Turkish Journal of Science & Technology
17(2), 343-356, 2022

Research Paper
https://doi.org/10.55525/tjst.1112582

Some Quantum Integral Inequalities Based on Left-Right Quantum Integrals

Mehmet KUNT[1], Abdul Wakil BAIDAR[2], Zeynep ŞANLI[3*]
[1] Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, Trabzon, Turkey
[2] Kabul University, Department of Mathematics, Kabul, Afghanistan
[3] Department of Mathematics, Faculty of Sciences and Letters, Mersin University, Mersin,Turkey
[1]mkunt@ktu.edu.tr, [2] baidarmath87@gmail.com, [3*]zeynepsanli@mersin.edu.tr

(Geliş/Received: 04/05/2022; Kabul/Accepted: 27/06/2022)

Abstract: In this work, the concepts of quantum derivative and quantum integral were renamed to be the left quantum derivative and the left definite quantum integral. Symmetrically to the left, a new quantum derivative (the right) and definite quantum integral (the right) were defined. Some properties of these new concepts were investigated and as well as according to do these new concepts some inaccuracies in quantum integral inequalities were corrected. Moreover, some new quantum Hermite-Hadamard type inequalities were established.

Key words: Quantum derivative, quantum integral, integral inequalities, convex functions.

Sol Sağ Kuantum İntegrallerine Dayalı Bazı Kuantum İntegral Eşitsizlikleri

Öz: Bu çalışmada, kuantum türevi ve kuantum integrali kavramları, sol kuantum türevi ve sol kuantum belirli integrali olarak yeniden adlandırıldı. Sola simetrik olarak, yeni bir kuantum türevi (sağ) ve belirli kuantum integrali (sağ) tanımlandı. Bu kavramların bazı özellikleri araştırıldı ve buna göre bu yeni kuantum integral eşitsizliklerindeki bazı yanlışlıklar düzeltildi. Ayrıca, bazı yeni kuantum Hermite-Hadamard tipi eşitsizlikler kuruldu.

Anahtar kelimeler: Kuantum türev, kuantum integral, integral eşitsizlikleri, konveks fonksiyonlar.

1.INTRODUCTION

A function $f: J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on J if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in J$ and $t \in [0,1]$.

One of the most useful inequalities for convex functions is Hermite-Hadamard's inequality, due to its geometrical importance and applications, which is described as follows:

Let $f: J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of real numbers and $a, b \in J$ with $a < b$. Then

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

This inequality has been widely studied in different ways and different forms in the theory of integral inequalities. It is indispensable to mention the papers [14], which are leading the studies over the past twenty years. In [14], the authors gave an identity to obtain trapezoid type error estimations. Following, researchers continued to work to obtain error estimates such as midpoint, Ostrowski, Simpson, Bullen, Newton, etc. and to improved existing inequalities. Many generalizations of this paper have been done by considering different types of functions or various types of integrals. In recent years, generalizations of these studies in quantum calculus theory have been widely researched. For some studies in this regard, the reader is refer to [1]-[4], [6]-[13] , [18] -[25], [27]-[36].

However, since the quantum integral is different from the Riemann integral in some properties, some inaccuracies in some of these studies arise. For instance, in [22, Lemma 3.1], the authors gave an identity to obtain quantum analogues of Ostrowski type inequalities. To prove this identity, using similar proving argument with its Riemann integral form, the authors proved the following identity:

(1.2) $\frac{(x-a)^2}{b-a}\int_0^1 t\ {}_aD_qf(tx+(1-t)a)\ \ {}_0d_qt=\frac{x-a}{q(b-a)}f(x)-\frac{1}{q(b-a)}\int_a^x f\ (u)\ \ {}_ad_q\,t\ .$

After that, they assumed symmetrically to (1.2) the following identity could be held:

(1.3) $\frac{(b-x)^2}{b-a}\int_0^1 t\ \ {}_aD_qf(tx+(1-t)b)\ \ {}_0d_qt=\frac{b-x}{q(b-a)}f(x)-\frac{1}{q(b-a)}\int_x^b f\ (u)\ \ {}_ad_q\,t\ .$

(see [22, equalities (3.2) and (3.3)]). But the current quantum derivative and integral definitions are not sufficient to satisfy (1.3). In fact, the reason for this problem is that there is no substitution rule in the quantum integral, and the quantum integral is not similar to the Riemann integral for some aspects.

In our investigations, we realized that the existing quantum derivative and integral consepts are not sufficent to overcome these problems. It is known that conjugate concepts of left and right fractional integrals are widely used to derive fractional integral inequalities. Considering that a similar stuation can be useful for quantum integral inequalities, we have found that it is necessary to give symmetric consepts which are conjugate to the concepts of quantum derivative and integral.

In this context, the main motivation of this study is to introduce new quantum derivative and integral consepts which are symmetrical to the known and to examine some of their properties. Using them, we will solve some inaccuracies encountered in quantum integral inequalities. In addition, similar to fractional integral inequalities, some quantum type integral inequalities will be obtained by using two symmetric quantum derivatives and integrals concepts together. Also, in the last section, we will use these two conjugate concepts together to obtain quantum analogs of some trapezoid type integral inequalities posed in paper [16]. We think that this new method will be used effectively and widely in future studies.

2.PRELIMINARIES

In this section, we recall some previously known concepts.

In [30,31], Tariboon and Ntouyas introduced the concepts of quantum derivative and definite quantum integral for the functions of defined on an arbitrary finite intervals as follows:

Definition 2.1. [30,31] A function $f(t)$ defined on $[a,b]$ is called quantum differentiable on $(a,b]$ with the following expression:

(2.1) ${}_aD_qf(t)=\frac{f(t)-f(qt+(1-q)a}{(1-q)(t-a)}\in\mathbb{R},\quad t\neq a,$

and quantum differentiable on $t=a$, if the following limit exists:

$${}_aD_qf(a)=\lim_{t\to a^+}{}_aD_qf(a)\ ,$$

for any $a<b$ (See also [19]).

