Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (2) (2024), 333 – 341
DOI : 10.15672/hujms.1112482

Research Article

On semi-cover-avoiding 2-maximal subgroups of finite groups

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Abstract

A subgroup H of a finite group G is said to be "semi-cover-avoiding in G", if there exists a chief series of G such that H covers or avoids every chief factor of the chief series. In this article, we will consider some 2-maximal subgroups with the property of semi-cover-avoiding of a group G and explore the structure of G.

Mathematics Subject Classification (2020). 20D10, 20D20

Keywords. semi-cover-avoiding proterty, soluble groups, maximal subgroups, 2-maximal subgroups, Sylow p-subgroups, p-solvable.

1. Introduction

All groups considered in this article will be finite and non-abelian. Our terminology and notation is standard and can be found in [4]. Let G be a group and H a subgroup of G. We use $|G|$ to denote the order of G and $\pi(G)$ denote the set of all primes dividing $|G|$. For every $p \in \pi(G)$, $|G|_p$ denotes the p-part of $|G|$. We write $M \lessdot G$ to express that M is a maximal subgroup of G. We use $\Phi(G)$ to denote the Frattini subgroup of G. We denote by $O_p(G)$ the product of all normal p-subgroups of G. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G. We denote by $Max(G,H)$ the set of all maximal subgroups M of G such that $H \leq M$. A subgroup H is called a 2-maximal subgroup if there exists $M \in Max(G,H)$ such that $H \lessdot M$. In particular, H is strictly 2-maximal subgroup if $H \lessdot M$ for all $M \in Max(G,H)$. For convenience, $Max(G)$ denotes the set of all maximal subgroups of G and $Max_2(G)$ denotes the set of all 2-maximal subgroups of G. We use $Max_2^*(G)$ to denote the set of all strictly 2-maximal subgroups of G. We use H_G to denote $\cap H^g$.

In the past, many scholars devoted themselves to explore the relationship between some 2-maximal subgroups of a finite group G and the structure of G. And they have got many meaningful results. One of the most classical results is due to B.Huppert. He [7] proved that if every 2-maximal subgroup of a group G is normal in G, the G is supersoluble.

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Received: 06.05.2022; Accepted: 23.04.2023

Guo [5] has characterized the solvable groups by it's 2-maximal subgroups with cover and avoidance properties. Obviously, cover and avoidance properties is some kind of normality. Inspired by this, Fan[3] first proposed the concept of semi cover-avoidance in 2006 and characterized the solvable groups by means of the maximal subgroups or Sylow subgroups. He proved that a group is solvable if and only if every maximal subgroup has semi cover-avoidance property.

At the same time, the development of the theory of formations of finite groups injected new vitality into the research on traditional group theory. In 2018, Miao [10] defined a class of groups $U_p^{\#}$:

$U_p^{\#} = \{G|\ H/K \le \Phi(G/K)\ or\ |H/K|_p \le p\ for\ every\ G\ chief\ factor\ H/K\}$

$U_p^{\#}$ contains not only all p-supersoluble groups but also part of non-solvable groups. With the deepening of research, we defined $U_{p^i}^{\#}$:

$U_{p^i}^{\#} = \{G|\ H/K \le \Phi(G/K)\ or\ |H/K|_p \le p^i\ for\ every\ G\ chief\ factor\ H/K\}$

It is noted that this class of groups is not a formation, because it only has the characteristics of quotient group inheritance.

First of all, we will continue Guo's work and characterize solvable groups by it's some 2-maximal subgroups with semi-cover-avoiding properties. Then we will give the semi-cover-avoiding properties to maximal subgroups of Sylow subgroups of G, and explore the structure of G on this basis.

Definition 1.1. Let G be a group and H a subgroup of G. We define

$T_1(G) = \{H|\ H \in Max_2(G),\ \forall M_1 \in Max(G,H)\ s.t.\ H_G = (M_1)_G\}$

$T_3(G) = \{H|\ H \in Max_2(G),\ \forall M_2 \in Max(G,H)\ s.t.\ H_G < (M_2)_G\}$

2. Preliminaries

Definition 2.1. Let G be a group and H be a subgroup of G, we call H is a 2-maximal subgroup of there exists a maximal subgroup M of G such that $H \lessdot M$.

Definition 2.2. [5, Definition 2.1] Let A be a subgroup of G and H/K a chief factor of G. We say that :

(1) A covers H/K if $HA = KA$;

(2) A avoids H/K if $H \cap A = K \cap A$;

(3) A has the cover and avoidance properties in G, in brevity, A is a CAP-subgroup of G, if A either covers or avoids every chief factor of G.

Definition 2.3. [6, Definition 2.2] Let H be a subgroup of a group G. H is said to be semi-cover-avoiding in G if there is chief series $1 = G_0 < G_1 < \cdots < G_t = G$ of G such that for every $j = 1, 2, \ldots, t$, either H covers G_j/G_{j-1} or H avoids G_j/G_{j-1}.

Lemma 2.4. [6, Lemma 2.6] *Let N be a normal subgroup of a group G and H a semi-cover-avoiding subgroup of G. Then HN/N is a semi-cover-avoiding subgroup of G/N if one of the following conditions holds:*

(1)$N \le H$;

(2)$gcd(|H|, |N|) = 1$, where $gcd(-,-)$ denotes the greatest common divisor.

