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Research Article

On the almost h-conformal semi-slant Riemannian maps

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Abstract

As a generalization of conformal semi-slant submersions, semi-slant Riemannian maps, almost h-semi-slant submersions and almost h-semi-slant Riemannian maps, we introduce almost h-conformal semi-slant submersions and almost h-conformal semi-slant Riemannian maps. We give some examples of such maps and also introduce some types of pluriharmonic maps, invariant maps and geodesic maps. We study the geometry of foliations, the integrability of distributions, the properties of pluriharmonic maps, invariant maps and geodesic maps. We also investigate the condition for such maps to be totally geodesic and the harmonicity of such maps.

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1. Introduction

In differential geometry, to investigate Riemannian manifolds, we usually use some types of maps between manifolds. Given a horizontally conformal submersion φ from a Riemannian manifold (N, g_N) into a Riemannian manifold (M, g_M), which is a generalization of a Riemannian submersion, we naturally obtain two distributions: a vertical distribution $\ker \varphi_*$ and a horizontal distribution $(\ker \varphi_*)^{\perp}$. It is interesting to study them as follows: the integrability of distributions and the geometry of foliations. If we add some geometrical structures to the Riemannian manifold (N, g_N) (i.e., an almost complex structure, a Kähler structure, a quaternionic Kähler structure, a hyperkähler structure, an almost contact metric structure, a Sasakian structure, a Kenmotsu structure, a cosymplectic structure), then we have some other distributions (i.e., $\mathcal{D}_1^P$, $\mathcal{D}_2^P$, $\omega_P \mathcal{D}_2^P$, μ^P) and it is also very interesting to investigate them. Similarly, given a conformal Riemannian map $\varphi : (N, g_N) \mapsto (M, g_M)$, which is a generalization of a Riemannian map, we also obtain some other distributions (i.e., $\text{range}\, \varphi_*$ and $(\text{range}\, \varphi_*)^{\perp}$). With some types of pluriharmonic maps, invariant maps and geodesic maps, we also investigate the properties of the manifold (N, g_N). Since the introduction of Riemannian submersions ([17], [10]) in 1960s, this area has been developed and studied by many geometers: horizontally conformal submersions ([9], [13]) in 1970s, Riemannian maps ([8]) in 1992, conformal Riemannian maps ([22]) in 2010, other conformal maps ([2], [3], [23]).

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With these maps, we have many applications: Kaluza-Klein theory, supergravity and superstring theories, Yang-Mills theory, computer vision, geometric modeling, medical imaging([5], [12], [15], [6], [14], [25], [26]).

As a generalization of conformal semi-slant submersions [1], semi-slant Riemannian maps [20], almost h-semi-slant submersions, h-semi-slant submersions [19], almost h-semi-slant Riemannian maps and h-semi-slant Riemannian maps [18], we introduce almost h-conformal semi-slant submersions, h-conformal semi-slant submersions, almost h-conformal semi-slant Riemannian maps and h-conformal semi-slant Riemannian maps.

We organized the paper as follows. In section 2 we remind some notions, which will be used later. In section 3 we define the notions of almost h-conformal semi-slant submersions, h-conformal semi-slant submersions, almost h-conformal semi-slant Riemannian maps and h-conformal semi-slant Riemannian maps and give some examples of such maps. In section 4 after introducing some types of pluriharmonic maps, invariant maps and geodesic maps, we study the geometry of foliations, the integrability of distributions and the condition for such maps to be horizontally homothetic. In section 5 we consider the conditions for such maps to be totally geodesic and the harmonicity of such maps.

2. Preliminaries

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds, where g_1 and g_2 are Riemannian metrics on the C^∞-manifolds N_1 and N_2, respectively.

Let $\varphi : (N_1, g_1) \mapsto (N_2, g_2)$ be a C^∞-map.

The *second fundamental form* of φ is given by

$$(\nabla\varphi_*)(X, Z) := \nabla^\varphi_X \psi_* Z - \varphi_*(\nabla_X Z) \quad \text{for } X, Z \in \Gamma(TN_1),$$

where ∇^φ is the pullback connection [4].

Recall that φ is *harmonic* if the tension field $\tau(\varphi) = \operatorname{trace}(\nabla\varphi_*) = 0$ and φ is *totally geodesic* if $(\nabla\varphi_*)(X, Z) = 0$ for $X, Z \in \Gamma(TN_1)$ [4].

Lemma 2.1 ([24])**.** *Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and $\varphi : (N_1, g_1) \mapsto (N_2, g_2)$ a C^∞-map. Then we obtain*

$$\nabla^\varphi_X \varphi_* Z - \nabla^\varphi_Z \varphi_* X - \varphi_*([X, Z]) = 0 \tag{2.1}$$

for $X, Z \in \Gamma(TN_1)$.

Remark 2.2. (1) From (2.1), $\nabla\varphi_*$ is symmetric.

(2) From (2.1),

$$[W, Z] \in \Gamma(\ker \varphi_*) \tag{2.2}$$

for $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$.

The map φ is said to be a *conformal Riemannian map* if there exists a positive function λ on N_1 (i.e., $\lambda(q) > 0$, $\forall q \in N_1$) such that

$$g_2(\varphi_* X, \varphi_* Z) = \lambda^2 g_1(X, Z) \quad \text{for } X, Z \in \Gamma((\ker \varphi_*)^\perp) \tag{2.3}$$

and $0 < \operatorname{rank} \varphi_{*p} = \operatorname{rank} \varphi_{*q} \leq \min(\dim N_1, \dim N_2)$ for $p, q \in N_1$, where $(\ker \varphi_*)^\perp$ is the orthogonal complement of $\ker \varphi_*$ in TN_1. We call λ *dilation*.

We call the map φ a *horizontally conformal submersion* if φ is surjective and φ is a conformal Riemannian map.

Theorem 2.3 ([22])**.** *Let $\varphi : (N_1, g_1) \mapsto (N_2, g_2)$ be a conformal Riemannian map with dilation λ. Then we get*

$$\begin{aligned}(\nabla\varphi_*)&(X, Z) \\ &= (\nabla\varphi_*)(X, Z)^{range\,\varphi_*} + (\nabla\varphi_*)(X, Z)^{(range\,\varphi_*)^\perp} \\ &= X(\ln\lambda)\varphi_* Z + Z(\ln\lambda)\varphi_* X - g_1(X, Z)\varphi_*(\nabla \ln\lambda) + (\nabla\varphi_*)(X, Z)^{(range\,\varphi_*)^\perp}\end{aligned} \tag{2.4}$$

for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, where $(range\, \varphi_*)^\perp$ is the orthogonal complement of $range\, \varphi_*$ in $\varphi^{-1}TN_2$.

In convenience, let $(\nabla\varphi_*)^r(X,Z) := (\nabla\varphi_*)(X,Z)^{\mathrm{range}\,\varphi_*}$, $(\nabla\varphi_*)^\perp(X,Z) := (\nabla\varphi_*)(X,Z)^{(\mathrm{range}\,\varphi_*)^\perp}$, $\nabla_X^{\varphi r}\varphi_* Z := (\nabla_X^\varphi \varphi_* Z)^{\mathrm{range}\,\varphi_*}$, $\nabla_X^{\varphi\perp}\varphi_* Z := (\nabla_X^\varphi \varphi_* Z)^{(\mathrm{range}\,\varphi_*)^\perp}$. We call the conformal Riemannian map φ *horizontally homothetic* if $Z(\lambda) = 0$ for $Z \in \Gamma((\ker \varphi_*)^\perp)$.

Let $\varphi : (N_1, g_1) \mapsto (N_2, g_2)$ be a conformal Riemannian map.

Given $W \in \Gamma(TN_1)$, we have

$$W = \mathcal{V}W + \mathcal{H}W, \tag{2.5}$$

where $\mathcal{V}W \in \Gamma(\ker \varphi_*)$ and $\mathcal{H}W \in \Gamma((\ker \varphi_*)^\perp)$.

We define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_V W = \mathcal{H}\nabla_{\mathcal{H}V}\mathcal{V}W + \mathcal{V}\nabla_{\mathcal{H}V}\mathcal{H}W, \tag{2.6}$$

$$\mathcal{T}_V W = \mathcal{H}\nabla_{\mathcal{V}V}\mathcal{V}W + \mathcal{V}\nabla_{\mathcal{V}V}\mathcal{H}W \tag{2.7}$$

for vector fields $V, W \in \Gamma(TN_1)$, where ∇ is the Levi-Civita connection of g_1 ([17], [7]).

Then we obtain

$$g_1(\mathcal{T}_U V, W) = -g_1(V, \mathcal{T}_U W), \tag{2.8}$$

$$g_1(\mathcal{A}_U V, W) = -g_1(V, \mathcal{A}_U W) \tag{2.9}$$

for $U, V, W \in \Gamma(TN_1)$.