Clearly, if $a=0$ in (2.1), then ${}_0D_qf(t)=D_qf(t)$ where $D_qf(t)$ is familiar quantum derivative given by the following expression:

(2.2) $D_qf(t)=\frac{f(t)-f(qt)}{(1-q)t},\ \ t\neq 0,$

$$D_qf(0)=\lim_{t\to 0^+}D_qf(0),$$

where f is defined on the interval $[a,0]$. (see also [5,17])

Definition 2.2. [30,31] Let $f(t)$ be a function defined on $[a,b]$. Then the definite quantum integral of $f(t)$ on $[a,b]$ is delineated as:

(2.3) $\int_a^b f(t)\ {}_a\,d_qt=(1-q)(b-a)\sum_{n=0}^{\infty}q^nf(q^nb+(1-q^n)a).$

If the series in the right-hand side of (2.3) is convergence, then $\int_a^b f(t)_a d_q t$ is exist, i.e., $f(t)$ is quantum integrable on $[a,b]$. For any $c \in (a,b)$

(2.4) $$\int_c^b f(t)\ {}_a d_q t = \int_a^b f(t)\ {}_a d_q t - \int_a^c f(t)\ {}_a d_q t\,,$$

if the series in the right-hand side of (2.4) is convergence. (See also [19]).

Clearly,if $a = 0$, in (2.3), then $\int_0^b f(t)\ d_q t$ is familiar definite quantum integral (see [5,17]) on $[0,b]$ such as

$$\int_0^b f(t)\ {}_0 d_q t = \int_0^b f(t)\ d_q t = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b).$$

Lemma 2.1. (19) Let $f:[a,b] \to \mathbb{R}$ be a differentiable funtion. Then we have

(2.5) $$\lim_{q\to 1^-} {}_a D_q f(t) = \frac{df(t)}{dt}.$$

Lemma 2.2. [19] Let $f:[a,b] \to \mathbb{R}$ be an arbitrary function. Provided that $\int_a^b f(t)\ dt$ converges, then we have

(2.6) $$\lim_{q\to 1^-} \int_a^b f(t)\ {}_a d_q t = \int_a^b f(t)\ dt\,.$$

By using the Definitions 2.1 and 2.2, in [7], the authors present the quantum Hermite-Hadamard inequality (in [32], the authors proved the same inequality with the fewer assumptions and shorter method) as follows:

Theorem 2.1. Let $f:[a,b] \to \mathbb{R}$ be a convex function and $0 < q < 1$. Then we have

(2.7) $$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(t)\ {}_a d_q t \le \frac{qf(a)+f(b)}{1+q}\,.$$

We will use the following notations and names to avoid confusion:
The left quantum derivative on $[a,b]$ for (2.1):

$${}_{a^+}D_q f(t) = {}_a D_q f(t) = \frac{f(t)-f(qt+(1-q)a)}{(1-q)(t-a)},\ t \neq a.$$

The left definite quantum integral on $[a,b]$ for (2.3):

$$\int_a^b f(t)\ {}_{a^+}d_q = \int_a^b f(t)\ {}_a d_q = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a).$$

The left definite quantum integral on $[0,b]$ for (2.1):

$$\int_a^b f(t)\ {}_{0^+}d_q = \int_0^b f(t)\ {}_0 d_q = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b).$$

The left quantum integral on Hermite-Hadamard ineququality for (2.7)

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(t)\ {}_{a^+}d_q t \le \frac{qf(a)+f(b)}{1+q}\,.$$

3.THE RIGHT QUANTUM DERVATIVE AND THE RIGHT QUANTUM INTEGRAL ON FINITE INTERVALS

To begin with, we should point out that definitions and some results in this section are given in [13] independently of us.

Let $f(t)$ be a function defined on $[a,b]$. It is clear that $qt+(1-q)b \in [a,b]$ for all $t \in [a,b]$. It means that $f(t)-f(qt+(1-q)b)$ is a real number for all $t \in [a,b]$. Hence, $\frac{f(t)-f(qt+(1-q)b)}{(1-q)(t-b)}$ is a real number and always exit for all $t \in [a,b)$. Symmetrically to Definition 2.1, we can introduce the following definition of quantum derivative:

Definition 3.1. A function $f(t)$ defined on $[a,b]$ is called the right quantum differentiable on $t \in [a,b)$ with the following expression:

(3.1) $$_{b}\text{-}D_q f(t) := \frac{f(t)-f(qt+(1-q)b)}{(1-q)(t-b)}, \ t \neq b,$$

and the right quantum differentiable on $t=b$, if the following limit exists:

$$_{b}\text{-}D_q f(b) = \lim_{t \to b^-} {}_{b}\text{-}D_q f(t).$$

Clearly, if $b=0$ in (3.1), then $_{0}\text{-}D_q f(t) = \ D_q f(t)$ where $D_q f(t)$ is familiar quantum derivative given in (2.2) of the function defined on the interval $[a,0]$.

