal{G}(S)$ the partially ordered set of identities forms a meet-semilattice. If $S$ is the Brandt semigroup $B_\omega$, whose set of elements is $\{(m, n) | m, n \in \omega = \{0, 1, 2, \cdots\}\} \cup \{0\}$ with the multiplication defined by:

$$(m, n) \cdot (m', n') = \begin{cases} (m, n') & \text{if } n = m' \\
0 & \text{if } n \neq m' \end{cases} \text{ and } 0 \cdot (m, n) = (m, n) \cdot 0 = 0 \cdot 0 = 0,$$
then $\mathcal{G}(B_s)$ is category isomorphic to the connected simple system with adjoined $0$: $G^0$. But $\mathcal{G}(B_s)$ is an ordered groupoid and the order $\leq_{B_s}$ on $\mathcal{G}(B_s)$ (that is the natural partial order on $B_s$) induces a partial order $\leq_{B_s}$ on $\text{Mor}(G^0)$ given by: $1_0 \leq_{B_s} f$ for all $f \in \text{Mor}(G^0)$, and $f \leq_{B_s} g$ iff $f = g$, otherwise. Note that $G^0_s$ (and $\mathcal{G}_s$) can be equipped as an ordered groupoid in many other ways.

Now, an ordered groupoid in which the set of identities forms a meet-semilattice (like in the case of the ordered groupoids $\mathcal{G}(S)$) is called inductive. If $f : X \to Y$ and $f' : X' \to Y'$ are two morphisms of an inductive groupoid $\mathcal{G}$ and $1_X \wedge 1_{X'} = 1_Z$ then the pseudoproduct $\otimes$:

$$f \otimes f' = f|_Z \circ f'|_Z$$

defines a binary operation on the set $\text{Mor}(\mathcal{G})$ such that $(\text{Mor}(\mathcal{G}), \otimes)$ is an inverse semigroup ([3, Proposition 4.1.7 (1)]). Note that if we denote this semigroup by $S(\mathcal{G})$, then $S(\mathcal{G}(S)) = S$ ([3, Proposition 4.1.7 (3)]), $S(\mathcal{G}(G \leq)) = (G, \leq)$ ([3, Proposition 4.1.7 (2)]), and $S(G^0, \leq) \cong B_s$ only if $s$ is the induced order $\leq_{B_s}$ on $\text{Mor}(G^0)$ considered above.

### 2.2. Normal inverse subsemigroup and the corresponding ordered groupoid

An inverse subsemigroup $N$ of an inverse semigroup $S$ is called normal if $E(S) = E(N)$ and if $s^{-1}Ns \subseteq N$ for all $s \in S$. A normal inverse subsemigroup $N$ of an inverse semigroup $S$ together with the defining concepts ($\leq$ and $\circ$) of the ordered groupoid $\mathcal{G}(S)$ determine a preorder $\leq_N$ on $S = \text{Mor}(\mathcal{G}(S))$, as follows:

$$s \leq_N t \iff$$

there exist two morphisms $a, b$ of $\mathcal{G}(S)$ such that $a, b \in N$, the compositions $a \circ s$ and $s \circ b$ are both defined, and

$$a \circ s \circ b \leq t.$$

Since $\leq_N$ is a preorder on the set $S$ then it defines an equivalence relation $\simeq_N$ on $S$ by $s \simeq_N t \iff s \leq_N t$ and $t \leq_N s$, and a partial order on the set of equivalence classes $S/ \simeq_N$. In [1] this quotient set is denoted by $S/N$ and the $\simeq_N$-class of $s \in S$ by $[s]_N$. The equivalence relation $\simeq_N$ needs not be a congruence on $S$. However, the quotient set $S/N$ leads us to an ordered groupoid $\overline{\mathcal{G}}(S/N)$: the objects are the classes $[e]_N$ where $e \in E(S)$, and $\text{Mor}(\overline{\mathcal{G}}(S/N)) = S/N$ with $[s]_N$ being a morphism from $[s^{-1}]_N$ to $[ss^{-1}]_N$. The composition of two morphisms $[s]_N \circ [t]_N$ (if $[s^{-1}]_N = [tt^{-1}]_N$) is given by $[s]_N \circ [t]_N = [sat]_N$, where $a \in N$ such that $a^{-1}a = tt^{-1}$ and $aa^{-1} \leq s^{-1}s$; and $[s]_N \leq_N [t]_N$ if $s \leq_N t$, is the partial order of $\overline{\mathcal{G}}(S/N)$. Now, if this ordered groupoid $\overline{\mathcal{G}}(S/N)$ is inductive then $S/N = \text{Mor}(\overline{\mathcal{G}}(S/N))$ forms an inverse semigroup $(S/N, \circ)$ (where $\circ$ is the pseudoproduct) called here the quotient inverse semigroup of $S$ by the normal inverse subsemigroup $N$.