We also have

$$\mathcal{T}_V W = \mathcal{T}_W V \quad \text{for } V, W \in \Gamma(\ker \varphi_*). \tag{2.10}$$

Throughout this paper, we will use these notations.

3. Almost h-conformal semi-slant Riemannian maps

Throughout the paper, we denote by (N, E, g_N) an almost quaternionic Hermitian manifold, where E is an almost quaternionic structure on N (See [21]). And we denote by (N, I, J, K, g_N) a hyperkähler manifold, where (I, J, K, g_N) is a hyperkähler structure on N (See [21]). In this section we define almost h-conformal semi-slant Riemannian maps, h-conformal semi-slant Riemannian maps, almost h-conformal semi-slant submersions, h-conformal semi-slant submersions and give some examples of such maps.

Definition 3.1. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be a horizontally conformal submersion. The map φ is said to be an *h-conformal semi-slant submersion* if given $x \in N$ with a neighborhood W, there is a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on W such that for any $P \in \{I, J, K\}$, there exist two orthogonal complementary distributions $\mathcal{D}_1, \mathcal{D}_2 \subset \ker \varphi_*$ on W such that

$$\ker \varphi_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \;\; P(\mathcal{D}_1) = \mathcal{D}_1$$

and the angle $\theta_P = \theta_P(Y)$ between PY and the space $(\mathcal{D}_2)_p$ is constant for nonzero $Y \in (\mathcal{D}_2)_p$ and $p \in W$.

We call the basis $\{I, J, K\}$ an *h-conformal semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-conformal semi-slant angles*.

Definition 3.2. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be a horizontally conformal submersion. The map φ is said to be an *almost h-conformal semi-slant submersion* if given $x \in N$ with a neighborhood W, there is a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on W such that for $P \in \{I, J, K\}$, there exist two orthogonal complementary distributions $\mathcal{D}_1^P, \mathcal{D}_2^P \subset \ker \varphi_*$ on W such that

$$\ker \varphi_* = \mathcal{D}_1^P \oplus \mathcal{D}_2^P, \;\; P(\mathcal{D}_1^P) = \mathcal{D}_1^P$$

and the angle $\theta_P = \theta_P(Y)$ between PY and the space $(\mathcal{D}_2^P)_p$ is constant for nonzero $Y \in (\mathcal{D}_2^P)_p$ and $p \in W$.

We call the basis $\{I, J, K\}$ an *almost h-conformal semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-conformal semi-slant angles.*

Definition 3.3. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be a conformal Riemannian map. The map φ is said to be an *h-conformal semi-slant Riemannian map* if given $x \in N$ with a neighborhood W, there is a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on W such that for any $P \in \{I, J, K\}$, there exist two orthogonal complementary distributions $\mathcal{D}_1, \mathcal{D}_2 \subset \ker \varphi_*$ on W such that

$$\ker \varphi_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ P(\mathcal{D}_1) = \mathcal{D}_1$$

and the angle $\theta_P = \theta_P(Y)$ between PY and the space $(\mathcal{D}_2)_p$ is constant for nonzero $Y \in (\mathcal{D}_2)_p$ and $p \in W$.

We call the basis $\{I, J, K\}$ an *h-conformal semi-slant Riemannian basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-conformal semi-slant Riemannian angles.*

Definition 3.4. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be a conformal Riemannian map. The map φ is said to be an *almost h-conformal semi-slant Riemannian map* if given $x \in N$ with a neighborhood W, there is a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on W such that for $P \in \{I, J, K\}$, there exist two orthogonal complementary distributions $\mathcal{D}_1^P, \mathcal{D}_2^P \subset \ker \varphi_*$ on W such that

$$\ker \varphi_* = \mathcal{D}_1^P \oplus \mathcal{D}_2^P, \ P(\mathcal{D}_1^P) = \mathcal{D}_1^P$$

and the angle $\theta_P = \theta_P(Y)$ between PY and the space $(\mathcal{D}_2^P)_p$ is constant for nonzero $Y \in (\mathcal{D}_2^P)_p$ and $p \in W$.

We call the basis $\{I, J, K\}$ an *almost h-conformal semi-slant Riemannian basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-conformal semi-slant Riemannian angles.*

Conveniently, we denote an h-conformal semi-slant submersion, an almost h-conformal semi-slant submersion, an h-conformal semi-slant Riemannian map, an almost h-conformal semi-slant Riemannian map, an h-conformal semi-slant basis, an almost h-conformal semi-slant basis, an h-conformal semi-slant Riemannian basis, an almost h-conformal semi-slant Riemannian basis, an h-conformal semi-slant angle, an almost h-conformal semi-slant angle, an h-conformal semi-slant Riemannian angle, an almost h-conformal semi-slant Riemannian angle by an hss submersion, an ahss submersion, an hssR map, an ahssR map, an hss basis, an ahss basis, an hssR basis, an ahssR basis, an hss angle, an ahss angle, an hssR angle, an ahssR angle, respectively.

Remark 3.5. (1) Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an hss submersion. Then the map φ is also an ahss submersion, an hssR map and an ahssR map.

(2) Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahss submersion. Then the map φ is also an ahssR map.

(3) Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an hssR map. Then the map φ is also an ahssR map.

Now, we give some examples of such maps. We consider a hyperkähler manifold $(\mathbb{R}^{4n}, I, J, K, \langle\ ,\ \rangle)$ such that

$$I(\tfrac{\partial}{\partial x_{4j+1}}) = \tfrac{\partial}{\partial x_{4j+2}}, I(\tfrac{\partial}{\partial x_{4j+2}}) = -\tfrac{\partial}{\partial x_{4j+1}}, I(\tfrac{\partial}{\partial x_{4j+3}}) = \tfrac{\partial}{\partial x_{4j+4}}, I(\tfrac{\partial}{\partial x_{4j+4}}) = -\tfrac{\partial}{\partial x_{4j+3}},$$
$$J(\tfrac{\partial}{\partial x_{4j+1}}) = \tfrac{\partial}{\partial x_{4j+3}}, J(\tfrac{\partial}{\partial x_{4j+2}}) = -\tfrac{\partial}{\partial x_{4j+4}}, J(\tfrac{\partial}{\partial x_{4j+3}}) = -\tfrac{\partial}{\partial x_{4j+1}}, J(\tfrac{\partial}{\partial x_{4j+4}}) = \tfrac{\partial}{\partial x_{4j+2}},$$
$$K(\tfrac{\partial}{\partial x_{4j+1}}) = \tfrac{\partial}{\partial x_{4j+4}}, K(\tfrac{\partial}{\partial x_{4j+2}}) = \tfrac{\partial}{\partial x_{4j+3}}, K(\tfrac{\partial}{\partial x_{4j+3}}) = -\tfrac{\partial}{\partial x_{4j+2}}, K(\tfrac{\partial}{\partial x_{4j+4}}) = -\tfrac{\partial}{\partial x_{4j+1}}$$

for $j \in \{0, 1, \cdots, n-1\}$, where $\langle\ ,\ \rangle$ is the Euclidean metric on $\mathbb{R}^{4n}$.

Example 3.6. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an almost h-semi-slant submersion [19]. Then the map φ is an ahss submersion with dilation $\lambda = 1$.

Example 3.7. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an almost h-semi-slant Riemannian map [18]. Then the map φ is an ahssR map with dilation $\lambda = 1$.

Example 3.8. Let (N, E, g_N) be an almost quaternionic Hermitian manifold. Let $\varphi : TN \mapsto N$ be the projection map [11]. Then the map φ is an hss submersion with $\mathcal{D}_1 = \ker \varphi_*$ and dilation $\lambda = 1$.

Example 3.9. Let (N_1, E_1, g_1) and (N_2, E_2, g_2) be almost quaternionic Hermitian manifolds. Let $\varphi : N_1 \mapsto N_2$ be a quaternionic submersion [11]. Then the map φ is an hss submersion with $\mathcal{D}_1 = \ker \varphi_*$ and dilation $\lambda = 1$.

Example 3.10. Let φ be a conformal Riemannian map from a $4n$-dimensional almost quaternionic Hermitian manifold (N, E, g_N) into a $(4n-1)$-dimensional Riemannian manifold (M, g_M) such that $\operatorname{range} \varphi_* = \varphi^{-1}TM$ and dilation a smooth function λ. Then the map φ is an hssR map such that $\mathcal{D}_2 = \ker \varphi_*$, hssR angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$ and dilation λ.