The question is that similar property as stated in Lemma 2.1 can be given or not for the right quantum derivative. The answer is positive and follows:

Lemma 3.1. Left $f:[a,b] \to \mathbb{R}$ be a differentiable function. Then

(3.2) $$\lim_{q \to 1^-} {}_{b}\text{-}D_q f(t) = \frac{df(t)}{dt}.$$

Proof. Since f is differentiable on $[a,b]$, clearly

(3.3) $$\lim_{h \to 0^+} \frac{f(t+h)-f(t)}{h} = \lim_{h \to 0^-} \frac{f(t+h)-f(t)}{h} = \lim_{h \to 0} \frac{f(t+h)-f(t)}{h} = \frac{df(t)}{dt},$$

for all $t \in (a,b)$ and

$$\frac{df(a^+)}{dt} = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h},$$

$$\frac{df(b^-)}{dt} = \lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h}.$$

Since $0<q<1, (1-q)(t-b)<0$, for all $a<t<b$. Changing variable in (3.1) as $(1-q)(t-b)=-h$, then $q \to 1^-$ we have $h \to 0^+$ and $qt+(1-q)b = t+h$. Using (3.3),

(3.4) $$\lim_{q \to 1^-} {}_{b}\text{-}D_q f(t) = \lim_{q \to 1^-} \frac{f(t)-f(qt+(1-q)b)}{(1-q)(t-b)}$$

$$= \lim_{h \to 0^+} \frac{f(t)-f(t+h)}{-h}$$

$$= \lim_{h \to 0^+} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$$

$$= \frac{df(t)}{dt},$$

for all $t \in (a,b)$.

For $t = a$, similarly to (3.4).

$$\lim_{q\to 1^-} {}_{b^-}D_q f(a) = \lim_{q\to 1^-} \frac{f(a)-f(qa+(1-q)b)}{(1-q)(a-b)}.$$

$$= \lim_{h\to 0^+} \frac{f(a)-f(a+h)}{-h}$$

$$= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$= \frac{df(a^+)}{dt},$$

On the other hand, for $t = b$,

$$\lim_{q\to 1^-} {}_{b^-}D_q f(a) = \lim_{q\to 1^-} \lim_{t\to b^-} {}_{b^-}D_q f(t)$$

$$= \lim_{q\to b^-} \lim_{q\to 1^-} {}_{b^-}D_q f(t)$$

$$= \lim_{t\to b^-} \left(\frac{df(t)}{dt}\right)$$

$$= \lim_{t\to b^-} \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{t\to b^-} \lim_{h\to 0^-} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{h\to 0^-} \lim_{t\to b^-} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

$$= \frac{df(b^-)}{dt},$$

which completes the proof. ■

It is known that in classical quantum derivative $D_q t^n = [n]_q t^{n-1}$ where $[n]_q = \frac{1-q^n}{1-q}$. The symmetric property in the right quantum derivative as follows:

Example 1. Let $f: [a,b] \to \mathbb{R}$, $f(t) = (t-b)^n$ for $n \in \mathbb{N}$, then

$${}_{b^-}D_q f(t) = {}_{b^-}D_q f(t-b)^n = \frac{(t-b)^n-((qt+(1-q)b)-b)^n}{(1-q)(t-b)}$$

$$= \frac{(b-t)^n - q^n(t-b)^n}{(1-q)(t-b)}$$

$$= \frac{1-q^n}{1-q}(t-b)^{n-1}$$

$$= [n]_q (t-b)^{n-1}$$

Theorem 3.1. Let $f, g: [a, b] \to \mathbb{R}$ arbitrary functions, $\lambda \in \mathbb{R}$ constant, then

(i) ${}_{b}\text{-}D_q[f(t) + g(t)] = {}_{b}\text{-}D_q f(t) + {}_{b}\text{-}D_q g(t)$, for all $t \in [a, b]$,

(ii) ${}_{b}\text{-}D_q \lambda f(t) = \lambda_{b}\text{-}D_q f(t)$, for all $t \in [a, b]$,

(iii) ${}_{b}\text{-}D_q(fg)(t) = f(t)\ {}_{b}\text{-}D_q g(t) + g(qt + (1-q)b)\ {}_{b}\text{-}D_q f(t)$, for all $t \in [a, b]$,

(iv) ${}_{b}\text{-}D_q\left(\frac{f}{g}\right)(t) = \frac{g(t)\ {}_{b}\text{-}D_q f(t) - f(t)\ {}_{b}\text{-}D_q g(t)}{g(t) g(qt+(1-q)b}$,

for all $t \in [a, b] \setminus \{t: g(t) g(qt + (1-q)b \neq 0\}$.

Proof. It is omitted because the proof is quite easy and similar to the proof of [30, Theorem 3.1].

Definition 3.2. Let $f(t)$ be a function defined on $[a, b]$. Then the nth order right quantum derivative of $f(t)$ for $n \in \mathbb{N}$ is defined as

(3.5) $${}_{b}\text{-}D_q^n f(t) = {}_{b}\text{-}D_q\left({}_{b}\text{-}D_q^{n-1} f(t)\ \right) .$$

For the definition of the right definite quantum integral the following shift operator is used.

$$K_q F(t) := F(qt + (1-q)b).$$

By Mathematical induction, we have $K_q^n F(t)$ can be written as

$$K_q^n F(t) = \begin{cases} K_q\ (K_q^{n-1} F)(t) = F(q^n t + (1-q^n)b & , \ n \in \mathbb{N} \\ F(t) & , \ n = 0 \end{cases} .$$

According to the Definition 3.1 and $K_q F(t)$

$${}_{b}\text{-}D_q F(t) = \frac{F(t) - F(qt+(1-q)b)}{(1-q)(t-b)} = \frac{1-K_q}{(1-q)(t-b)} F(t) = f(t).$$

It implies that the right quantum antiderivative of $F(t)$ can be expressed as :

$$F(t) = (1-q) \frac{1}{1-K_q} [(t-b) f(t)].$$

By using the geometric series expansion, the right quantum antiderivative of $f(t)$ can be written as:

(3.6) $$F(t) = (1-q) \sum_{n=0}^{\infty} K_q^n [(t-b) f(t)]$$

$$= (1-q) \sum_{n=0}^{\infty} [(q^n t + (1-q^n)b) - b] f(q^n t + (1-q^n)b)$$

$$= (1-q)(t-b) \sum_{n=0}^{\infty} q^n f(q^n t + (1-q^n)b) ,$$

provided that if the series in the right-hand side of (3.6) converges. Symmetrically to Definition 2.2, definite quantum integral can be defined as follows:

Definition 3.3. Let $f(t)$ be a function defined on $[a, b]$. Then the right definite quantum integral of $f(t)$ on $[a, b]$ is described as:

(3.7) $$\int_a^b f(t)\ {}_{b}\text{-}d_q t = F(b) - F(a) = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) .$$

If the series in right hand side of (3.7) is convergence, then $\int_a^b f(t)\ {}_{b}\text{-}d_q t$ is exist, i.e., $f(t)$ is right quantum integrable on $[a, b]$. For any $c \in (a, b)$

(3.8) $$\int_a^c f(t)\ {}_{b}\text{-}d_q t = \int_a^b f(t)\ {}_{b}\text{-}d_q t - \int_c^b f(t)\ {}_{b}\text{-}d_q t ,$$

if the series in right hand side of (3.8) is convergence.
If $b = 0$ in (3.7), then

(3.9) $$\int_a^0 f(t)\ {}_{0^-}d_q t = (1-q)(-a)\sum_{n=0}^{\infty} q^n f(q^n a).$$

It is called as the right definite quantum integral of f on $[a, 0]$.

The question is that similar property as stated in Lemma 2.2 can be given or not for the right definite quantum integral on $[a, b]$. The answer is positive and as follows:

Lemma 3.2. Let $f: [a, b] \to \mathbb{R}$ be an arbitrary function. Provided that if $\int_a^b f(t)dt$ converges, then

(3.10) $$\lim_{q\to 1^-} \int_a^b f(t)\ {}_{b^-}d_q t = \int_a^b f(t)dt\,.$$

Proof. Let $\int_a^b f(t)dt$ converges, then $\int_0^1 f(ta + (1-t)b)dt$ converges also. Hence, by using Lemma 2.2, we have

$$\begin{aligned}\lim_{q\to 1^-} \int_a^b f(t)\ {}_{b^-}d_q t &= \lim_{q\to 1^-} (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b)\\ &= (b-a)\lim_{q\to 1^-} \int_0^1 f(ta + (1-t)b)\ {}_{0^+}d_q t\\ &= (b-a)\int_0^1 f(ta + (1-t)b)dt\\ &= \int_a^b f(t)dt.\end{aligned}$$

■

Some properties of the right definite quantum integral similar to the [30, Theorem 3.2, Theorem 3.3] are discussed in the following theorem.

Theorem 3.2. Let $f: [a, b] \to \mathbb{R}$ be arbitrary functions, $\lambda \in \mathbb{R}$ be a constant, then

(i) $\int_a^b [f(t) + g(t)]\ {}_{b^-}d_q t = \int_a^b f(t)\ {}_{b^-}d_q t + \int_a^b g(t)\ {}_{b^-}d_q t$,
(ii) $\int_a^b \lambda f(t)\ {}_{b^-}d_q t = \lambda \int_a^b f(t)\,{}_{b^-}d_q t$,
(iii) ${}_{b^-}D_q \int_t^b f(s)\,{}_{b^-}d_q s = -f(t)$, if f is right quantum integrable on $[t, b]$ for all $t \in [a, b]$,
(iv) $\int_t^b {}_{b^-}D_q f(s)\ {}_{b^-}d_q s = f(b) - f(t)$, for all $t \in [a, b]$, if f is continuous on $[a, b]$,
(v) $\int_t^b f(s)\,{}_{b^-}D_q g(s)\,{}_{b^-}d_q s = (fg)(b) - (fg)(t) - \int_t^b g(qs + (1-q)b)\ {}_{b^-}D_q f(s)\ {}_{b^-}d_q s$,
if f is right quantum integrable on $[t, b]$ for all $t \in [a, b]$.

Proof. The proof of (i) and (ii) follows by direct computation by Definition 3.3.

(iii) Using Definitions 3.1 and 3.3, then

$$\begin{aligned}&{}_{b^-}D_q \int_t^b f(s)\,{}_{b^-}d_q s\\ &= \frac{\int_t^b f(s)\ {}_{b^-}d_q s - \int_{qt+(1-q)b}^b f(s)\ {}_{b^-}d_q s}{(1-q)(t-b)}\\ &= \frac{\begin{bmatrix}(1-q)(b-t)\sum_{n=0}^{\infty} q^n f(q^n t + (1-q^n)b)\\ -(1-q)\big(b-(qt+(1-q)b)\big)\sum_{n=0}^{\infty} q^n f(q^n(qt+(1-q)b) + (1-q^n)b)\end{bmatrix}}{(1-q)(t-b)}\end{aligned}$$

$$= \frac{(b-t)\sum_{n=0}^{\infty} q^n f(q^n t+(1-q^n)b) - q(b-t)\sum_{n=0}^{\infty} q^n f(q^{n+1}t+(1-q^{n+1})b)}{(t-b)}$$

$$= \sum_{n=1}^{\infty} q^n f(q^n\, t + (1-q^n)b) - \sum_{n=0}^{\infty} q^n f(q^n\, t + (1-q^n)b)$$

$$= -f(t).$$

(iv) Similarly to (iii), we have

$$\int_t^b {}_{b^-}D_q f(s)\, {}_{b^-}d_q s$$

$$= \int_t^b \frac{f(s)-f(qs+(1-q)b)}{(1-q)(s-b)}\ {}_{b^-}d_q s$$

$$= \frac{1}{(1-q)}\left[\int_t^b \frac{f(s)}{s-b}\ {}_{b^-}d_q s - \int_t^b \frac{f(qs+(1-q)b)}{s-b}\ {}_{b^-}d_q s\right]$$