Lemma 2.5. [6, Theorem 3.6] *If there is a 2-maximal subgroup L of G such that L is a solvable semi-cover-avoiding subgroup of G, then G is solvable.*

Lemma 2.6. [9, Theorem 2.4] *Let G be a group and H be a second maximal subgroup of G. If $H = 1$, then G is solvable.*

Lemma 2.7. [11, Lemma 2.13] *Let H be a second maximal subgroup of a group G and $X \in Max(G,H)$. Assume that N is a normal subgroup of G such that $N \le X$. If $N \nleq H$, then $X = HN$.*

Lemma 2.8. [4, lemma 2.3,4] *A subgroup H of a group G is a minimal supplement of N in G if and only if $HN = G$ and $H \cap N \leq \Phi(H)$.*

Lemma 2.9. [1, Theorem 2] *Let G be a finite group G such that, for all primes p, $N_G(P)$ is nilpotent where P is a Sylow p-subgroup of G. Then G is nilpotent.*

Lemma 2.10. [2, lemma 9.11] *Let K be a nilpotent normal subgroup of G and N a normal subgroup of G. If $N \leq K$ and $K/N \leq \Phi(G/N)$, then $K \leq \Phi(G)N$.*

Lemma 2.11. [8, Lemma 2.1] *If $G \neq 1$ is a group of nonprime order, then $Max_2^*(G) \neq \emptyset$.*

3. Main results

Theorem 3.1. *Let G be a group. If H is semi-cover-avoiding in G for every strictly 2-maximal subgroup H of G, then G is solvable.*

Proof. We assume that the result is not true and let G be a counterexample with minimal order. We will complete the proof in the following steps.

In this case, G is not a simple group. In fact, if G were simple, then $G/1$ would be the only chief factor of G. By Lemma 2.11, we know that $Max_2^*(G) \neq \emptyset$. Hence we can pick a 2-maximal subgroup $H_0 \in Max_2^*(G)$. By hypothesis, we have either $H_0G = H_0$ or $H_0 \cap G = 1$. Obviously, the former case is impossible. On the other hand, the latter case implies $H_0 = 1$. By Lemma 2.6, we get that G is solvable, a contradiction. Let L be a minimal normal subgroup of G, we consider the quotient group G/L. By Lemma 2.11, $Max_2^*(G/L) \neq \emptyset$. It is easy to see that $H_1 \in Max_2^*(G)$ for any 2-maximal subgroup $H_1/L \in Max_2^*(G/L)$. Therefore, by hypothesis, H_1 is a semi-cover-avoiding subgroup of G. Noticing that $L \leq H_1$, by Lemma 2.4, we can see that H_1/L is semi-cover-avoiding in G/L. By induction, we have that G/L is solvable. Since the class of solvable groups is a saturated formation, we get that L is a unique minimal normal subgroup of G.

Since L is a unique minimal normal subgroup of G and therefore L is contained in every chief series of G. Thus, for any 2-maximal subgroup $K \in Max_2^*(G)$, we have either $K \cap L = 1$ or $KL = K$. We get that $K \cong KL/L \leq G/L$ is solvable from the former case. Noticing that K is semi-cover-avoiding in G, by Lemma 2.5, we see that G is solvable, a contradiction. Hence, we have $KL = K$ for any 2-maximal subgroup $K \in Max_2^*(G)$, which means $\forall K \in Max_2^*(G)$, $L \leq K$.

Obviously, there exists a maximal subgroup M of G such that $L \nleq M$. Otherwise, $L \leq \Phi(G)$ and therefore L is solvable. Hence G is solvable, a contradiction. Let H^M be a maximal subgroup of M. We assert that $H^M \notin Max_2^*(G)$. If not, by the above discussion, we have $L \leq H^M \leq M$, a contradiction. Thus, there exists a 2-maximal subgroup $H^{M_0} \in Max_2^*(G)$ such that $H^M \lessdot \cdots \lessdot H^{M_0} \lessdot M_0 \lessdot G$. By the above discussion, we have $L \leq H^{M_0} \leq M_0$. Hence, by Lemma 2.7, $M_0 = LH^M$. Therefore, $H^{M_0} = H^{M_0} \cap M_0 = H^{M_0} \cap LH^M = L(H^{M_0} \cap H^M) = LH^M = M_0$, a contradiction. Now, our proof is complete. □

Corollary 3.2. [6, Theorem 3.5] *If every 2-maximal subgroup of a group G is a semi-cover-avoiding subgroup of G, then G is solvable.*

Theorem 3.3. *Let G be a group. If $T_1(G) \cup T_3(G) = \emptyset$, then G is solvable.*

Proof. We assume that the theorem is not true and let G be a counterexample with the minimal order.

We claim that G is not a simple group. If not, then for any 2-maximal subgroup H_0 of G, we have $H_0 \in T_1(G) \cup T_3(G)$, which contradicts $T_1(G) \cup T_3(G) = \emptyset$. Let L be a minimal normal subgroup of G, now we consider the quotient group G/L. We assert that $T_1(G/L) \cup T_3(G/L) = \emptyset$. If not, we can easily get $H_1 \in T_1(G) \cup T_3(G)$ for any 2-maximal subgroup $H_1/L \in T_1(G/L) \cup T_3(G/L)$, a contradiction. By using the induction, we get that G/L is solvable. Since the class of solvable groups is a saturated formation, we have that L is a unique minimal normal subgroup of G.