### 2.3. Gauge inverse submonoids

Following [5], a nontrivial right cancellative monoid $M$ is a RILL monoid if $1_M$ is indecomposable and any two elements $s, t \in M$ that admit a common left multiple admit at least common left multiple $s \lor t$. In the RILL monoid $M$, we shall denote $s \ll t$ if $t$ is a left multiple of $s$, $t = rs$, and by $\frac{t}{s}$ the "left quotient" $r$. Since $M$ is right cancellative and 1 is indecomposable, the *right divisibility* relation $\ll$ is a partial order on $M$. A length function on the RILL monoid $M$ is a monoid homomorphism $\ell : M \to (\mathbb{N}, +)$ such that $\ell^{-1}(0) = 1_M$. A non-trivial monoid with a length function is atomic (every non-units element is a product of finitely many atoms). A length function $\ell$ is said to be normalized if $\ell(s) = 1 \iff s$ is an atom. An $\ell$-RILL monoid is a RILL monoid equipped with a normalized length function $\ell$.

If $M$ is an $\ell$-RILL monoid then the set $\mathbb{S}(M) = \{ M \times M \mid \text{if } Ms \cap Mt \neq \emptyset \text{ for any } s, t \in M \}$, 

\begin{align*}
\mathbb{S}(M) = & \{ (M \times M) \cup \{ \emptyset \} \mid \text{if there exist } s, t \in M \text{ such that } Ms \cap Mt = \emptyset \}
\end{align*}
(that is $M \times M$, adjoining an extra element $\theta$ if necessary), together with the product $\odot$ defined by
\[
(s, t) \odot (s', t') = \begin{cases} 
\frac{\ell(s)}{\ell(t)} s \cdot \frac{\ell(s')}{\ell(t')} t' & \text{if } t \text{ and } s' \text{ admit a common left multiple} \\
\theta & \text{otherwise}
\end{cases}
\]
and
\[
\theta \odot (s, t) = (s, t) \odot \theta = \theta \odot \theta = \theta
\]
is an inverse monoid (the inverse of $(s, t)$ is $(t, s)$; the element $(s, t)$ is an idempotent if and only if $s = t$, and $(1_M, 1_M)$ is the identity element). The submonoid of $\mathbb{S}(M)$:
\[
G_M = \mathbb{S}(M) = \left\{(s, t) \in M \times M \mid \ell(s) = \ell(t)\right\} \quad \text{if } \mathbb{S}(M) = M \times M
\]
\[
\mathbb{S}(M) = \left\{(s, t) \in M \times M \mid \ell(s) = \ell(t)\right\} \cup \{\theta\} \quad \text{if } \mathbb{S}(M) = (M \times M) \cup \{\theta\}
\]
is the gauge inverse submonoid of $\mathbb{S}(M)$ induced by the $\ell$-RILL monoid $M$. This submonoid of $\mathbb{S}(M)$ is a normal submonoid ([5, Proposition 5.6]).

In [5] the first example of a gauge inverse submonoid is the submonoid of idempotents $E(B)$ of the bicyclic semigroup $B$. The bicyclic semigroup $B$ is the monoid of all pairs of non-negative integers equipped with the multiplication defined by:
\[
(m, n) \cdot (m', n') = \begin{cases} 
(m, n + n') & \text{if } n \geq n' \\
(m + m', n') & \text{if } n \leq n'.
\end{cases}
\]
In this paper $(B^0, \cdot)$ denotes the bicyclic semigroup with adjoined zero 0.

3. Main results. The quotient inverse monoid $\mathbb{S}(M)/G_M$

Let $M$ be an $\ell$-RILL monoid and $(\mathbb{S}(M), \odot)$ the corresponding inverse monoid.