$$= \frac{1}{(1-q)}\left[\begin{array}{c}(1-q)(b-t)\sum_{n=0}^{\infty} q^n \frac{f(q^n t+(1-q^n)b)}{(q^n t+(1-q^n)b)-b} \\ -(1-q)(b-t)\sum_{n=0}^{\infty} q^n \frac{f(q(q^n t+(1-q^n)b)+(1-q)b)}{(q^n t+(1-q^n)b)-b}\end{array}\right]$$

$$= (b-t)\sum_{n=0}^{\infty} q^n \frac{f(q^n t+(1-q^n)b)}{q^n(t-b)} - (b-t)\sum_{n=0}^{\infty} q^n \frac{f(q^{n+1}t+(1-q^{n+1})b)}{(q^n(t-b)}$$

$$= \sum_{n=0}^{\infty} f(q^{n+1}t + (1-q^{n+1})b) - \sum_{n=0}^{\infty} f(q^n t + (1-q^n)b)$$

$$= \lim_{k\to\infty}[f(q^{k+1}t + (1-q^{k+1})b - f(t)] \quad \text{(Since } f(t) \text{ is continuous)}$$

$$= f(b) - f(t).$$

(v) Using Theorem 3.1 (ii), we have

(3.11) $\quad f(s)\ {}_{b^-}D_q g(s) = {}_{b^-}D_q(fg)(s) - g(qs+(1-q)b)\ {}_{b^-}D_q f(s)\,.$

Taking the right definite quantum integral of (3.11) on $[t,b]$, it gives that

(3.12) $\quad \int_t^b\ f(s)\ {}_{b^-}D_q g(s)\ {}_{b^-}d_s = \int_t^b\ {}_{b^-}D_q(fg)(s)\ {}_{b^-}d_s - \int_t^b\ g(qs+(1-q)b)\ \ {}_{b^-}D_q f(s)\ {}_{b^-}d_s\,.$

If the previous result is used in (3.12), then the desired result is obtained. ∎

4.CORRECTION OF "QUANTUM OSTROWSKI INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS"

In this section the right quantum derivative and the right quantum integral are used to correct the identity given in [22, Lemma 3.1].

Lemma 4.1. Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function with ${}_{a^+}D_q f$ and ${}_{b^-}D_q f$ are left quantum integrable on $[a,b]$, then we have

$$(4.1)\qquad f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_{a^+}d_qt+\int_x^b f(t)\ {}_{b^-}d_qt\right]$$
$$=\frac{q(x-a)^2}{b-a}\int_0^1 t\ {}_{a^+}D_q\, f(tx+(1-t)a)\quad {}_{0^+}d_qt$$
$$-\frac{q(b-x)^2}{b-a}\int_0^1 t\ {}_{b^-}D_q\, f(tx+(1-t)b)\quad {}_{0^+}d_qt$$

for all $x\in[a,b]$.

Proof. From [22, Lemma 3.1] , we have

$$(4.2)\qquad =\frac{q(x-a)^2}{b-a}\int_0^1 t\ {}_{a^+}D_q\, f(tx+(1-t)a)\ {}_{0^+}d_qt\ =\frac{x-a}{b-a}f(x)-\frac{1}{b-a}\int_a^x f(t)\ {}_{a^+}d_qt\ .$$

Similarly, we have

$$(4.3)\qquad -\frac{q(b-x)^2}{b-a}\int_0^1 t\ {}_{b^-}D_q\, f(tx+(1-t)b)\quad {}_{0^+}d_qt$$
$$=-\frac{q(b-x)^2}{b-a}\int_0^1 t\ \frac{f(tx+(1-t)b)-f(tqx+(1-tq)b)}{(1-q)t(x-b)}\quad {}_{0^+}d_qt$$
$$=\frac{q(b-x)}{(b-a)(1-q)}\left[\int_0^1 f(tx+(1-t)b)\ {}_{0^+}d_qt-\int_0^1 f(tqx+(1-tq)b)\ {}_{0^+}d_qt\ \right]$$
$$=\frac{q(b-x)}{(b-a)(1-q)}\left[\begin{array}{c}(1-q)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)\\-(1-q)\sum_{n=0}^{\infty}q^nf(q^{n+1}x+(1-q^{n+1})b)\end{array}\right]$$
$$=\frac{q(b-x)}{(b-a)}\left[\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)-\frac{1}{q}\sum_{n=1}^{\infty}q^nf(q^nx+(1-q^n)b)\right]$$
$$=\frac{q(b-x)}{(b-a)}\left[\sum_{n=0}^{\infty}q^nf(\ q^nx+(1-q^n)b)-\frac{1}{q}\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)+\frac{f(x)}{q}\right]$$
$$=\frac{q(b-x)}{(b-a)}\left[\left(1-\frac{1}{q}\right)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)+\frac{f(x)}{q}\right]$$
$$=\frac{(b-x)}{(b-a)}\ f(x)-\frac{1}{b-a}\left[(1-q)(b-x)\ \sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)\quad\right]$$
$$=\frac{(b-x)}{(b-a)}f(x)-\frac{1}{b-a}\int_x^b f(t)\ {}_{b^-}d_qt\ .$$

Summing (4.2) and (4.3), we get (4.1). ■

By using Lemma 4.1, [22, Theorem 3.1 and Theorem 3.2] are corrected as follows:

Theorem 4.1. (Correction to [22, Theorem 3.1]) Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function with ${}_{a^+}D_qf$ and ${}_{b^-}D_qf$ are left quantum integrable on $[a,b]$. If $\left|{}_{a^+}D_qf\right|$ and $\left|{}_{b^-}D_qf\right|$ are convex functions, $\left|{}_{a^+}D_qf\ (x)\right|\le M$ and $\left|{}_{b^-}D_qf(x)\ \right|\le M$ for all $x\in[a,b]$, then we have

$$(4.4)\qquad \left|f(x)-\frac{1}{b-a}\left[\ \int_a^x f(t)\ \ {}_{a^+}d_q\,t+\int_x^b f(t)\ \ {}_{b^-}d_qt\right]\ \right|\le\frac{qM\left[(x-a)^2+(b-x)^2\right]}{(b-a)(1+q)}.$$