For any $p \in \pi(L)$, by using the Frattini argument, we get $G = LN_G(L_p)$ with $L_p \in Syl_p(L)$. If $N_G(L_p) = G$, then $L_p \trianglelefteq G$. Noticing that $L_p \le L$, by the minimality of normal subgroup L, we see that $L_p = L$. Thus, L is a p-group and therefore L is solvable. Further, G is solvable, a contradiction. Hence, $N_G(L_p) < G$ and thus there exists a maximal subgroup M of G such that $N_G(L_p) \le M$. It follows from $G = LN_G(L_p) \le LM \le G$ that $G = LM$. Obviously, $M_G = 1$. Otherwise, by the uniqueness of minimal normal subgroup L, we have $L \le M_G \le M$, which contradicts $G = LM$.

Since $M_G = 1$, for any maximal subgroup H of M, thus $H_G = 1$. It follows from $T_1(G) \cup T_3(G) = \emptyset$ that there exists a maximal subgroup $M_1 \in Max(G, H)$ such that $(M_1)_G > 1$. Again by the uniqueness of L, we have that $L \le (M_1)_G \le M_1$. It's easy to prove that $L \nleq H$. Then by Lemma 2.7, $M_1 = LH < G$, which implies that M is a minimal supplement of L in G. Thus, $L \cap M \le \Phi(M)$ by Lemma 2.8. Hence, $N_L(L_p) = L \cap N_G(L_p) \le L \cap M \le \Phi(M)$ is nilpotent. By the arbitrariness of p and Lemma 2.9, L is nilpotent. Hence, L is solvable. Further, G is solvable, a contradiction. Now, the proof is complete. □

Corollary 3.4. *Let G be a group. If G is not solvable, then $T_1(G) \cup T_3(G) \neq \emptyset$.*

Theorem 3.5. *Let G be a group. If H is semi-cover-avoiding in G for every 2-maximal subgroup $H \in T_1(G) \cup T_3(G)$, then G is solvable.*

Proof. We assume that the theorem is not true and let G be a counterexample with the minimal order. If $T_1(G) \cup T_3(G) = \emptyset$, by Theorem 3.3, G is solvable. Now we suppose that $T_1(G) \cup T_3(G) \neq \emptyset$. We will complete the proof in the following steps.

By using the arguments similar to the proof of Theorem 3.1, we can deduce that G is not a simple group. Let L be a minimal normal subgroup of G, we consider the quotient group G/L. If $T_1(G/L) \cup T_3(G/L) = \emptyset$, then by Theorem 3.3, G/L is solvable. We assume that $T_1(G/L) \cup T_3(G/L) \neq \emptyset$. It's easy to prove that $H_1 \in T_1(G) \cup T_3(G)$ for any 2-maximal subgroup $H_1/L \in T_1(G/L) \cup T_3(G/L)$. By hypotheses, we know that H_1 is a semi-cover-avoiding subgroup of G. By Lemma 2.4, H_1/L is semi-cover-avoiding in G/L. We get G/L is solvable by using induction. Since the class of solvable groups is a saturated formation, we have that L is a unique minimal normal subgroup of G. Using the arguments similar to the proof of Theorem 3.1, for any 2-maximal subgroup $K \in T_1(G) \cup T_3(G)$, we have $L \le K$.

For any $p \in \pi(L)$, by using the Frattini argument, we get $G = LN_G(L_p)$ with $L_p \in Syl_p(L)$. Using the similar arguments as Theorem 3.3, we have that $N_G(L_p) < G$ and therefore there exists a maximal M subgroup of G such that $N_G(L_p) \le M$. Now we have $G = LM$ and $M_G = 1$. Hence, for any maximal subgroup H of M, $H_G = 1$. We claim that $H \notin T_1(G) \cup T_3(G)$. If not, by the discussion as above, we have $L \le H$, which contradicts $H_G = 1$. Then, there exists a maximal subgroup $M_1 \in Max(G, H)$ such that $(M_1)_G > 1$. Hence, by the uniqueness of minimal normal subgroup L, we have $L \le (M_1)_G \le M_1$. Obviously, $L \nleq H$. Otherwise, $L \le H_G = 1$, a contradiction. By Lemma 2.7, we have $M_1 = LH < G$, which means that M is a minimal supplement of L in G. By Lemma 2.8, we have $L \cap M \le \Phi(M)$ is nilpotent. Noticing that $N_L(L_p) = L \cap N_G(L_p) \le L \cap M$, we can see that $N_L(L_p)$ is nilpotent. By the arbitrariness of p and Lemma 2.9, we have that L is solvable. So G is solvable, a contradiction. Thus, our proof is complete. □

Theorem 3.6. *Let G be a group. If $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) = \emptyset$, then G is solvable.*

Proof. We suppose that the theorem is false and let G be a counterexample with the minimal order. We will complete the proof in the following steps.