Example 3.11. Let $\varphi : \mathbb{R}^8 \mapsto \mathbb{R}^7$ be a conformal Riemannian map such that $\operatorname{range} \varphi_* = \varphi^{-1}T\mathbb{R}^7$ and dilation a smooth function λ. Then the map φ is an hssR map such that $\mathcal{D}_2 = \ker \varphi_*$, hssR angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$ and dilation λ.

Example 3.12. Define a map $\varphi : \mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$\varphi(x_1, \cdots, x_8) = (y_1, y_2, y_3) = \pi^2(x_7, \frac{\sqrt{3}}{2}x_5 - \frac{1}{2}x_8, x_6).$$

Then the map φ is an hss submersion such that $\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $\mathcal{D}_2 =< \frac{1}{2}\frac{\partial}{\partial x_5} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_8} >$, $(\ker \varphi_*)^\perp =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_5} - \frac{1}{2}\frac{\partial}{\partial x_8} >$, $\omega_I \mathcal{D}_2 =< \frac{1}{2}\frac{\partial}{\partial x_6} - \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_7} >$, $\omega_J \mathcal{D}_2 =< \frac{1}{2}\frac{\partial}{\partial x_7} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_6} >$, $\omega_K \mathcal{D}_2 =< \frac{1}{2}\frac{\partial}{\partial x_8} - \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_5} >$, hss angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$ and dilation $\lambda = \pi$.

Example 3.13. Define a map $\varphi : \mathbb{R}^{12} \mapsto \mathbb{R}^7$ by

$$\varphi(x_1, \cdots, x_{12}) = (y_1, \cdots, y_7) = e^2(x_{10}, \frac{x_5 - \sqrt{3}x_7}{2}, 68, \frac{\sqrt{3}x_9 - x_{11}}{2}, x_8, 78, 34).$$

Then the map φ is an hssR map such that $\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $\mathcal{D}_2 =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_{12}}, \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_5} + \frac{1}{2}\frac{\partial}{\partial x_7}, \frac{1}{2}\frac{\partial}{\partial x_9} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_{11}} >$, $(\ker \varphi_*)^\perp =< \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{1}{2}\frac{\partial}{\partial x_5} - \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_7}, \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_9} - \frac{1}{2}\frac{\partial}{\partial x_{11}} >$, $\operatorname{range} \varphi_* =< \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_5} >$, $(\operatorname{range} \varphi_*)^\perp =< \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_6}, \frac{\partial}{\partial y_7} >$, hssR angles $\{\theta_I = \frac{\pi}{6}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{3}\}$ and dilation $\lambda = e$.

Example 3.14. Define a map $\varphi : \mathbb{R}^{12} \mapsto \mathbb{R}^6$ by

$$\varphi(x_1, \cdots, x_{12}) = (y_1, \cdots, y_6) = \pi^4(x_{12}, x_9, x_1, x_{10}, x_2, x_{11}).$$

Then the map φ is an ahss submersion such that $\mathcal{D}_1^I =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} > = \ker \varphi_*$, $\mathcal{D}_1^J = \mathcal{D}_1^K =< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} >$, $\mathcal{D}_2^J = \mathcal{D}_2^K =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $(\ker \varphi_*)^\perp = < \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >$, ahss angles $\{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$ and dilation $\lambda = \pi^2$.

Example 3.15. Define a map $\varphi : \mathbb{R}^{12} \mapsto \mathbb{R}^5$ by

$$\varphi(x_1, \cdots, x_{12}) = (y_1, \cdots, y_5) = e^6(x_3, x_{11}, 56, x_1, x_{10}).$$

Then the map φ is an ahssR map such that $\mathcal{D}_1^I =< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} >$, $\mathcal{D}_1^J =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} >$, $\mathcal{D}_1^K =< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} >$, $\mathcal{D}_2^I =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} >$, $\mathcal{D}_2^J =< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} >$, $\mathcal{D}_2^K =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} >$, $(\ker \varphi_*)^\perp =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}} >$, $\operatorname{range} \varphi_* =< \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_5} >$, $(\operatorname{range} \varphi_*)^\perp =< \frac{\partial}{\partial y_3} >$, ahssR angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$ and dilation $\lambda = e^3$.

Example 3.16. Define a map $\varphi : \mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$\varphi(x_1, \cdots, x_8) = (y_1, y_2, y_3) = \pi^8(x_1 \cos\alpha - x_3 \sin\alpha, x_2, 56)$$

with $0 \leq \alpha \leq \frac{\pi}{2}$. Then the map φ is an hssR map such that $\mathcal{D}_1 =< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} >$, $\mathcal{D}_2 =< \sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $(\ker \varphi_*)^\perp =< \frac{\partial}{\partial x_2}, \cos\alpha \frac{\partial}{\partial x_1} - \sin\alpha \frac{\partial}{\partial x_3} >$, $\mathrm{range}\, \varphi_* =< \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} >$, $(\mathrm{range}\, \varphi_*)^\perp =< \frac{\partial}{\partial y_3} >$, hssR angles $\{\theta_I = \alpha, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2} - \alpha\}$ and dilation $\lambda = \pi^4$.

4. Geometry of distributions

Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K).

Given $W \in \Gamma(\ker \varphi_*)$, we have

$$PW = \phi_P W + \omega_P W, \tag{4.1}$$

where $\phi_P W \in \Gamma(\ker \varphi_*)$ and $\omega_P W \in \Gamma((\ker \varphi_*)^\perp)$ for $P \in \{I, J, K\}$.

Given $Z \in \Gamma((\ker \varphi_*)^\perp)$, we write

$$PZ = B_P Z + C_P Z, \tag{4.2}$$

where $B_P Z \in \Gamma(\ker \varphi_*)$ and $C_P Z \in \Gamma((\ker \varphi_*)^\perp)$ for $P \in \{I, J, K\}$.

We easily get

$$\phi_P^2 W + B_P \omega_P W = -W, \qquad \omega_P \phi_P W + C_P \omega_P W = 0,$$
$$\phi_P B_P Z + B_P C_P Z = 0, \qquad \omega_P B_P Z + C_P^2 Z = -Z$$

for $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$.

Then

$$(\ker \varphi_*)^\perp = \omega_P \mathcal{D}_2^P \oplus \mu^P \quad \text{for } P \in \{I, J, K\}, \tag{4.3}$$

where μ^P is the orthogonal complement of $\omega_P \mathcal{D}_2^P$ in $(\ker \varphi_*)^\perp$. It is easy to see that μ^P is P-invariant.

Define $\widehat{\nabla}_V W := \mathcal{V} \nabla_V W$ for $V, W \in \Gamma(\ker \varphi_*)$.

We define

$$(\nabla_V \phi_P) W := \widehat{\nabla}_V \phi_P W - \phi_P \widehat{\nabla}_V W \tag{4.4}$$

and

$$(\nabla_V \omega_P) W := \mathcal{H} \nabla_V \omega_P W - \omega_P \widehat{\nabla}_V W \tag{4.5}$$

for $V, W \in \Gamma(\ker \varphi_*)$ and $P \in \{I, J, K\}$.

Then we get

$$(\nabla_V \phi_P) W = B_P \mathcal{T}_V W - \mathcal{T}_V \omega_P W, \tag{4.6}$$

$$(\nabla_V \omega_P) W = C_P \mathcal{T}_V W - \mathcal{T}_V \phi_P W \tag{4.7}$$

for $V, W \in \Gamma(\ker \varphi_*)$ and $P \in \{I, J, K\}$.

Define

$$(\nabla_X B_P) Z := \mathcal{V} \nabla_X B_P Z - B_P \mathcal{H} \nabla_X Z \tag{4.8}$$

and

$$(\nabla_X C_P) Z := \mathcal{H} \nabla_X C_P Z - C_P \mathcal{H} \nabla_X Z \tag{4.9}$$

for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$ and $P \in \{I, J, K\}$.

Then we obtain

$$(\nabla_X B_P) Z = \phi_P \mathcal{A}_X Z - \mathcal{A}_X C_P Z, \tag{4.10}$$

$$(\nabla_X C_P) Z = \omega_P \mathcal{A}_X Z - \mathcal{A}_X B_P Z \tag{4.11}$$

for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$ and $P \in \{I, J, K\}$.

Remark 4.1. (1) Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). Then we have the orthogonal decompositions

$$TN = \ker \varphi_* \oplus (\ker \varphi_*)^\perp = \mathcal{D}_1^P \oplus \mathcal{D}_2^P \oplus \omega_P \mathcal{D}_2^P \oplus \mu^P$$

and

$$\varphi^{-1}TM = \operatorname{range} \varphi_* = \varphi_*(\omega_P \mathcal{D}_2^P) \oplus \varphi_* \mu^P$$

for $P \in \{I, J, K\}$.