**Proposition 3.1.** The natural partial order $\leq$, the preorder $\leq_{G_M}$ and the equivalence relation $\simeq_{G_M}$ on $\mathbb{S}(M)$ are given by:

(i) $(s, t) \leq (s', t') \iff s' \leq_{\mathbb{S}(M)} s$, $t' \leq_{\mathbb{S}(M)} t$ and $\frac{s}{t} = \frac{s'}{t'}$ ([4, Proposition 2.6 (1)])

(ii) $(s, t) \leq_{G_M} (s', t') \iff$ there exists $(p, q) \in \mathbb{S}(M)$ such that $\ell(p) = \ell(s)$, $\ell(q) = \ell(t)$ and $(p, q) \leq (s', t')$

(iii) $(s, t) \simeq_{G_M} (s', t') \iff \ell(s) = \ell(s')$ and $\ell(t) = \ell(t')$

(if $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$ then the $\simeq_{G_M}$-class $[\theta]_{G_M}$ is a singleton).

**Proof.** (i). We have

\[
(s, t) \leq (s', t') \iff (s, t) = (s, t) \odot (t, s) \odot (s', t') \iff (s, t) = (s, s) \odot (s', t') \iff
\]

\[
(s, t) = \left(\frac{s \vee s'}{\triangleright s}, \frac{s \vee s'}{\triangleright s'} t'\right) \iff (s, t) = (s \vee s', \frac{s \vee s'}{\triangleright s'} t') \iff s' \leq_{\mathbb{S}(M)} s \text{ and } \frac{s}{\triangleright s'} t' = t
\]

\[
\iff s' \leq_{\mathbb{S}(M)} s, t' \leq_{\mathbb{S}(M)} t \text{ and } \frac{s}{\triangleright s'} = \frac{t}{\triangleright t'}.
\]
(ii). We have
\[(s, t) \leq_{G_M} (s', t') \iff \text{there exist } (p, u), (v, q) \in G_M \text{ such that}
\]
\[\begin{align*}
(p, u)^{-1} \circ (p, u) &= (s, t) \circ (s, t)^{-1}, \\
(s, t)^{-1} \circ (s, t) &= (v, q) \circ (v, q)^{-1}
\end{align*}
\]
\[\text{and } (p, u) \circ (s, t) \circ (v, q) \leq (s', t').
\]
Since
\[(p, u)^{-1} \circ (p, u) = (u, u) \text{ and } (s, t) \circ (s, t)^{-1} = (s, s),
\]
it follows \(u = s\). Analogously, \(v = t\). Now, we have:
\[(p, u) \circ (s, t) \circ (v, q) = (p, s) \circ (s, t) \circ (t, q) = (p, q)
\]
and taking into account that \((p, s), (t, q) \in G_M\) we obtain:
\[(s, t) \leq_{G_M} (s', t') \iff \text{there exist } p, q \in M \text{ such that}
\]
\[\ell(p) = \ell(s), \ell(q) = \ell(t) \text{ and } (p, q) \leq (s', t').
\]
(iii). The assertion follows from (i) and (ii).

**Remark 3.2.** The equivalence relation \(\simeq_{G_M}\) is not necessarily a congruence on \(S(M)\). For example, if \(M\) is the multiplicative \(\ell\)-RILL monoid of positive integers \((\mathbb{Z}^+, \cdot)\) ([5, Example 4.2]), where \(\ell(1) = 0\) and \(\ell(n) = \text{the total number of prime divisors of } n\) counted with their multiplicities if \(n > 1\), then \(S(\mathbb{Z}^+)\) is the multiplicative analogue of the bicyclic semigroup:
\[S(\mathbb{Z}^+) = \mathbb{Z}^+ \times \mathbb{Z}^+; \quad (m, n) \cdot (m', n') = \left(\frac{nm'}{n}, \frac{mn'}{m'}\right),
\]
\([n, m']\) being the least common multiple of \(n\) and \(m'\). Now, if \(p\) and \(q\) are two distinct primes then \((p, q) \simeq_{G_M} (p, q)\) and \((p, q) \simeq_{G_M} (q, p)\) (since \(\ell(p) = \ell(q) = 1\)), but \((p, p) \cdot (p, q) = (p^2, q)\) and \((p, q) \cdot (p, p) = (p, q)\), that is \((p, q) \cdot (p, q) \neq G_M (p, q) \cdot (q, p)\). Thus \(\simeq_{G_M}\) is not a congruence on the multiplicative analogue of the bicyclic semigroup.