First we assume G is a simple group. By Lemma 2.11, $Max_2^*(G) \neq \emptyset$ and therefore we can pick a 2-maximal subgroup $H_0 \in Max_2^*(G)$. It is clear that $(H_0)_G = (M_0)_G = 1$ for any maximal subgroup $M_0 \in Max(G, H_0)$. Thus, $H_0 \in T_1(G) \cup T_3(G)$. Hence, we have $H_0 \in Max_2^*(G) \cap (T_1(G) \cup T_3(G))$, which contradicts $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) = \emptyset$. Therefore, the assumption is not tenable. Let L be a minimal normal subgroup of G, we consider the quotient group G/L. We asset that $Max_2^*(G/L) \cap (T_1(G/L) \cup T_3(G/L)) = \emptyset$. If not, we can choose a 2-maximal subgroup $H_1/L \in Max_2^*(G/L) \cap (T_1(G/L) \cup T_3(G/L))$. We can easily prove that $H_1 \in Max_2^*(G) \cap (T_1(G) \cup T_3(G))$, which contradicts $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) = \emptyset$. By induction, G/L is solvable. Since the class of solvable groups is a saturated formation, we get that L is a unique minimal normal subgroup of G.

For any $p \in \pi(L)$, by the Frattini argument, we have $G = LN_G(L_p)$ with $L_p \in Syl_p(L)$. By using the arguments similar to the proof of Theorem 3.3, we get $N_G(L_p) < G$. Thus there exist a maximal subgroup M of G such that $N_G(L_p) \leq M$. Hence we have $G = LM$ and $M_G = 1$. Next we will show that M is a minimal supplement of L in G. It is clear that $H_G = 1$ for every maximal subgroup H of M and therefore $L \nleq H$. Now we consider the following cases separately.

(a) $H \in Max_2^*(G)$: It follows from $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) = \emptyset$ that $H \notin T_1(G) \cup T_3(G)$. Thus, there exists a maximal subgroup $M_1 \in Max(G, H)$ such that $(M_1)_G > 1$. By the uniqueness of minimal normal subgroup L, we have $L \leq (M_1)_G \leq M_1$. By Lemma 2.7, we get that $LH = M_1 < G$.

(b) $H \notin Max_2^*(G)$: Then there exists a 2-maximal subgroup $H^{M_2} \in Max_2^*(G)$ such that $H \lessdot \cdots \lessdot H^{M_2} \lessdot M_2 \lessdot G$. Obviously, $H^{M_2} \notin T_1(G) \cup T_3(G)$. If $(H^{M_2})_G = 1$, then there exists a maximal subgroup $M_3 \in Max(G, H^{M_2})$ such that $(M_3)_G > 1$. By the uniqueness L, we have $L \leq (M_3)_G \leq M_3$. Noticing that $H \leq H^{M_2} \leq M_3$, by Lemma 2.7, we see that $LH = M_3 < G$; If $(H^{M_2})_G > 1$, by the uniqueness of L again, then we have $L \leq (H^{M_2})_G \leq M_2$. By Lemma 2.7 again, we get that $LH = M_2 < G$.

Now we have proved that M is a minimal supplement of L in G. Hence, by Lemma 2.8, we get $L \cap M \leq \Phi(M)$ is nilpotent. Noticing that $N_L(L_p) = L \cap N_G(L_p) \leq L \cap M$, we can immediately see that $N_L(L_p)$ is nilpotent. By the arbitrariness of p and Lemma 2.9, we have that L is nilpotent. Further, L is solvable. Since G/L is solvable, we have that G is solvable, a contradiction. Now the proof is complete. □

Corollary 3.7. *Let G be a group. If G is not solvable, then $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) \neq \emptyset$.*

Theorem 3.8. *Let G be a group. If H is semi-cover-avoiding in G for every 2-maximal subgroup $H \in Max_2^*(G) \cap (T_1(G) \cup T_3(G))$, then G is solvable.*

Proof. We suppose that the theorem is not true and let G be a counterexample with the minimal order. If $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) = \emptyset$, by Theorem 3.6, G is solvable. Now we may assume that $Max_2^*(G) \cap (T_1(G) \cup T_3(G)) \neq \emptyset$. We will complete the proof in the following steps.

By using the arguments similar to the Theorem 3.1, we can deduce that G is not a simple group. Let L be a minimal normal subgroup of G, we consider the quotient group G/L. If $Max_2^*(G/L) \cap (T_1(G/L) \cup T_3(G/L)) = \emptyset$, by Theorem 3.6, G/L is solvable. If $Max_2^*(G/L) \cap (T_1(G/L) \cup T_3(G/L)) \neq \emptyset$, for every 2-maximal subgroup $H_1/L \in Max_2^*(G/L) \cap (T_1(G/L) \cup T_3(G/L))$, we know that $H_1 \in Max_2^*(G) \cap (T_1(G) \cup T_3(G))$. By hypothesis, H_1 is semi-cover-avoiding in G. Hence, by Lemma 2.4, H_1/L is semi-cover-avoiding in G/L. By using the induction, we get that G/L is solvable. Since the class of solvable groups is a saturated formation, we have that L is a unique minimal normal subgroup of G. Using the arguments similar to the Theorem 3.1, for any 2-maximal subgroup $K \in Max_2^*(G) \cap (T_1(G) \cup T_3(G))$, we can get $L \leq K$

For any $p \in \pi(L)$, by the Frattini argument, we have $G = LN_G(L_p)$ with $L_p \in Syl_p(L)$. By using the arguments similar to the proof of Theorem 3.3, we get $N_G(L_p) < G$. Thus, there exist a maximal subgroup M of G such that $N_G(L_p) \leq M$. Hence we have $G = LM$ and $M_G = 1$. Next we will show that M is a minimal supplement of L in G. It is clear that $H_G = 1$ for every maximal subgroup H of M and therefore $L \nleq H$. Now we consider the following cases separately.