Theorem 4.2. (Correction to [22, Theorem 3.2]) Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function with ${}_{a^+}D_qf$ and ${}_{b^-}D_qf$ are left quantum integrable on $[a,b]$. If $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex functions for $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, $\left|{}_{a^+}D_qf(x)\right|\le M$ and $\left|{}_{b^-}D_qf(x)\right|\le M$ for all $x\in[a,b]$, then we have

(4.5) $$\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_{a^+}d_q t+\int_x^b f(t)\ {}_{b^-}d_q t\right]\right|\le\frac{qM[(x-a)^2+(b-x)^2]}{(b-a)}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}.$$

5. SOME NEW QUANTUM INTEGRAL INEQUALITIES

In this section, new quantum Hermite-Hadamard typr inequalities are investigated with respect to the left and right definite quantum integrals.

Theorem 5.1. Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have

(5.1) $$f\left(\frac{a+b}{2}\right)\le\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_{a^+}d_q t+\int_a^b f(t)\ {}_{b^-}d_q t\right]\le\frac{f(a)+f(b)}{2}.$$

Proof. Clearly

(5.2) $$\int_0^1 f(tb+(1-t)a)\ {}_{0^+}d_q=(1-q)\sum_{n=0}^{\infty}q^nf(q^nb+(1-q^n)a)$$

$$=\frac{1}{b-a}\left[(1-q)(b-a)\sum_{n=0}^{\infty}q^nf(q^nb+(1-q^n)a)\right]$$

$$=\frac{1}{b-a}\int_a^b f(t)\ {}_{a^+}d_q t,$$

(5.3) $$\int_0^1 f(ta+(1-t)b)\ {}_{0^+}d_q=(1-q)\sum_{n=0}^{\infty}q^nf(q^na+(1-q^n)b)$$

$$=\frac{1}{b-a}\left[(1-q)(b-a)\sum_{n=0}^{\infty}q^nf(q^na+(1-q^n)b)\right]$$

$$=\frac{1}{b-a}\int_a^b f(t)\ {}_{b^-}d_q t,$$

Since $f(t)$ is a convex function on $[a,b]$, then we have

(5.4) $$f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]\le\frac{f(a)+f(b)}{2},$$

for all $t\in[0,1]$. Taking the left definite quantum integral on all sides of the inequalities (5.4) over [0,1], using (5.2) and (5.3), we have (5.1). ■

Similarly to Theorem 2.1, the right quantum Hermite –Hadamard inequality can be given as follows:

Theorem 5.2. Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have

(5.5) $$f\left(\frac{a+qb}{1+q}\right)\le\frac{1}{b-a}\int_a^b f(t)\ {}_{b^-}d_q t\le\frac{f(a)+qf(b)}{1+q}.$$

Proof. It is omitted because the proof is similar to the proof of Theorem 2.1.

In the following quantum Hermite-Hadamard type inequalities , the left and right quantum integrals are used together.

Theorem 5.3. Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have

(5.6) $$\frac{f\left(\frac{qa+b}{1+q}\right)+f\left(\frac{a+qb}{1+q}\right)}{2} \leq \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_{a^+}d_q t+\int_a^b f(t)\ {}_{b^-}d_q t\right] \leq \frac{f(a)+f(b)}{2}.$$

Proof. Adding (2.7) and (5.5) side by side and multiplying the final inequality by $\frac{1}{2}$, implies (5.6) ∎

Remark 5.1. Since any convex function $f(t)$ and $0<q<1$. we have $f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{qa+b}{1+q}\right)+f\left(\frac{a+qb}{1+q}\right)}{2}$, i.e.., (5.6) is better than (5.1).

Thearem 5.4. Let $f,g:[a,b]\to[0,\infty)$ be the right quantum integrable functions on $[a,b]$ for $0<q<1,\ r,s>1$ and $\frac{1}{r}+\frac{1}{s}=1$. Then we have :

(5.7) $$\int_a^b f(t)g(t)\ {}_{b^-}d_q t \leq \left(\int_a^b f^r(t)\ {}_{b^-}d_q t\right)^{1/r}\left(\int_a^b g^s(t)\ {}_{b^-}d_q t\right)^{1/s}.$$

Proof. Using classical Hölder inequality for sum and Definition 3.3,

$$\int_a^b f(t)g(t)\ {}_{b^-}d_q = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n a+(1-q^n)b)g(q^n a+(1-q^n)b)$$

$$=(1-q)(b-a)$$

$$\times\sum_{n=0}^{\infty}\left[(q^n)^{1/r} f(q^n a+(1-q^n)b)\right]\left[(q^n)^{1/s} g(q^n a+(1-q^n)b)\right]$$

$$\leq (1-q)(b-a)\left(\sum_{n=0}^{\infty} q^n f^r(q^n a+(1-q^n)b)\right)^{1/r}$$

$$\times\left(\sum_{n=0}^{\infty} q^n g^s(q^n a+(1-q^n)b)\right)^{1/s}$$

$$=\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n f^r(q^n a+(1-q^n)b)\right)^{1/r}$$

$$\times\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n g^s(q^n a+(1-q^n)b)\right)^{1/s}$$

$$=\left(\int_a^b f^r(t)\ {}_{b^-}d_q t\right)^{1/r}\left(\int_a^b g^s(t)\ {}_{b^-}d_q t\right)^{1/s}.$$

The proof is completed. ∎

Theorem 5.5. Let $f:[a,b]\to\mathbb{R}$ be a right quantum differentiable function for $0<q<1$ and ${}_{b^-}D_q f\in L_\infty[a,b]$, then we have

(5.8) $$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(qt+(1-q)b)\ {}_{b^-}d_q t\right| \leq \frac{(b-a)^2}{2(1+q)}\left\|{}_{b^-}D_q f\right\|_\infty,$$

where $\left\|{}_{b^-}D_q f\right\|_\infty = \sup_{t\in[a,b]}\left|{}_{b^-}D_q f(t)\right|$.