(2) Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then we get the orthogonal decompositions

$$TN = \ker \varphi_* \oplus (\ker \varphi_*)^\perp = \mathcal{D}_1^P \oplus \mathcal{D}_2^P \oplus \omega_P \mathcal{D}_2^P \oplus \mu^P$$

and

$$\varphi^{-1}TM = \operatorname{range} \varphi_* \oplus (\operatorname{range} \varphi_*)^\perp = \varphi_*(\omega_P \mathcal{D}_2^P) \oplus \varphi_* \mu^P \oplus (\operatorname{range} \varphi_*)^\perp$$

for $P \in \{I, J, K\}$.

Proposition 4.2. *Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahssR map. Then we have*

$$\phi_P^2 W = -\cos^2 \theta_P \; W \tag{4.12}$$

for $W \in \Gamma(\mathcal{D}_2^P)$ and $P \in \{I, J, K\}$, where $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. Given nonzero $W \in \Gamma(\mathcal{D}_2^P)$, we obtain

$$\cos \theta_P = \frac{g_N(PW, \phi_P W)}{|PW| \, |\phi_P W|} = \frac{|\phi_P W|^2}{|W| \, |\phi_P W|} = \frac{|\phi_P W|}{|W|}$$

so that

$$\cos^2 \theta_P g_N(W, W) = g_N(\phi_P W, \phi_P W) = -g_N(\phi_P^2 W, W).$$

By polarization,

$$\cos^2 \theta_P g_N(W_1, W_2) = -g_N(\phi_P^2 W_1, W_2)$$

for $W_1, W_2 \in \Gamma(\mathcal{D}_2^P)$, which implies the result. □

Remark 4.3. Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahssR map. From (4.12), we easily have

$$g_N(\phi_P V, \phi_P W) = \cos^2 \theta_P \; g_N(V, W),$$
$$g_N(\omega_P V, \omega_P W) = \sin^2 \theta_P \; g_N(V, W)$$

for $V, W \in \Gamma(\mathcal{D}_2^P)$.

Lemma 4.4. *Let $\varphi : (N, E, g_N) \mapsto (M, g_M)$ be an ahssR map. If the tensor ω_P is parallel, then we get*

$$\mathcal{T}_{\phi_P V} \phi_P W = -\cos^2 \theta_P \; \mathcal{T}_V W \tag{4.13}$$

for $V, W \in \Gamma(\mathcal{D}_2^P)$ and $P \in \{I, J, K\}$, where $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. Given $V, W \in \Gamma(\mathcal{D}_2^P)$ and $P \in \{I, J, K\}$, by (4.7), (2.10), (4.12), we obtain

$$\begin{aligned}
\mathcal{T}_{\phi_P V} \phi_P W &= C_P \mathcal{T}_{\phi_P V} W = C_P \mathcal{T}_W \phi_P V \\
&= \mathcal{T}_W \phi_P^2 V = -\cos^2 \theta_P \; \mathcal{T}_W V \\
&= -\cos^2 \theta_P \; \mathcal{T}_V W,
\end{aligned}$$

which implies the result. □

Theorem 4.5. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then given $P \in \{I, J, K\}$, the assertions are equivalent:*
(a) The distribution $\mathcal{D}_1^P$ is integrable.
(b) $(\nabla\varphi_*)(V, PW) = (\nabla\varphi_*)(PV, W)$ for $V, W \in \Gamma(\mathcal{D}_1^P)$.

Proof. Given $V, W \in \Gamma(\mathcal{D}_1^P)$, we have

$$\begin{aligned}\varphi_* P[V, W] &= \varphi_* \nabla_V PW - \varphi_* \nabla_W PV \\ &= -(\nabla\varphi_*)(V, PW) + (\nabla\varphi_*)(W, PV).\end{aligned}$$

If $\theta_P = 0$, then since $\mathcal{D}_1^P = \ker\varphi_*$ and $\nabla\varphi_*$ is symmetric, the result follows.
If $0 < \theta_P \leq \frac{\pi}{2}$, then since $\varphi_* P[V, W] = 0 \Leftrightarrow [V, W] \in \Gamma(\mathcal{D}_1^P)$, the result follows. □

Corollary 4.6. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an hssR map with an hssR basis (I, J, K). Then the assertions are equivalent:*
(a) The distribution $\mathcal{D}_1$ is integrable.
(b) $(\nabla\varphi_*)(V, IW) = (\nabla\varphi_*)(IV, W)$ for $V, W \in \Gamma(\mathcal{D}_1)$.
(c) $(\nabla\varphi_*)(V, JW) = (\nabla\varphi_*)(JV, W)$ for $V, W \in \Gamma(\mathcal{D}_1)$.
(d) $(\nabla\varphi_*)(V, KW) = (\nabla\varphi_*)(KV, W)$ for $V, W \in \Gamma(\mathcal{D}_1)$.

Theorem 4.7. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then given $P \in \{I, J, K\}$, the assertions are equivalent:*
(a) The distribution $\mathcal{D}_2^P$ is integrable.
(b) $g_M(U, \widehat{\nabla}_W \phi_P V + \mathcal{T}_W \omega_P V - \widehat{\nabla}_V \phi_P W - \mathcal{T}_V \omega_P W) = 0$ for $U \in \Gamma(\mathcal{D}_1^P)$ and $V, W \in \Gamma(\mathcal{D}_2^P)$.

Proof. Given $V, W \in \Gamma(\mathcal{D}_2^P)$ and $U \in \Gamma(\mathcal{D}_1^P)$, we have

$$\begin{aligned}g_N(PU, [V, W]) &= g_N(U, \nabla_W PV - \nabla_V PW) \\ &= g_N(U, \widehat{\nabla}_W \phi_P V + \mathcal{T}_W \phi_P V + \mathcal{T}_W \omega_P V + \mathcal{H}\nabla_W \omega_P V \\ &\quad - \widehat{\nabla}_V \phi_P W - \mathcal{T}_V \phi_P W - \mathcal{T}_V \omega_P W - \mathcal{H}\nabla_V \omega_P W) \\ &= g_N(U, \widehat{\nabla}_W \phi_P V + \mathcal{T}_W \omega_P V - \widehat{\nabla}_V \phi_P W - \mathcal{T}_V \omega_P W).\end{aligned}$$

Therefore, the result follows. □

Corollary 4.8. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an hssR map with an hssR basis (I, J, K). Then the assertions are equivalent:*
(a) The distribution $\mathcal{D}_2$ is integrable.
(b) $g_M(U, \widehat{\nabla}_W \phi_I V + \mathcal{T}_W \omega_I V - \widehat{\nabla}_V \phi_I W - \mathcal{T}_V \omega_I W) = 0$ for $U \in \Gamma(\mathcal{D}_1)$ and $V, W \in \Gamma(\mathcal{D}_2)$.
(c) $g_M(U, \widehat{\nabla}_W \phi_J V + \mathcal{T}_W \omega_J V - \widehat{\nabla}_V \phi_J W - \mathcal{T}_V \omega_J W) = 0$ for $U \in \Gamma(\mathcal{D}_1)$ and $V, W \in \Gamma(\mathcal{D}_2)$.
(d) $g_M(U, \widehat{\nabla}_W \phi_K V + \mathcal{T}_W \omega_K V - \widehat{\nabla}_V \phi_K W - \mathcal{T}_V \omega_K W) = 0$ for $U \in \Gamma(\mathcal{D}_1)$ and $V, W \in \Gamma(\mathcal{D}_2)$.

Theorem 4.9. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then the assertions are equivalent:*
(a) The distribution $(\ker\varphi_*)^\perp$ is integrable.
(b) $B_I(\mathcal{A}_Z B_I X - \mathcal{A}_X B_I Z + \mathcal{H}\nabla_Z C_I X - \mathcal{H}\nabla_X C_I Z) + \phi_I(\mathcal{V}\nabla_Z B_I X - \mathcal{V}\nabla_X B_I Z + \mathcal{A}_Z C_I X - \mathcal{A}_X C_I Z) = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.
c) $B_J(\mathcal{A}_Z B_J X - \mathcal{A}_X B_J Z + \mathcal{H}\nabla_Z C_J X - \mathcal{H}\nabla_X C_J Z) + \phi_J(\mathcal{V}\nabla_Z B_J X - \mathcal{V}\nabla_X B_J Z + \mathcal{A}_Z C_J X - \mathcal{A}_X C_J Z) = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.
d) $B_K(\mathcal{A}_Z B_K X - \mathcal{A}_X B_K Z + \mathcal{H}\nabla_Z C_K X - \mathcal{H}\nabla_X C_K Z) + \phi_K(\mathcal{V}\nabla_Z B_K X - \mathcal{V}\nabla_X B_K Z + \mathcal{A}_Z C_K X - \mathcal{A}_X C_K Z) = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.