(a) $H \in T_1(G) \cup T_3(G)$: Obviously, $H \notin Max_2^*(G)$. Otherwise, by the discussion as above, $L \leq H$, which contradicts $H_G = 1$. So there exists a 2-maximal subgroup $H^{M_1} \in Max_2^*(G)$ such that $H \lessdot \cdots \lessdot H^{M_1} \lessdot M_1 \lessdot G$. Noticing that $H \in T_1(G) \cup T_3(G)$, we can see that $(M_1)_G = 1$. Since $H^{M_1} \leq M_1$, thus $(H^{M_1})_G = 1$. We claim that $H^{M_1} \notin T_1(G) \cup T_3(G)$. If not, we have $H^{M_1} \leq Max_2^*(G) \cap (T_1(G) \cup T_3(G))$. By the discussion as above, $L \leq H^{M_1}$, which contradicts $(H^{M_1})_G = 1$. Hence, there exists a maximal subgroup $M_2 \in Max(G, H^{M_1})$ such that $(M_2)_G > 1$. By the uniqueness of minimal normal subgroup L, we have $L \leq (M_2)_G \leq M_2$. We also have $H \leq H^{M_1} \leq M_2$. Now, by Lemma 2.7, $M_2 = LH < G$.

(b) $H \notin T_1(G) \cup T_3(G)$: Since $H \notin T_1(G) \cup T_3(G)$ and $H_G = M_G = 1$, then there exists a maximal subgroup $M_3 \in Max(G, H)$ such that $(M_3)_G > 1$. By the uniqueness of minimal normal subgroup L, we have $L \leq (M_3)_G \leq M_3$. Noticing that $H \leq M_3$, by Lemma 2.7, we have $M_3 = LH < G$.

Now we have shown that M is a minimal supplement of L in G. Then by Lemma 2.8, we have that $L \cap M \leq \Phi(M)$ is nilpotent. Then $N_L(L_p) = L \cap N_G(L_p) \leq L \cap M$ is nilpotent. By the arbitrariness of p and Lemma 2.9, we have that L is nilpotent. Further, L is solvable. Noticing that G/L is solvable, we see that G is solvable, a contradiction. Thereby, our proof is complete. □

Theorem 3.9. *Let G be a group. If every maximal subgroup of each Sylow p-subgroup of G is semi-cover-avoiding in G, then $G \in U_p^{\#}$.*

Proof. We suppose that the theorem is not true and let G be a counterexample with the minimal order. We will complete our proof in the following steps.

We claim that G is not a simple group. If not, then $G/1$ would be the only chief factor of G. Let P_0 be a maximal subgroup of $S_p^{(0)}$, where $S_p^{(0)}$ is a Sylow p-subgroup of G. Then by hypothesis, we have either $P_0G = P_0$ or $P_0 \cap G = 1$. The former case is clearly impossible. On the other hand, the latter case means $P_0 = 1$. Obviously, $|S_p^{(0)}| = p$. Thus, $|G|_p = p$. Further, $G \in U_p^{\#}$, a contradiction.

Let $S_p^{(1)}$ be a Sylow p-subgroup of G and P_1 a maximal subgroup of $S_p^{(1)}$. By hypothesis, we can choose a minimal normal subgroup N_1 of G such that either $P_1N_1 = P_1$ or $P_1 \cap N_1 = 1$. Next we will consider the two cases separately.

(1) $P_1N_1 = P_1$.

By using the Lemma 2.4 (1) and induction, we can deduce that $G/N_1 \in U_p^{\#}$. Obviously, $N_1 \nleq \Phi(G)$. Otherwise, $G \in U_p^{\#}$, a contradiction. Hence, there exists a maximal subgroup M_1 of G such that $G = N_1M_1$ and $N_1 \cap M_1 = 1$. Noticing that $|G|_p = |N_1||M_1|_p$, we can see that $1 < |M_1|_p < |G|_p$. Let $(M_1)_p$ be a Sylow p-subgroup of M_1. Now we can pick a Sylow p-subgroup $S_p^{(2)}$ of G and a maximal subgroup P_2 of $S_p^{(2)}$ that satisfies $(M_1)_p \leq P_2 \lessdot S_p^{(2)}$. By hypothesis, there exists a minimal normal subgroup N_2 of G such that either $P_2N_2 = P_2$ or $P_2 \cap N_2 = 1$. Next we will consider the two cases separately.