The \(\simeq_{G_M}\)-class
\[[(s, t)]_{G_M} = \{(u, v) \in S(M) \mid \ell(u) = \ell(s) \text{ and } \ell(v) = \ell(t)\}
\]
is a morphism in the ordered groupoid \(G(S(M) \parallel G_M)\) from \([(t, t)]_{G_M}\) to \([(s, s)]_{G_M}\). If \([(s, t)]_{G_M}\) and \([(s', t')]_{G_M}\) are two morphisms of \(G(S(M) \parallel G_M)\) such that \(\ell(s') = \ell(t)\) (that is \([(s', t')]_{G_M} = [(t, t)]_{G_M}\),
\[[(t', t')]_{G_M} \xrightarrow{[(s', t')]_{G_M}} [(s', s')]_{G_M} = [(t, t)]_{G_M} \xrightarrow{[(s, t)]_{G_M}} [(s, s)]_{G_M},
\]
then the composition of these two morphisms, \([(s, t)]_{G_M} \circ [(s', t')]_{G_M}\) is given by
\[[(s, t)]_{G_M} \circ [(s', t')]_{G_M} = [(s, t) \circ (a, b) \circ (s', t')]_{G_M},
\]
where \((a, b) \in G_M\) such that \((a, b)^{-1} \circ (a, b) = (s', t') \circ (s', t')^{-1}\) and \((a, b) \circ (a, b)^{-1} \leq (s, t)^{-1} \circ (s, t)\). We choose \((a, b) = (t, s')\) which is an element of \(G_M\) since \(\ell(t) = \ell(s')\). Thus the composition \([(s, t)]_{G_M} \circ [(s', t')]_{G_M}\) in \(G(S(M) \parallel G_M)\) such that \(\ell(t) = \ell(s')\) is given by:
\[[(s, t)]_{G_M} \circ [(s', t')]_{G_M} = [(s, t) \circ (t, s') \circ (s', t')]_{G_M} = [(s, t')]_{G_M}.
\]
The ordering $\preceq_{G_M}$ of $\simeq_{G_M}$-classes in the ordered groupoid $\mathcal{G}(\mathcal{S}(M)/G_M)$ is given by:

$$[(s,t)]_{G_M} \preceq_{G_M} [(s',t')]_{G_M} \iff \text{there exists } (p,q) \in [(s,t)]_{G_M} \text{ such that } s' \preceq p, \ t' \preceq q \text{ and } \frac{p}{\triangleright s'} = \frac{q}{\triangleright t'}.$$ 

and

$[\theta]_{G_M} \preceq_{G_M} [x]_{G_M}$ for any morphism $[x]_{G_M}$ of $\mathcal{G}(\mathcal{S}(M)/G_M)$ if $\mathcal{S}(M) = (M \times M) \cup \{\theta\}$.

**Remark 3.3.** The objects of $\mathcal{G}(\mathcal{S}(M)/G_M)$ other than $[\theta]_{G_M}$ (that is the $\simeq_{G_M}$-classes $[(s,s)]_{G_M}$) can be indexed by non-negative integers (namely $[(s,s)]_{G_M}$ by $\ell(s)$), then the set of morphisms from $m$ to $n$ is a singleton (for any pair $(m,n)$ of non-negative integers) and, it goes without saying the composition of two morphisms.

It follows:

**Theorem 3.4.** The (ordered) groupoid $\mathcal{G}(\mathcal{S}(M)/G_M)$ is category isomorphic either to the connected simple system $\mathcal{G}_N$ (if $\mathcal{S}(M) = M \times M$) or to the connected simple system with adjoined 0: $\mathcal{G}_{N0}^+$ (if $\mathcal{S}(M) = (M \times M) \cup \{\theta\}$).

**Theorem 3.5.** The ordered groupoid $\mathcal{G}(\mathcal{S}(M)/G_M)$ is inductive.

**Proof.** It is straightforward to see that in the set of identities of $\mathcal{G}(\mathcal{S}(M)/G_M)$ we have:

$$[(s,s)]_{G_M} \preceq_{G_M} [(t,t)]_{G_M} \iff \ell(t) \leq \ell(s).$$

It follows that the partially ordered set of identities of $\mathcal{G}(\mathcal{S}(M)/G_M)$ forms a meet-semilattice:

$$[(s,s)]_{G_M} \land [(t,t)]_{G_M} = \begin{cases} [(s,s)]_{G_M} & \text{if } \ell(s) \geq \ell(t) \\ [(t,t)]_{G_M} & \text{if } \ell(s) \leq \ell(t) \end{cases}$$

and

$[\theta]_{G_M} \land [x]_{G_M} = [\theta]_{G_M}$ for any identity morphism $[x]_{G_M}$ of $\mathcal{G}(\mathcal{S}(M)/G_M)$ if $\mathcal{S}(M) = (M \times M) \cup \{\theta\}$.