Proof. Given $X, Z \in \Gamma((\ker \varphi_*)^\perp)$ and $P \in \{I, J, K\}$, we get

$$\begin{aligned}
&\nabla_X Z \\
&= -P\nabla_X PZ = -P\nabla_X(B_P Z + C_P Z) \\
&= -(B_P \mathcal{A}_X B_P Z + C_P \mathcal{A}_X B_P Z + \phi_P \mathcal{V}\nabla_X B_P Z + \omega_P \mathcal{V}\nabla_X B_P Z \\
&+\phi_P \mathcal{A}_X C_P Z + \omega_P \mathcal{A}_X C_P Z + B_P \mathcal{H}\nabla_X C_P Z + C_P \mathcal{H}\nabla_X C_P Z.
\end{aligned} \tag{4.14}$$

Interchanging X and Z, we obtain

$$\begin{aligned}
&\nabla_Z X \\
&= -(B_P \mathcal{A}_Z B_P X + C_P \mathcal{A}_Z B_P X + \phi_P \mathcal{V}\nabla_Z B_P X + \omega_P \mathcal{V}\nabla_Z B_P X \\
&+\phi_P \mathcal{A}_Z C_P X + \omega_P \mathcal{A}_Z C_P X + B_P \mathcal{H}\nabla_Z C_P X + C_P \mathcal{H}\nabla_Z C_P X.
\end{aligned} \tag{4.15}$$

From (4.14) and (4.15),

$$\begin{aligned}
[X, Z]|_{\ker \varphi_*} = {} & B_P(\mathcal{A}_Z B_P X - \mathcal{A}_X B_P Z + \mathcal{H}\nabla_Z C_P X - \mathcal{H}\nabla_X C_P Z) \\
& + \phi_P(\mathcal{V}\nabla_Z B_P X - \mathcal{V}\nabla_X B_P Z + \mathcal{A}_Z C_P X - \mathcal{A}_X C_P Z).
\end{aligned}$$

Therefore, we get the result. □

In a similar way with the notion of pluriharmonicity [16], we give some notions.

Definition 4.10. Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Given $P \in \{I, J, K\}$, we call the map φ *P-pluriharmonic, $(\ker \varphi_*)^\perp$-P-pluriharmonic, $\ker \varphi_*$-P-pluriharmonic, $\mathcal{D}_1^P$-P-pluriharmonic, $\mathcal{D}_2^P$-P-pluriharmonic, $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-P-pluriharmonic* if

$$(\nabla\varphi_*)(X, Z) + (\nabla\varphi_*)(PX, PZ) = 0 \tag{4.16}$$

for $X, Z \in \Gamma(TN)$, for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, for $X, Z \in \Gamma(\ker \varphi_*)$, for $X, Z \in \Gamma(\mathcal{D}_1^P)$, for $X, Z \in \Gamma(\mathcal{D}_2^P)$, for $X \in \Gamma((\ker \varphi_*)^\perp), Z \in \Gamma(\ker \varphi_*)$, respectively.

Definition 4.11. Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Given $P \in \{I, J, K\}$, we call the map φ *P-invariant, $(\ker \varphi_*)^\perp$-P-invariant, $\ker \varphi_*$-P-invariant, $\mathcal{D}_1^P$-P-invariant, $\mathcal{D}_2^P$-P-invariant, $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-P-invariant* if

$$(\nabla\varphi_*)(X, Z) = (\nabla\varphi_*)(PX, PZ) \tag{4.17}$$

for $X, Z \in \Gamma(TN)$, for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, for $X, Z \in \Gamma(\ker \varphi_*)$, for $X, Z \in \Gamma(\mathcal{D}_1^P)$, for $X, Z \in \Gamma(\mathcal{D}_2^P)$, for $X \in \Gamma((\ker \varphi_*)^\perp), Z \in \Gamma(\ker \varphi_*)$, respectively.

Definition 4.12. Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Given $P \in \{I, J, K\}$, we call the map φ *totally geodesic, $(\ker \varphi_*)^\perp$-geodesic, $\ker \varphi_*$-geodesic, $\mathcal{D}_1^P$-geodesic, $\mathcal{D}_2^P$-geodesic, $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-geodesic* if

$$(\nabla\varphi_*)(X, Z) = 0 \tag{4.18}$$

for $X, Z \in \Gamma(TN)$, for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, for $X, Z \in \Gamma(\ker \varphi_*)$, for $X, Z \in \Gamma(\mathcal{D}_1^P)$, for $X, Z \in \Gamma(\mathcal{D}_2^P)$, for $X \in \Gamma((\ker \varphi_*)^\perp), Z \in \Gamma(\ker \varphi_*)$, respectively.

Remark 4.13. Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K).

(1) Given $X, Z \in \Gamma(TN)$ and $P \in \{I, J, K\}$, we have

$$\begin{aligned}
&(\nabla\varphi_*)(X, Z) + (\nabla\varphi_*)(PX, PZ) = 0 \\
&\Leftrightarrow \left(\begin{array}{l} \nabla_X^{\varphi r} \varphi_* Z + \nabla_{PX}^{\varphi r} \varphi_* PZ = \varphi_*(\nabla_X Z + \nabla_{PX} PZ) \\ \nabla_X^{\varphi\perp} \varphi_* Z = -\nabla_{PX}^{\varphi\perp} \varphi_* PZ. \end{array} \right.
\end{aligned}$$

(2) Given $X, Z \in \Gamma(TN)$ and $P \in \{I, J, K\}$, we get

$$(\nabla\varphi_*)(X, Z) = (\nabla\varphi_*)(PX, PZ)$$
$$\Leftrightarrow \left(\begin{array}{l} \nabla^{\varphi r}_X \varphi_* Z - \nabla^{\varphi r}_{PX} \varphi_* PZ = \varphi_*(\nabla_X Z - \nabla_{PX} PZ) \\ \nabla^{\varphi\perp}_X \varphi_* Z = \nabla^{\varphi\perp}_{PX} \varphi_* PZ. \end{array} \right.$$

(3) Given $X, Z \in \Gamma(TN)$ and $P \in \{I, J, K\}$, we obtain

$$(\nabla\varphi_*)(X, Z) = 0$$
$$\Leftrightarrow \left(\begin{array}{l} \nabla^{\varphi r}_X \varphi_* Z = \varphi_* \nabla_X Z \\ \nabla^{\varphi\perp}_X \varphi_* Z = 0. \end{array} \right.$$

(4) We see that the map φ is $\ker\varphi_*$-geodesic if and only if the distribution $\ker\varphi_*$ gives a totally geodesic foliation.

(5) If the map φ is P-invariant, then we have

$$(\nabla\varphi_*)(PX, Z) = -(\nabla\varphi_*)(X, PZ)$$

for $X, Z \in \Gamma(TN)$.

(6) If the map φ is P-pluriharmonic, then we have

$$(\nabla\varphi_*)(PX, Z) = (\nabla\varphi_*)(X, PZ)$$

for $X, Z \in \Gamma(TN)$.

Lemma 4.14. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). If the map φ is P-pluriharmonic and either $\theta_P = 0$ or $\theta_P = \frac{\pi}{2}$ for some $P \in \{I, J, K\}$, then φ is harmonic.*

Proof. If $\theta_P = 0$, then we have a local orthonormal frame $\{u_1, \cdots, u_s, Pu_1, \cdots, Pu_s, v_1, \cdots, v_t, Pv_1, \cdots, Pv_t, w_1, \cdots, w_r, Pw_1, \cdots, Pw_r\}$ of TN such that $\{u_1, \cdots, u_s, Pu_1, \cdots, Pu_s\} \subset \Gamma(\mathcal{D}_1^P)$, $\{v_1, \cdots, v_t, Pv_1, \cdots, Pv_t\} \subset \Gamma(\mathcal{D}_2^P)$ and $\{w_1, \cdots, w_r, Pw_1, \cdots, Pw_r\} \subset \Gamma(\mu^P)$.