Proof. Using Theorem 3.2 (v) (The right quantum partial integration formula), we get

(5.9) $$\int_a^b\left(\frac{a+b}{2}-t\right)\ {}_{b^-}D_q f(t)\ {}_{b^-}d_q t$$

$$=\left[\left(\frac{a+b}{2}-t\right)f(t)\right](b)-\left[\left(\frac{a+b}{2}-t\right)f(t)\right](a)$$

$$-\int_a^b f(qt+(1-q)b)\ _{b^-}D_q(\tfrac{a+b}{2}-t)\ _{b^-}d_q s$$

$$=-\tfrac{(b-a)}{2}f(a)-\tfrac{(b-a)}{2}f(b)+\int_a^b f(qt+(1-q)b)\ _{b^-}d_q s$$

$$=-(b-a)\tfrac{f(a)+f(b)}{2}+\int_a^b f(qt+(1-q)b)\ _{b^-}d_q s\ .$$

Then from (5.9), we have

(5.10) $$\left|(b-a)\tfrac{f(a)+f(b)}{2}-\int_a^b f\big(qt+(1-q)b\big)\ _{b^-}d_q s\big)\right|$$

$$\le\int_a^b\left|\tfrac{a+b}{2}-t\right|\ \left|_{b^-}D_q f(t)\right|\ _{b^-}d_q t$$

$$\le\int_a^b\left|\tfrac{a+b}{2}-t\right|\ \left\|_{b^-}D_q f\right\|_\infty\ _{b^-}d_q s$$

$$=\left\|_{b^-}D_q f\right\|_\infty\int_a^b\left|\tfrac{a+b}{2}-t\right|\ _{b^-}d_q s\ .$$

Calculating the following right quantum integral,

(5.11) $$\int_a^b\left|\tfrac{a+b}{2}-t\right|\ _{b^-}d_q t$$

$$=\left[\int_a^{\frac{a+b}{2}}\left(\tfrac{a+b}{2}-t\right)\ _{b^-}d_q t+\int_{\frac{a+b}{2}}^b\left(t-\tfrac{a+b}{2}\right)\ _{b^-}d_q t\right]$$

$$=\left[\int_a^b\left(\tfrac{a+b}{2}-t\right)\ _{b^-}d_q t+2\int_{\frac{a+b}{2}}^b\left(t-\tfrac{a+b}{2}\right)\ _{b^-}d_q t\right]$$

$$=\begin{bmatrix}(1-q)(b-a)\sum_{n=0}^{\infty}q^n\left[(\tfrac{a+b}{2}-(q^n a+(1-q^n)b)\right]\\+2(1-q)\left(b-\tfrac{a+b}{2}\right)\sum_{n=0}^{\infty}q^n\left[\left(q^n\tfrac{a+b}{2}+(1-q^n)b\right)-\tfrac{a+b}{2}\right]\end{bmatrix}$$

$$=\begin{bmatrix}(1-q)(b-a)\left[\tfrac{a+b}{2}\sum_{n=0}^{\infty}q^n-a\sum_{n=0}^{\infty}q^{2n}-b\sum_{n=0}^{\infty}q^n+b\sum_{n=0}^{\infty}q^{2n}\right]\\+2(1-q)\left(b-\tfrac{a+b}{2}\right)\left[\tfrac{a+b}{2}\sum_{n=0}^{\infty}q^{2n}+b\sum_{n=0}^{\infty}q^n-b\sum_{n=0}^{\infty}q^{2n}-\tfrac{a+b}{2}\sum_{n=0}^{\infty}q^n\right]\end{bmatrix}$$

$$=\begin{bmatrix}(b-a)\left[\tfrac{a+b}{2}-\tfrac{a}{1+q}-b+\tfrac{b}{1+q}\right]\\+2\left(b-\tfrac{a+b}{2}\right)\left[\tfrac{a+b}{2(1+q)}+b-\tfrac{b}{1+q}-\tfrac{a+b}{2}\right]\end{bmatrix}$$

$$=\left[(b-a)^2\left[\tfrac{1}{1+q}-\tfrac{1}{2}\right]+(b-a)^2\left[\tfrac{1}{2}-\tfrac{1}{2(1+q)}\right]\right]$$

$$=\tfrac{(b-a)^2}{2(1+q)}\ .$$

The desired inequality is obtained from combining (5.10) and (5.11), we have (5.8). This completes the proof. ■

6. CONCLUSION

Quantum integral inequalities on an arbitrary finite interval is widely studied in the papers [6,7], [18]-[36] and some others. From now on, they should be considered as the left quantum integral inequalites. Definitions of the right quantum derivative and the right definite quantum integral are bear a torch to many aymmetric invertigations. Using the left and right quantum integrals together could give interesting results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper.The authors read and approved the final manuscript.