Therefore the ordered groupoid $\mathcal{G}(\mathcal{S}(M)/G_M)$ is inductive.

**Theorem 3.6.** The corresponding inverse semigroup $(\mathcal{S}(M)/G_M, \odot)$ is isomorphic either to the bicyclic semigroup $(B^0, \cdot)$ (if $\mathcal{S}(M) = M \times M$) or to the bicyclic semigroup with adjoined zero $(B^0, \cdot)$ (if $\mathcal{S}(M) = (M \times M) \cup \{\theta\}$).

**Proof.** Let $[(s,t)]_{G_M}, [(s',t')]_{G_M} \in \mathcal{S}(M)/G_M$. As morphisms of $\mathcal{G}(\mathcal{S}(M)/G_M)$, we have:

$$[(s,t)]_{G_M} : [(t,t)]_{G_M} \rightarrow [s,s]_{G_M} \quad \text{and} \quad [(s',t')]_{G_M} : [(t',t')]_{G_M} \rightarrow [s',s']_{G_M}.$$ 

Since

$$[(t,t)]_{G_M} \land [(s',s')]_{G_M} = \begin{cases} [(t,t)]_{G_M} & \text{if } \ell(t) \geq \ell(s') \\ [(s',s')]_{G_M} & \text{if } \ell(t) \leq \ell(s') \end{cases},$$

we shall consider two cases:
Remark 3.7. Two categories (since both are category isomorphic to the connected simple system with adjoined gauge inverse submonoid is the semilattice of idempotents. The ordered groupoid $G$ is isomorphic to the ordered groupoid $S$ if the monoid $G$ is isomorphic to the ordered groupoid $S$ due to the two partial orders $\vartriangleleft$ and of $G$ to $S$. The corestriction $\ell(\theta) = \ell(s) - \ell(t) + \ell(s)$.

In this case, $\ell(y) = \ell(t) - \ell(s) + \ell(t')$, since $[(t,y)]_{G_M} \cong [(s',t')]_{G_M}$.

2) $[(t,t)]_{G_M} \wedge [(s',s')]_{G_M} = [(s',s')]_{G_M}$. Then the restriction $[(s,t)]_{G_M} \wedge [(s',s')]_{G_M}$ of $[(s,t)]_{G_M}$ to $[(s',s')]_{G_M}$ is the morphism $[(s,t)]_{G_M} : [(s',s')]_{G_M} \to [(x,x)]_{G_M}$, where $x \in M$ such that $\ell(x) = \ell(s') - \ell(t) + \ell(s)$ since $[(x,x)]_{G_M} \leq [(s',s')]_{G_M}$. The corestriction $[(s,t)]_{G_M} \wedge [(s',s')]_{G_M}$ of $[(s,t)]_{G_M}$ to $[(s',s')]_{G_M}$ is just $[(s',t')]_{G_M}$. So, in this case, the product $[(s,t)]_{G_M} \wedge [(s',s')]_{G_M}$ is given by:

$$[(s,t)]_{G_M} \wedge [(s',s')]_{G_M} = [(s,t)]_{G_M} \wedge [(s',s')]_{G_M} = [(s',t')]_{G_M}.$$ (If $S(M) = (M \times M) \cup \{\theta\}$ then it is straightforward to check that $[(s,t)]_{G_M}$ is the zero element of $(S(M)/G_M, \wedge)$.)

Now, a careful examination shows that $\tilde{\ell} : (S(M)/G_M, \wedge) \to (B^\omega, \cdot)$ if $S(M) = (M \times M)$ defined by

$$\tilde{\ell}([(s,t)]_{G_M}) = (\ell(s), \ell(t))$$

is a monoid isomorphism.

Remark 3.7. What is happening if the $\ell$-RILL monoid $M$ is the additive monoid of non-negative integers? (that is if the monoid $(S(M)/G_M, \wedge)$ is the bicyclic semigroup $B^\omega$?) The gauge inverse submonoid of $B^\omega$ is the semilattice of idempotents $E(B)$. (If $E(B)$ is the trivial relation (the equality) on $B$ and of course $B = B$.) It is straightforward to check that $\approx_{E(B)}$ is the trivial relation (the equality) on $B$ (and $\overline{G}(B)/E(B) = G(B)$).