(a) $P_2N_2 = P_2$: Obviously, $N_2 \leq P_2$. By using the Lemma 2.4 (1) and induction, we also have $G/N_2 \in U_p^{\#}$. We assert that $N_1 = N_2$. If not, by the minimality of normal subgroup N_1, we have $N_1 \cap N_2 = 1$. It's easy to prove that N_1N_2/N_1 is a minimal normal subgroup of G/N_1. Since $G/N_1 \in U_p^{\#}$, thus $|N_1N_2/N_1|_p \leq p$ or $N_1N_2/N_1 \leq \Phi(G/N_1)$. Because $N_2 \cong N_1N_2/N_1$, so we can infer $|N_2|_p \leq p$ from the former case. Hence, $G \in U_p^{\#}$, a contradiction. By Lemma 2.10, we get $N_1N_2 \leq \Phi(G)N_1$ from the latter case. Meanwhile, $N_1N_2 \leq O_p(G)$. Therefore $N_1N_2 \leq \Phi(G)N_1 \cap O_p(G) = (\Phi(G) \cap O_p(G))N_1$. We claim that $\Phi(G) \cap O_p(G) = 1$. If not, we can pick a minimal normal subgroup N_3 of G that holds $N_3 \leq \Phi(G) \cap O_p(G)$. Since $N_3 \leq O_p(G)$, by using the Lemma 2.4 (1) and induction, we have $G/N_3 \in U_p^{\#}$. It follows from $N_3 \leq \Phi(G)$ that $G \in U_p^{\#}$, a contradiction. Thus, $N_1N_2 \leq (\Phi(G) \cap O_p(G))N_1 = N_1$, which is impossible. Since $|G|_p = |N_1||M_1|_p$, then we have $N_1(M_1)_p$ is a Sylow p-subgroup of G. Noticing that $N_1 = N_2$, we see that $N_2(M_1)_p$ is a Sylow p-subgroup of G, which contradicts $N_2(M_1)_p \leq P_2$.

(b) $P_2 \cap N_2 = 1$: Since $P_2N_2 \leq S_p^{(2)}N_2$, then $|P_2N_2|||S_p^{(2)}N_2|$. It's easy to prove $|S_p^{(2)} \cap N_2||p$, which implies $|N_2|_p \leq p$. If $|N_2|_p = 1$, by Lemma 2.4 (2) and induction, we have $G/N_2 \in U_p^{\#}$. Therefore, $G \in U_p^{\#}$, a contradiction. Now we consider $|N_2|_p = p$. Since $P_2 \cap N_2 = 1$, so $(M_1)_p \cap N_2 = 1$. Hence, we have $N_2 \nleq M_1$. Further, $G = N_2M_1$. Let $(N_2 \cap M_1)_p$ be a Sylow p-subgroup of $N_2 \cap M_1$. There exists an element $x \in M_1$ such that $x(N_2 \cap M_1)_p x^{-1} \leq (M_1)_p \leq P_2$. Since $x(N_2 \cap M_1)_p x^{-1} \leq N_2$, thus $x(N_2 \cap M_1)_p x^{-1} \leq P_2 \cap N_2 = 1$. Hence, $|N_2 \cap M_1|_p = 1$. It follows from $|G|_p = \frac{|N_2|_p|M_1|_p}{|N_2 \cap M_1|_p} = |N_1||M_1|_p$ that $|N_1| = |N_2|_p = p$. Therefore, $G \in U_p^{\#}$, a contradiction.

(2) $P_1 \cap N_1 = 1$.

Using the same method described in (b), we can deduce that $|N_1|_p \leq p$. If $|N_1|_p = 1$, by Lemma 2.4 (2) and induction, we have $G/N_1 \in U_p^{\#}$. Thus, $G \in U_p^{\#}$, a contradiction. Now we may assume that $|N_1|_p = p$ and $|G|_p > p$. Let $(N_1)_p$ be a Sylow p-subgroup of N_1. Thus, there exists a Sylow p-subgroup $S_p^{(3)}$ of G and a maximal subgroup P_3 of $S_p^{(3)}$ such that $(N_1)_p \leq P_3 \lessdot S_p^{(3)}$. By hypothesis, we can pick a minimal normal subgroup N_3 of G that satisfies either $P_3N_3 = P_3$ or $P_3 \cap N_3 = 1$. The former case is clearly impossible by case (1). We can deduce that $|N_3|_p = p$ from the latter case. If $N_3 \leq \Phi(G)$, then $|N_3| = p$. By the induction, we have $G/N_3 \in U_p^{\#}$. Hence, $G \in U_p^{\#}$, a contradiction. Now we have $N_3 \nleq \Phi(G)$. Then there exists a maximal subgroup $M_3^{'}$ of G such that $G = N_3M_3^{'}$.

We set that $(M_3^{'})_p$ is a Sylow p-subgroup of $M_3^{'}$. Noticing that $|N_3|_p = p$, we can immediately get that $(M_3^{'})_p$ is a Sylow p-subgroup of G or $(M_3^{'})_p$ is a maximal subgroup of $S_p^{(4)}$, where $S_p^{(4)} \in Syl_p(G)$. If $(M_3^{'})_p$ is a Sylow p-subgroup of G, then we get $M_3^{'} \in U_p^{\#}$ by using the induction. It is easy to prove that $G/N_3 \cong M_3^{'}/M_3^{'} \cap N_3 \in U_p^{\#}$, and therefore $G \in U_p^{\#}$, a contradiction. If $(M_3^{'})_p$ is a maximal subgroup of $S_p^{(4)}$, by hypothesis, $(M_3^{'})_p$ is semi-cover-avoiding in G. Hence, there exists a chief series $1 = G_0 < G_1 < \cdots < G_{t-1} < G_t = G$ such that either $(M_3^{'})_pG_i = (M_3^{'})_pG_{i-1}$ or $(M_3^{'})_p \cap G_i = (M_3^{'})_p \cap G_{i-1}$ for every $i = 1, 2 \cdots t$. It's clear that $1 = G_0 \cap M_3^{'} \leq G_1 \cap M_3^{'} \leq \cdots \leq G_{t-1} \cap M_3^{'} \leq G_t \cap M_3^{'} = M_3^{'}$ is a normal series of $M_3^{'}$. If $(M_3^{'})_pG_i = (M_3^{'})_pG_{i-1}$, then we have that $(M_3^{'})_p(G_i \cap M_3^{'}) = (M_3^{'})_p(G_{i-1} \cap M_3^{'})$, which means $G_i \cap M_3^{'}/G_{i-1} \cap M_3^{'}$ is a p-group. If $(M_3^{'})_p \cap G_i = (M_3^{'})_p \cap G_{i-1}$, then $(M_3^{'})_p \cap (G_i \cap M_3^{'}) = (M_3^{'})_p \cap (G_{i-1} \cap M_3^{'})$, which implies $G_i \cap M_3^{'}/G_{i-1} \cap M_3^{'}$ is a $p^{'}$-group. In summary, we have $M_3^{'}$ is p-solvable. Thus, $G/N_3 \cong M_3^{'}/M_3^{'} \cap N_3$ is p-solvable. Hence, G is p-solvable and therefore $|N_3| = p$. By induction, $G/N_3 \in U_p^{\#}$, thus $G \in U_p^{\#}$, a contradiction. Now our proof is complete. □