Since the map φ is P-pluriharmonic, we get

$$\begin{aligned} \tau(\varphi) &= \operatorname{trace}(\nabla\varphi_*) \\ &= \sum_{i=1}^{s} ((\nabla\varphi_*)(u_i, u_i) + (\nabla\varphi_*)(Pu_i, Pu_i)) \\ &+ \sum_{j=1}^{t} ((\nabla\varphi_*)(v_j, v_j) + (\nabla\varphi_*)(Pv_j, Pv_j)) \\ &+ \sum_{k=1}^{r} ((\nabla\varphi_*)(w_k, w_k) + (\nabla\varphi_*)(Pw_k, Pw_k)) \\ &= 0. \end{aligned}$$

If $\theta_P = \frac{\pi}{2}$, then we get a local orthonormal frame $\{u_1, \cdots, u_s, Pu_1, \cdots, Pu_s, v_1, \cdots, v_t, Pv_1, \cdots, Pv_t, w_1, \cdots, w_r, Pw_1, \cdots, Pw_r\}$ of TN such that $\{u_1, \cdots, u_s, Pu_1, \cdots, Pu_s\} \subset \Gamma(\mathcal{D}_1^P)$, $\{v_1, \cdots, v_t\} \subset \Gamma(\mathcal{D}_2^P)$, $\{Pv_1, \cdots, Pv_t\} \subset \Gamma(P\mathcal{D}_2^P)$ and $\{w_1, \cdots, w_r, Pw_1, \cdots, Pw_r\} \subset \Gamma(\mu^P)$.

Since the map φ is P-pluriharmonic, we obtain

$$\begin{aligned}\tau(\varphi) &= \operatorname{trace}(\nabla\varphi_*)\\ &= \sum_{i=1}^{s}((\nabla\varphi_*)(u_i,u_i)+(\nabla\varphi_*)(Pu_i,Pu_i))\\ &+\sum_{j=1}^{t}((\nabla\varphi_*)(v_j,v_j)+(\nabla\varphi_*)(Pv_j,Pv_j))\\ &+\sum_{k=1}^{r}((\nabla\varphi_*)(w_k,w_k)+(\nabla\varphi_*)(Pw_k,Pw_k))\\ &= 0.\end{aligned}$$

Therefore, the result follows. □

From Example 3.10 and Lemma 4.14, we get

Corollary 4.15. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be a conformal Riemannian map with $\dim N = 4n$, $\dim M = 4n-1$, range $\varphi_* = \varphi^{-1}TM$ and dilation a smooth function λ. Assume that $(\nabla\varphi_*)(X,Z) + (\nabla\varphi_*)(PX,PZ) = 0$ for some $P \in \{I, J, K\}$. Then the map φ is harmonic.*

Let $\varphi : (N, g_N) \mapsto (M, g_M)$ be a conformal Riemannian map. We call the map φ a conformal Riemannian map *with totally umbilical fibers* if

$$\mathcal{T}_V W = g_N(V, W)H \tag{4.19}$$

for $V, W \in \Gamma(\ker\varphi_*)$, where H is the mean curvature vector field of the fiber.

Lemma 4.16. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with totally umbilical fibers such that (I, J, K) is an ahssR basis. Then we get*

$$H \in \Gamma(\omega_I \mathcal{D}_2^I) \cap \Gamma(\omega_J \mathcal{D}_2^J) \cap \Gamma(\omega_K \mathcal{D}_2^K). \tag{4.20}$$

Proof. Given $V, W \in \Gamma(\mathcal{D}_1^P)$, $Z \in \Gamma(\mu^P)$ and $P \in \{I, J, K\}$, we have

$$\mathcal{T}_V PW + \widehat{\nabla}_V PW = \nabla_V PW = P\nabla_V W = P\mathcal{T}_V W + \phi_P \widehat{\nabla}_V W + \omega_P \widehat{\nabla}_V W$$

so that

$$g_N(\mathcal{T}_V PW, Z) = g_N(P\mathcal{T}_V W, Z). \tag{4.21}$$

From (4.21), by (4.19), we obtain

$$g_N(V, PW)g_N(H, Z) = -g_N(V, W)g_N(H, PZ). \tag{4.22}$$

Interchanging V and W, we get

$$g_N(W, PV)g_N(H, Z) = -g_N(W, V)g_N(H, PZ). \tag{4.23}$$

From (4.22) and (4.23), we derive

$$g_N(V, W)g_N(H, PZ) = 0,$$

which means

$$H \in \Gamma(\omega_P \mathcal{D}_2^P).$$

Therefore, the result follows. □

Corollary 4.17. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with totally umbilical fibers such that (I, J, K) is an ahssR basis. Suppose that $\omega_I \mathcal{D}_2^I \cap \omega_J \mathcal{D}_2^J \cap \omega_K \mathcal{D}_2^K = \{0\}$. Then all the fibers are minimal.*

Theorem 4.18. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $\ker \varphi_*$-P-pluriharmonic map for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The distribution $\ker \varphi_*$ gives a totally geodesic foliation on N.

(b) $\nabla^{\varphi r}_{PV} \varphi_* PW = \varphi_* \nabla_{PV} PW$ for $V, W \in \Gamma(\ker \varphi_*)$.

Proof. Given $V, W \in \Gamma(\ker \varphi_*)$, we get

$$\begin{aligned}0 &= (\nabla \varphi_*)(V, W) + (\nabla \varphi_*)(PV, PW)\\ &= -\varphi_* \nabla_V W + \nabla^{\varphi}_{PV} \varphi_* PW - \varphi_* \nabla_{PV} PW\end{aligned}$$

so that

$$\nabla^{\varphi \perp}_{PV} \varphi_* PW = 0$$

and

$$-\varphi_* \nabla_V W + \nabla^{\varphi r}_{PV} \varphi_* PW - \varphi_* \nabla_{PV} PW = 0.$$

The latter implies the result. □

From the proof of Theorem 4.18, we have

Corollary 4.19. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). Suppose that φ is a $\ker \varphi_*$-P-pluriharmonic map for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The distribution $\ker \varphi_*$ gives a totally geodesic foliation on N.

(b) $\nabla^{\varphi}_{PV} \varphi_* PW = \varphi_* \nabla_{PV} PW$ for $V, W \in \Gamma(\ker \varphi_*)$.

In a similar way, we obtain

Lemma 4.20. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $\ker \varphi_*$-P-invariant map for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The distribution $\ker \varphi_*$ gives a totally geodesic foliation on N.

(b) $\nabla^{\varphi r}_{PV} \varphi_* PW = \varphi_* \nabla_{PV} PW$ for $V, W \in \Gamma(\ker \varphi_*)$.

Theorem 4.21. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $(\ker \varphi_*)^\perp$-P-pluriharmonic map with $\dim \mu^P \geq 4$ for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The map φ is horizontally homothetic.

(b) $\omega_P \mathcal{T}_{B_P X} Z + C_P \mathcal{H} \nabla_{B_P X} Z + \mathcal{A}_{C_P X} B_P X = 0$ for $X, Z \in \Gamma((\ker \varphi_*)^\perp)$.

Proof. Given $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, we get

$$\begin{aligned}0 &= (\nabla \varphi_*)(X, Z) + (\nabla \varphi_*)(PX, PZ)\\ &= X(\ln \lambda)\varphi_* Z + Z(\ln \lambda)\varphi_* X - g_N(X, Z)\varphi_*(\nabla \ln \lambda) + \nabla^{\varphi \perp}_X \varphi_* Z\\ &\quad - \varphi_*(\nabla_{B_P X} PZ + \nabla_{C_P X} B_P Z) + C_P X(\ln \lambda)\varphi_* C_P Z + C_P Z(\ln \lambda)\varphi_* C_P X\\ &\quad - g_N(C_P X, C_P Z)\varphi_*(\nabla \ln \lambda) + \nabla^{\varphi \perp}_{C_P X} \varphi_* C_P Z\end{aligned}$$

so that

$$\nabla^{\varphi \perp}_X \varphi_* Z + \nabla^{\varphi \perp}_{C_P X} \varphi_* C_P Z = 0$$

and

$$\begin{aligned}0 \quad &= -\varphi_*(\omega_P \mathcal{T}_{B_P X} Z + C_P \mathcal{H} \nabla_{B_P X} Z + \mathcal{A}_{C_P X} B_P X)\\ &\quad + X(\ln \lambda)\varphi_* Z + Z(\ln \lambda)\varphi_* X - g_N(X, Z)\varphi_*(\nabla \ln \lambda)\\ &\quad + C_P X(\ln \lambda)\varphi_* C_P Z + C_P Z(\ln \lambda)\varphi_* C_P X\\ &\quad - g_N(C_P X, C_P Z)\varphi_*(\nabla \ln \lambda).\end{aligned} \tag{4.24}$$

We claim that φ is horizontally homothetic if and only if

$$0 = X(\ln\lambda)\varphi_*Z + Z(\ln\lambda)\varphi_*X - g_N(X,Z)\varphi_*(\nabla\ln\lambda) \qquad (4.25)$$
$$+C_PX(\ln\lambda)\varphi_*C_PZ + C_PZ(\ln\lambda)\varphi_*C_PX - g_N(C_PX, C_PZ)\varphi_*(\nabla\ln\lambda)$$

for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.

One direction is obvious. So, we prove the other one.