Now, since for any inverse semigroup $S$ the relation $\approx_{E(S)}$ is the trivial relation on $S$ (If $S(M) = (M \times M) \cup \{\theta\}$) it follows that

Corollary 3.8. The bicyclic semigroup is the only combinatorial bisimple inverse monoid for which the gauge inverse submonoid is the semilattice of idempotents.

Remark 3.9. The ordered groupoid $\overline{G}(S(M)/G_M)$ is isomorphic either to the ordered groupoid $G(B)$ or to the ordered groupoid $G(B^\omega)$. Of course, the groupoids $G(P_n//G_n)$ and $G(B_n)$ are also isomorphic as two categories (since both are category isomorphic to the connected simple system with adjoined 0: $G_n^0$), but they are not isomorphic as two ordered groupoids due to the two partial orders $\cong_{G_n}$ and $\leq_{B_n}$ on $G(P_n//G_n)$ and $G(B_n)$, respectively.
4. Supplements. The quotient group $S(M)/G_M$

If $\rho$ is a relation on an inverse semigroup $S$, the kernel $ker\rho$ is the set

$$ker\rho = \{s \in S \mid s \rho e \text{ for some } e \in E(S)\}.$$ 

If $\rho$ is a group congruence on $S$ then we agree to write $S/ker\rho$ for the quotient group $S/\rho$.

In what follows assume that $S(M) = M \times M$ (that is, $Ms \cap Mt \neq \emptyset$ for any $s, t \in M$). We have:

**Proposition 4.1.** The relation $\approx_M$ on $S(M)$ defined by

$$(x, y) \approx_M (x', y') \text{ if and only if } \ell(x) - \ell(y) = \ell(x') - \ell(y'),$$

is a group congruence on $S(M)$. The gauge inverse submonoid $G_M$ is the kernel of $\approx_M$, and it is the identity element of the quotient group $S(M)/G_M = S(M)/\approx_M$. This quotient group is isomorphic to the additive group of integers $(\mathbb{Z}, +)$.

**Proof.** The relation $\approx_M$ is an equivalence relation on $S(M)$. Obviously, $G_M$ is the kernel of $\approx_M$. If $(s, t) \approx_M (s', t')$ and $(u, v) \approx_M (u', v')$, then $(s, t) \circ (u, v) = (\ell(t) - \ell(s), \ell(u) - \ell(v))$. Then $(s', t') \circ (u', v') = (\ell(t') - \ell(s'), \ell(u') - \ell(v'))$ and

$$\ell(t \vee u) = \ell(t) + \ell(u) - \ell(t \wedge u) = \ell(t \vee u).$$

It follows that $\approx_M$ is a congruence relation on $S(M)$. The quotient monoid $S(M)/\approx_M$ is again an inverse monoid. Since $G_M$ is the only idempotent of $S(M)/\approx_M$ it follows that this inverse monoid is a group (the quotient group $S(M)/G_M$). The map $\overline{t} : S(M)/G_M \to \mathbb{Z}$ defined by

$$\overline{[x, y]} \in S(M)/\approx_M \quad \overline{t([x, y])} = \ell(x) - \ell(y)$$

is an isomorphism from the group $S(M)/G_M$ onto the additive group of integers $(\mathbb{Z}, +)$.

**Remark 4.2.** It is straightforward to see that the kernel of $\approx_{G_M}$ is also the gauge inverse submonoid $G_M$. However, the differences between the relations $\approx_{G_M}$ and $\approx_M$ are significant:

(a) in general, the equivalence relation $\approx_{G_M}$ is not a congruence on $S(M)$ (Remark 3.2), but $\approx_M$ is a group congruence on $S(M)$;

(b) the gauge inverse submonoid $G_M$ is not a $\approx_{G_M}$-equivalence class in $S(M)$, but it is an $\approx_M$-equivalence class in $S(M)$;

(c) there is not a $\approx_{G_M}$-equivalence class $[(s, t)]_{G_M}$ such that $E(S(M)) \subseteq [(s, t)]_{G_M}$, but the $\approx_M$-equivalence class $G_M$ contains the set of all idempotents of $S(M)$;

(d) the group $S(M)/G_M$ is equipped with the product $\odot$ via the inverse monoid $S(M)$; the product in the inverse monoid $S(M)/G_M$ is the pseudoproduct $\odot$ via the inductive groupoid $\overline{G}(S(M)/G_M)$;

(e) the following inclusion holds: $\approx_{G_M} \subset \approx_M$.

**Acknowledgment:** The author would like to thank the referee for helpful suggestions.
References