Theorem 3.10. *Let G be a group. If every 2-maximal subgroup of each Sylow p-subgroup of G is semi-cover-avoiding in G, then $G \in U_{p^2}^{\#}$.*

Proof. We suppose that the theorem is not true and let G be a counterexample with the minimal order. We will complete our proof in the following steps.

We assert that G is not a simple group. If not, then $G/1$ would be the only chief factor of G. Let S_p be a Sylow p-subgroup of G and P a 2-maximal subgroup of S_p. Then by hypothesis, we have either $PG = P$ or $P \cap G = 1$. Obviously, the former case is impossible. The latter case means $P = 1$. Thus, $|G|_p = p^2$. Therefore, $G \in U_{p^2}^{\#}$, a contradiction.

Let $S_p^{(a)}$ be a Sylow p-subgroup of G and P_{11} a 2-maximal subgroup of $S_p^{(a)}$. By hypothesis, we can pick a minimal normal subgroup N_{11} of G such that either $P_{11}N_{11} = P_{11}$ or $P_{11} \cap N_{11} = 1$. Next we will consider the two cases separately.

(1) $P_{11}N_{11} = P_{11}$.

It's clear to see that $N_{11} \leq P_{11}$. By using the Lemma 2.4(1) and induction, we have $G/N_{11} \in U^{\#}_{p^2}$. Obviously, $N_{11} \nleq \Phi(G)$. Otherwise, $G \in U^{\#}_{p^2}$, a contradiction. Hence we can choose a maximal subgroup M_{11} of G that holds $G = N_{11}M_{11}$ and $N_{11} \cap M_{11} = 1$. Noticing that $|G|_p = |N_{11}||M_{11}|_p$, we can see that $1 < |M_{11}|_p < |G|_p$. Let $(M_{11})_p$ be a Sylow p-subgroup of M_{11}. Thus there exists a Sylow p-subgroup $S_p^{(b)}$ of G and a maximal subgroup P_1 of $S_p^{(b)}$ such that $(M_{11})_p \leq P_1$. If $(M_{11})_p = P_1$, we can immediately get $|N_{11}| = p$ from $|G|_p = |N_{11}||M_{11}|_p$. Thus, $G \in U^{\#}_{p^2}$, a contradiction. Hence, $(M_{11})_p < P_1$ and therefore there exist a maximal subgroup P_{12} of P_1 such that $(M_{11})_p \leq P_{12}$. Then, by hypothesis, we can pick a minimal normal subgroup N_{12} of G that satisfies either $P_{12}N_{12} = P_{12}$ or $P_{12} \cap N_{12} = 1$. Next we will consider the two cases separately.

(a) $P_{12}N_{12} = P_{12}$: Obviously, $N_{12} \leq P_{12}$. By Lemma 2.4 (1) the induction, we have $G/N_{12} \in U^{\#}_{p^2}$. We claim that $N_{11} = N_{12}$. If not, by the minimality of N_{11}, we have $N_{11} \cap N_{12} = 1$. It's easy to prove $N_{11}N_{12}/N_{11}$ is a minimal normal subgroup of G/N_{11}. It follows from $G/N_{11} \in U^{\#}_{p^2}$ that $|N_{11}N_{12}/N_{11}|_p \leq p^2$ or $N_{11}N_{12}/N_{11} \leq \Phi(G/N_{11})$. Since $N_{12} \cong N_{11}N_{12}/N_{11}$, then we get $|N_{12}|_p \leq p^2$ from the former case. Therefore, $G \in U^{\#}_{p^2}$, a contradiction. By Lemma 2.10, we can deduce that $N_{11}N_{12} \leq \Phi(G)N_{11}$ from the latter case. Meanwhile, $N_{11}N_{12} \leq O_p(G)$. Therefore we have $N_{11}N_{12} \leq \Phi(G)N_{11} \cap O_p(G) = (\Phi(G) \cap O_p(G))N_{11}$. We claim that $\Phi(G) \cap O_p(G) = 1$. If not, there exists a minimal normal subgroup N of G such that $N \leq \Phi(G) \cap O_p(G)$. Since $N \leq O_p(G)$, therefore $G/N \in U^{\#}_{p^2}$ by induction. Noticing that $N \leq \Phi(G)$, we see that $G \in U^{\#}_{p^2}$, a contradiction. Now we have $N_{11}N_{12} \leq N_{11}$, which is impossible. Because $N_{11}(M_{11})_p$ is a Sylow p-subgroup of G and $N_{11} = N_{12}$, so $N_{12}(M_{11})_p \in Syl_p(G)$, which contradicts $N_{12}(M_{11})_p \leq P_{12} < S_p^{(b)}$.