If $\mathcal{D}_2^P = \{0\}$ (i.e., $\theta_P = 0$), then we choose any orthonormal vector fields $\{X_1, X_2\} \subset \Gamma(\mu^P)$ with $g_N(X_1, X_2) = g_N(PX_1, X_2) = 0$. Applying $X = Z = X_1$ at (4.25), we get

$$0 = X_1(\ln\lambda)\varphi_*X_1 + PX_1(\ln\lambda)\varphi_*PX_1 - \varphi_*(\nabla\ln\lambda)$$

so that

$$\begin{aligned}0 &= g_N(X_1(\ln\lambda)\varphi_*X_1 + PX_1(\ln\lambda)\varphi_*PX_1 - \varphi_*(\nabla\ln\lambda), \varphi_*X_2)\\ &= -\lambda^2 X_2(\ln\lambda),\end{aligned}$$

which implies that φ is horizontally homothetic.

If $\mathcal{D}_2^P \neq \{0\}$, then we choose any unit vector field $V \in \Gamma(\mathcal{D}_2^P)$.

Applying $X = Z = \omega_P V$ at (4.25), we obtain

$$2\omega_P V(\ln\lambda)\varphi_*\omega_P V - \sin^2\theta_P\ \varphi_*(\nabla\ln\lambda) = 0 \qquad (4.26)$$

From (4.26),

$$\begin{aligned}0 &= g_N(2\omega_P V(\ln\lambda)\varphi_*\omega_P V - \sin^2\theta_P\ \varphi_*(\nabla\ln\lambda), \varphi_*\omega_P V)\\ &= \lambda^2\sin^2\theta_P\ \omega_P V(\ln\lambda)\end{aligned}$$

and

$$\begin{aligned}0 &= g_N(2\omega_P V(\ln\lambda)\varphi_*\omega_P V - \sin^2\theta_P\ \varphi_*(\nabla\ln\lambda), \varphi_*Y)\\ &= -\lambda^2\sin^2\theta_P\ Y(\ln\lambda)\end{aligned}$$

for $Y \in \Gamma(\mu^P)$, which implies that φ is horizontally homothetic.

From (4.24), we obtain the result. □

From the proof of Theorem 4.21, we get

Corollary 4.22. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). Suppose that φ is a $(\ker\varphi_*)^\perp$-P-pluriharmonic map with $\dim\mu^P \geq 4$ for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The map φ is horizontally homothetic.

(b) $\omega_P\mathcal{T}_{B_PX}Z + C_P\mathcal{H}\nabla_{B_PX}Z + \mathcal{A}_{C_PX}B_PX = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.

Similarly, we obtain

Lemma 4.23. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $(\ker\varphi_*)^\perp$-P-invariant map with $\dim\mu^P \geq 4$ for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The map φ is horizontally homothetic.

(b) $\omega_P\mathcal{T}_{B_PX}Z + C_P\mathcal{H}\nabla_{B_PX}Z + \mathcal{A}_{C_PX}B_PX = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$.

Lemma 4.24. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). If the map φ is $(\ker\varphi_*)^\perp$-geodesic, then φ is horizontally homothetic.*

Proof. Given $Z \in \Gamma((\ker\varphi_*)^\perp)$, by (2.4), we have

$$\begin{aligned}0 &= g_M((\nabla\varphi_*)(Z,Z), \varphi_*Z)\\ &= g_M(2Z(\ln\lambda)\varphi_*Z - |Z|^2\varphi_*(\nabla\ln\lambda), \varphi_*Z)\\ &= \lambda^2|Z|^2 Z(\ln\lambda).\end{aligned}$$

Therefore, the result follows. □

From (2.4), with polarization in the proof of Lemma 4.24, we obtain

Corollary 4.25. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). Then the assertions are equivalent.*

(a) The map φ is $(\ker \varphi_*)^\perp$-geodesic.

(b) The map φ is horizontally homothetic.

Theorem 4.26. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-P-pluriharmonic map for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The map φ is horizontally homothetic.

(b) $\mathcal{A}_Z W + \mathcal{T}_{B_P Z} \phi_P W + \mathcal{A}_{C_P Z} \phi_P W + \mathcal{H} \nabla_{B_P Z} \omega_P W = 0$ for $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$.

Proof. Given $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$, we have

$$\begin{aligned}
0 &= (\nabla \varphi_*)(Z, W) + (\nabla \varphi_*)(PZ, PW) \\
&= -\varphi_* \nabla_Z W - \varphi_*(\nabla_{PZ} \phi_P W + \nabla_{B_P Z} \omega_P W) + C_P Z(\ln \lambda) \varphi_* \omega_P W \\
&+ \omega_P W(\ln \lambda) \varphi_* C_P Z - g_N(C_P Z, \omega_P W) \varphi_*(\nabla \ln \lambda) + \nabla^{\varphi \perp}_{C_P Z} \varphi_* \omega_P W \\
&= -\varphi_*(\mathcal{A}_Z W + \mathcal{T}_{B_P Z} \phi_P W + \mathcal{A}_{C_P Z} \phi_P W + \mathcal{H} \nabla_{B_P Z} \omega_P W) \\
&+ C_P Z(\ln \lambda) \varphi_* \omega_P W + \omega_P W(\ln \lambda) \varphi_* C_P Z + \nabla^{\varphi \perp}_{C_P Z} \varphi_* \omega_P W
\end{aligned}$$

so that

$$\nabla^{\varphi \perp}_{C_P Z} \varphi_* \omega_P W = 0$$

and

$$\begin{gathered}
-\varphi_*(\mathcal{A}_Z W + \mathcal{T}_{B_P Z} \phi_P W + \mathcal{A}_{C_P Z} \phi_P W + \mathcal{H} \nabla_{B_P Z} \omega_P W) \\
+ C_P Z(\ln \lambda) \varphi_* \omega_P W + \omega_P W(\ln \lambda) \varphi_* C_P Z = 0.
\end{gathered} \tag{4.27}$$

We claim that φ is horizontally homothetic if and only if

$$C_P Z(\ln \lambda) \varphi_* \omega_P W + \omega_P W(\ln \lambda) \varphi_* C_P Z = 0 \tag{4.28}$$

for $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$. Since (4.28) means $C_P Z(\ln \lambda) = 0$ and $\omega_P W(\ln \lambda) = 0$, it is clear.

From (4.27), we obtain the result. □

In a similar way, we get

Lemma 4.27. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is a $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-P-invariant map for some $P \in \{I, J, K\}$. Then the assertions are equivalent.*

(a) The map φ is horizontally homothetic.

(b) $-\mathcal{A}_Z W + \mathcal{T}_{B_P Z} \phi_P W + \mathcal{A}_{C_P Z} \phi_P W + \mathcal{H} \nabla_{B_P Z} \omega_P W = 0$ for $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$.

Theorem 4.28. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). If the map φ is a $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-geodesic map, then the distribution $(\ker \varphi_*)^\perp$ gives a totally geodesic foliation.*

Proof. Given $W \in \Gamma(\ker \varphi_*)$ and $Z \in \Gamma((\ker \varphi_*)^\perp)$, we have

$$\begin{aligned}
0 &= (\nabla \varphi_*)(Z, W) \\
&= -\varphi_* \nabla_Z W,
\end{aligned}$$

which means

$$\nabla_Z W \in \Gamma(\ker \varphi_*). \tag{4.29}$$

Given $Z_1, Z_2 \in \Gamma((\ker \varphi_*)^\perp)$ and $W \in \Gamma(\ker \varphi_*)$, by (4.29), we obtain

$$g_N(\nabla_{Z_1} Z_2, W) = -g_N(Z_2, \nabla_{Z_1} W) = 0.$$

Therefore, the result follows. □

Lemma 4.29. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). If the distribution $\mathcal{D}_1^P$ is integrable for some $P \in \{I, J, K\}$, then the map φ is a $\mathcal{D}_1^P$-P-pluriharmonic map.*

Proof. Given $V, W \in \Gamma(\mathcal{D}_1^P)$, since $\mathcal{D}_1^P$ is integrable, we obtain

$$\begin{aligned}
&(\nabla \varphi_*)(V, W) + (\nabla \varphi_*)(PV, PW) \\
&= -\varphi_* \nabla_V W - \varphi_* \nabla_{PV} PW \\
&= -\varphi_*(\nabla_V W + P\nabla_{PV} W) \\
&= -\varphi_*(\nabla_V W + P(\nabla_W PV + [PV, W])) \\
&= -\varphi_*(\nabla_V W - \nabla_W V) \\
&= -\varphi_*(\nabla_V W - (\nabla_V W + [W, V])) \\
&= 0.
\end{aligned}$$

Therefore, we get the result. □

Lemma 4.30. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that the distribution $\mathcal{D}_1^P$ is integrable for some $P \in \{I, J, K\}$. Then the map φ is $\mathcal{D}_1^P$-P-invariant if and only if the map φ is $\mathcal{D}_1^P$-geodesic.*

Proof. Since $\mathcal{D}_1^P$ is integrable, given $V, W \in \Gamma(\mathcal{D}_1^P)$, by using the proof of Lemma 4.29, we have

$$\begin{aligned}
&(\nabla \varphi_*)(V, W) - (\nabla \varphi_*)(PV, PW) \\
&= -2\varphi_* \nabla_V W \\
&= 2(\nabla \varphi_*)(V, W),
\end{aligned}$$

which means the result. □

Theorem 4.31. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then given $P \in \{I, J, K\}$, the assertions are equivalent.*

(a) The distribution $\mathcal{D}_1^P$ gives a totally geodesic foliation.