Funding
Not applicable

Acknowledgements
Not applicable

Availatibility of Data and Materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

REFERENCES

[1] Ali MA, Abbas M, Budak H, Agarwal P, Murtaza G, Chu YM. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Advances in Difference Equations 2021; 2021(2): 1-21.
[2] Ali M, Budak H, Abbas M, Chu YM. Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^b-derivaties. Advances in Difference Equations 2021; 2021(1): 1-12.
[3] Ali MA, Budak H, Zhang Z, Yıldırım H. Some new Simpson's type inequalities for coordinated convex functions in quantum calculus. Mathematical Methods in the Applied Sciences 2021; 44(6): 4515-4540.
[4] Ali MA, Chu YM, Budak H, Akkurt A, Yıldırım H, Zahid MA. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Advance in Difference Equations, 2021; 2021(1): 1-26.
[5] Annaby MH, Mansour ZS. q-Fractional Calculus and Eguations: Springer, Heidelberg, 2012.
[6] Alp N, Sarıkaya MZ. A new definition and properties of quantum intrgral which calls $\bar{q}$-integral. Konuralp J Math 2017; 5 (2):146-159.
[7] Alp N, Sarıkaya MZ, Kunt M, İşcan İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J King Saud Univ Sci 2018; 30 (2): 193-203.
[8] Awan MU, Talib S, Kashuri A, Noor MA, Chu YM. Estimates of quantum bounds pertaining to new q-integral identity with applications. Advances in Difference Equations 2020; 2020(1): 1-15.
[9] Awan MU, Talib S, Kashuri A, Noor MA, Noor KI, Chu YM. A new q-integral identity and estimation of its bounds involving generalized exponentially μ-preinvex functions. Advances in Difference Equations 2020; 2020(1): 1-12.
[10] Budak H, Ali MA, Tarhanacı M. Sone new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. Journal of Optimization Theary and Applications 2020; 189(3): 899-910.
[11] Budak H, Ali MA, Tunç T. Quantum Ostrowski-type integral inequalities for functions of two variables. Mathematical Methods in the Applied Sciences 2020.
[12] Budak H, Erden S, Ali MA. Simpson and Newton Type inequalities for convex functions via newly defined quantum integrals. Mathematical Methods in the Applied Sciences2021; 44(1): 378-390.
[13] Bermudo S, Korus P, Valdes JEN. On q-Hermite-Hadamard inequalities for general convex functions, Acta Math Hungar 2020; 162 (1): 364-374
[14] Dragomir SS, Agarwal R. Two inequalities for diffirentiable mappings and applications to special means of real numbers and to trapezoidal formula. Applied Mathematics Letters 1998; 11(5): 91-95
[15] Erden S, Iftikhar S, Delavar MR, Kumam P, Thounthong P, Kumam W. On generalications of some inequalities for convex functions via quantum integrals. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 2020; 114(3): 1-15.
[16] Kavurmacı H, Avci M, Özdemir ME. New inequalities of Hermite-Hadamad type for convex functions with applications. Journal of Inequalities and Applications 2011; 2011(1): 1-11.
[17] Kac V, Cheung P. Quantum calculus: Springer 2001.
[18] Kunt M, İşcan İ, Alp N, Sarıkaya MZ. (p,q) −Hermite-Hadamard inequalities and (p,q)-estimates for midpoint type

inequalities via convex and quasi-cunvex functions, RACSAM 2018; 112: 969-992.
[19] Kunt M, Kashuri A, Du TS, Baidar AW. Quantum Montgomery identity and some new quantum integral inequalities, AIMS Math. 2020; 5(6): 5439-5457.
[20] Kunt M, Latif MA, İşcan İ, Dragomir SS. Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals. Sigma J Eng Nat Sci. 2019; 37(1): 207-223.
[21] Khan MA, Mohammad N, Nwaeze ER, Chu YM. Quantum Hermite-Hadamard inequality by means of a Green function. Advances in Difference Equations 2020; 2020(1): 1-20.
[22] Noor MA, Awan MU, Noor KI, Quantum Ostrowski inequalities for q-differentiable convex functions. J Math Inequal. 2016; 10(4):1013-1018.
[23] Noor MA, Noor KI, Awan MU. Some quantum estimates for Hermite-Hadamard inequalities. Appl Marth Comput 2015; 251: 675-679.
[24] Noor MA, Noor KI, Awan MU. Some quantum integral inequalities via preinvex functions. Appl Math Comput 2015; 269: 242-251.
[25] Prabseang J, Nonlaopon K, Ntouyas SK. On the refinement of quantum Hermite-Hadamard inequalities for continuous convex functions. J Math Inequal 2020; 14(3): 875-885.
[26] Pearce CE, Pearic J. Inequalities for differentiable mappings with application to special means and quadrature formulae. Applied Mathematics Letters 2000; 13(2): 51-55.
[27] Rashid S, Butt SI, Kanwal S, Ahmad H, Wang MK. Quantum Integral Inequalities with Respect to Raina's Function via Coordinated Generalized-Convex Functions with Applications. Journal of Function Spaces 2021.
[28] Sudsutad , Ntouyas SK, Tariboon T. Quantum integral inequalities for convex fonctions. J Math Inequal 2015; 9(3): 781-793.
[29] Tunç M, Gov E, Balgeçti S. Simpson type quantum integral inequalities for convex functions. Miskolc Math Notes 2018; 19 (1): 649-664.
[30] Tariboon J, Ntouyas SK. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv Difference Equ 2013; 282:1-19.
[31] Tariboon J, Ntouyas SK. Quantum integral inequalities on finite intervals. J Inequal Appl 2014; 121: 1-13.
[32] Zhang T, Du TS, Wang H, Shen YJ. Different types of quantum integral inequalities via (α, m)-convexity. J Inequal Appl 2018; 264: 1-24.
[33] Vivas-Cortez M, Aamir Ali, M, Kashuri A, Bashir Sial I, Zhang Z. Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus. Symmetry 2020; 12(9): 1476.
[34] Vivas-Cortez M, Kashuri A, Liko R, Hernandez JE. Some new q-integral inequalities using generalized quantum Montgomery identity via preinvex functions. Symmetry 2020; 12(4): 553.
[35] You X, Kara H, Budak H, Kalsoom H. Quantum Inequalities of Hermite-Hadamard Type for-Convex Functions. Journal of Mathematics 2021.
[36] Zhuang H, Liu W, Park J. Some quantum estimates of Hermite-Hadamard inequalities for quasi-convex functions. Math 2019; 7(2): 1-18.