(b) $P_{12} \cap N_{12} = 1$: Using the same method described in Theorem 3.9(b), we get $|N_{12}|_p \leq p^2$. Since $P_{12} \cap N_{12} = 1$, thus $(M_{11})_p \cap N_{12} = 1$, which implies that $N_{12} \nleq M_{11}$. Hence we have $G = N_{12}M_{11}$. Using the argument similar to the Theorem 3.9(b), we have $|N_{12} \cap M_{11}|_p = 1$. Noticing that $|G|_p = \frac{|N_{12}|_p|M_{11}|_p}{|N_{12} \cap M_{11}|_p} = |N_{11}||M_{11}|_p$, we can see that $|N_{11}| = |N_{12}|_p \leq p^2$. Hence, $G \in U^{\#}_{p^2}$, a contradiction.

(2) $P_{11} \cap N_{11} = 1$.

It's easy to know that $|N_{11}|_p \leq p^2$. Let's first discuss the quantitative relationship between $|N_{11}|_p$ and $|G|_p$. If $|N_{11}|_p = |G|_p$, then $|G|_p \leq p^2$. Hence $G \in U^{\#}_{p^2}$, a contradiction; If $p|N_{11}|_p = |G|_p$, then we know that $|G/N_{11}|_p = p$. Hence, $G/N_{11} \in U^{\#}_{p^2}$, and therefore $G \in U^{\#}_{p^2}$, a contradiction; Next we consider $p|N_{11}|_p < |G|_p$. Let $(N_{11})_p$ be a Sylow p-subgroup of N_{11}. Obviously, there exists a Sylow p-subgroup $S_p^{(c)}$ of G and a 2-maximal subgroup P_{13} of $S_p^{(c)}$ such that $(N_{11})_p \leq P_{13}$. Then, by hypothesis, we can choose a minimal normal subgroup N_{13} of G such that $P_{13}N_{13} = P_{13}$ or $P_{13} \cap N_{13} = 1$. The former case is impossible by case (1). Next we focus on the latter case. We also have $|N_{13}|_p \leq p^2$. If $N_{13} \leq \Phi(G)$, then $|N_{13}| \leq p^2$. By using the induction, we have $G/N_{13} \in U^{\#}_{p^2}$. Hence, $G \in U^{\#}_{p^2}$, a contradiction. Therefore, $N_{13} \nleq \Phi(G)$. There exists a maximal subgroup M'_{13} of G such that $G = N_{13}M'_{13}$.

We set $(M'_{13})_p$ is a Sylow p-subgroup of M'_{13} and $|G|_p = p^n$, where $n \in N$. Since $|N_{13}|_p \leq p^2$ and the relation between the order of maximal subgroup and 2-maximal subgroup of p-group, then we have $|(M'_{13})_p| = p^n, p^{n-1}$ *or* p^{n-2}. If $|(M'_{13})_p| = p^n$, then $(M'_{13})_p$ is a Sylow p-subgroup of G. By using the induction, we have $M'_{13} \in U^{\#}_{p^2}$. Since $G/N_{13} \cong M'_{13}/M'_{13} \cap N_{13}$, then $G/N_{13} \in U^{\#}_{p^2}$. Therefore $G \in U^{\#}_{p^2}$, a contradiction. If

$|(M_{13}^{'})_p| = p^{n-1}$, then $(M_{13}^{'})_p$ is a maximal subgroup of $S_p^{(d)}$ where $S_p^{(d)} \in Syl_p(G)$. It's easy to prove that $M_{13}^{'} \in U_p^{\#}$ by Theorem 3.9. Because $G/N_{13} \cong M_{13}^{'}/M_{13}^{'} \cap N_{13}$, thus we have $G/N_{13} \in U_p^{\#} \subseteq U_{p^2}^{\#}$. Hence $G \in U_{p^2}^{\#}$, a contradiction. If $|(M_{13}^{'})_p| = p^{n-2}$, then $(M_{13}^{'})_p$ is a 2-maximal subgroup of $S_p^{(e)}$ where $S_p^{(e)} \in Syl_p(G)$. Then by hypothesis, we know that $(M_{13}^{'})_p$ is a semi-cover-avoiding subgroup of G. Similarly to Theorem 3.9(2), we have that $M_{13}^{'}$ is p-solvable. Thus, $G/N_{13} \cong M_{13}^{'}/M_{13}^{'} \cap N_{13}$ is p-solvable. Hence, G is p-solvable and therefore $|N_{13}| \leq p^2$. By using the induction, we have $G/N_{13} \in U_{p^2}^{\#}$. Thus, $G \in U_{p^2}^{\#}$, a contradiction. Now our proof is complete. □

Acknowledgment. We are grateful to the editor and anonymous reviewers for their valuable comments. This research is supported by NSFC(Grant # 11871062), NSFC-RFBR(Grant # 12011530061), the Natural Science Foundation of Jiangsu Province (Grant # BK20181451), and the NSF of the JiangSu Higher Education Institutions (Grant # 22KJB110024).

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