(b) The map φ is $\mathcal{D}_1^P$-geodesic and $\widehat{\nabla}_V PW \in \Gamma(\mathcal{D}_1^P)$ for $V, W \in \Gamma(\mathcal{D}_1^P)$.

Proof. The case $(a) \Rightarrow (b)$ is obvious.

Conversely, since φ is $\mathcal{D}_1^P$-geodesic, we obtain

$$0 = (\nabla \varphi_*)(W_1, W_2) = -\varphi_* \nabla_{W_1} W_2$$

for $W_1, W_2 \in \Gamma(\mathcal{D}_1^P)$, which implies

$$\mathcal{T}_{W_1} W_2 = 0 \quad \text{for } W_1, W_2 \in \Gamma(\mathcal{D}_1^P). \tag{4.30}$$

Since $\widehat{\nabla}_{W_1} PW_2 \in \Gamma(\mathcal{D}_1^P)$ for $W_1, W_2 \in \Gamma(\mathcal{D}_1^P)$, given $V, W \in \Gamma(\mathcal{D}_1^P)$, by (4.30), we have

$$\begin{aligned}
\nabla_V W &= -P(\nabla_V PW) = -P(\widehat{\nabla}_V PW + \mathcal{T}_V W) \\
&= -P\widehat{\nabla}_V PW \in \Gamma(\mathcal{D}_1^P).
\end{aligned}$$

Therefore, the result follows. □

Theorem 4.32. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then given $P \in \{I, J, K\}$, the assertions are equivalent.*

(a) The distribution $\mathcal{D}_2^P$ gives a totally geodesic foliation.

(b) The map φ is $\mathcal{D}_2^P$-geodesic and $\phi_P(\widehat{\nabla}_V \phi_P W + \mathcal{T}_V \omega_P W) + B_P(\mathcal{T}_V \phi_P W + \mathcal{H}\nabla_V \omega_P W) \in \Gamma(\mathcal{D}_2^P)$ for $V, W \in \Gamma(\mathcal{D}_2^P)$.

Proof. Given $V, W \in \Gamma(\mathcal{D}_2^P)$, we get

$$\begin{aligned}\nabla_V W &= -P(\nabla_V PW)\\ &= -P(\nabla_V \phi_P W + \nabla_V \omega_P W)\\ &= -(\phi_P \widehat{\nabla}_V \phi_P W + B_P \mathcal{T}_V \phi_P W + \phi_P \mathcal{T}_V \omega_P W + B_P \mathcal{H}\nabla_V \omega_P W)\\ &\quad - (\omega_P \widehat{\nabla}_V \phi_P W + C_P \mathcal{T}_V \phi_P W + \omega_P \mathcal{T}_V \omega_P W + C_P \mathcal{H}\nabla_V \omega_P W)\end{aligned}$$

and

$$\begin{aligned}(\nabla\varphi_*)(V, W) &= -\varphi_* \nabla_V W\\ &= \varphi_*(\omega_P \widehat{\nabla}_V \phi_P W + C_P \mathcal{T}_V \phi_P W + \omega_P \mathcal{T}_V \omega_P W + C_P \mathcal{H}\nabla_V \omega_P W)\end{aligned}$$

so that we obtain $(a) \Leftrightarrow (b)$. □

Lemma 4.33. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Given $P \in \{I, J, K\}$, if the map φ is $\mathcal{D}_2^P$-P-pluriharmonic, then we have*

$$\nabla^{\varphi\perp}_{\omega_P V} \varphi_* \omega_P W = 0 \quad \textit{for } V, W \in \Gamma(\mathcal{D}_2^P).$$

Proof. Given $V, W \in \Gamma(\mathcal{D}_2^P)$, by (2.4), we have

$$\begin{aligned}0 &= (\nabla\varphi_*)(V, W) + (\nabla\varphi_*)(PV, PW)\\ &= -\varphi_* \nabla_V W - \varphi_*(\nabla_{\phi_P V} PW + \nabla_{\omega_P V} \phi_P W) + \omega_P V(\ln\lambda)\varphi_* \omega_P W\\ &\quad + \omega_P W(\ln\lambda)\varphi_* \omega_P V - g_N(\omega_P V, \omega_P W)\varphi_*(\nabla \ln\lambda) + \nabla^{\varphi\perp}_{\omega_P V} \varphi_* \omega_P W,\end{aligned}$$

which means

$$\nabla^{\varphi\perp}_{\omega_P V} \varphi_* \omega_P W = 0.$$

□

Similarly, we have

Lemma 4.34. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Given $P \in \{I, J, K\}$, if the map φ is $\mathcal{D}_2^P$-P-invariant, then we have*

$$\nabla^{\varphi\perp}_{\omega_P V} \varphi_* \omega_P W = 0 \quad \textit{for } V, W \in \Gamma(\mathcal{D}_2^P).$$

Theorem 4.35. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that φ is horizontally homothetic. Then the assertions are equivalent.*

(a) The distribution $(\ker\varphi_*)^\perp$ gives a totally geodesic foliation.

(b) $g_N(\mathcal{V}\nabla_X B_I Z + \mathcal{A}_X C_I Z, \phi_I W) = \frac{1}{\lambda^2} g_M((\nabla\varphi_*)(X, B_I Z) - \nabla^\varphi_X C_I Z,\ \varphi_* \omega_I W)$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$ and $W \in \Gamma(\ker\varphi_*)$.

(c) $g_N(\mathcal{V}\nabla_X B_J Z + \mathcal{A}_X C_J Z, \phi_J W) = \frac{1}{\lambda^2} g_M((\nabla\varphi_*)(X, B_J Z) - \nabla^\varphi_X C_J Z,\ \varphi_* \omega_J W)$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$ and $W \in \Gamma(\ker\varphi_*)$.

(d) $g_N(\mathcal{V}\nabla_X B_K Z + \mathcal{A}_X C_K Z, \phi_K W) = \frac{1}{\lambda^2} g_M((\nabla\varphi_*)(X, B_K Z) - \nabla^\varphi_X C_K Z,\ \varphi_* \omega_K W)$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$ and $W \in \Gamma(\ker\varphi_*)$.

Proof. Given $X, Z \in \Gamma((\ker \varphi_*)^\perp)$, $W \in \Gamma(\ker \varphi_*)$ and $P \in \{I, J, K\}$, we have

$$\begin{aligned}
& g_N(\nabla_X Z, W) \\
& = g_N(\nabla_X(B_P Z + C_P Z), \phi_P W + \omega_P W) \\
& = g_N(\mathcal{V}\nabla_X B_P Z + \mathcal{A}_X C_P Z, \phi_P W) + g_N(\nabla_X B_P Z + \nabla_X C_P Z, \omega_P W) \\
& = g_N(\mathcal{V}\nabla_X B_P Z + \mathcal{A}_X C_P Z, \phi_P W) + \frac{1}{\lambda^2} g_M(\varphi_* \nabla_X B_P Z + \varphi_* \nabla_X C_P Z, \varphi_* \omega_P W) \\
& = g_N(\mathcal{V}\nabla_X B_P Z + \mathcal{A}_X C_P Z, \phi_P W) + \frac{1}{\lambda^2} g_M(-(\nabla \varphi_*)(X, B_P Z) - (\nabla \varphi_*)(X, C_P Z) \\
& + \nabla_X^\varphi C_P Z, \varphi_* \omega_P W) \\
& = g_N(\mathcal{V}\nabla_X B_P Z + \mathcal{A}_X C_P Z, \phi_P W) + \frac{1}{\lambda^2} g_M(-(\nabla \varphi_*)(X, B_P Z) + \nabla_X^\varphi C_P Z \\
& - (X(\ln \lambda)\varphi_* C_P Z + C_P Z(\ln \lambda)\varphi_* X - g_N(X, C_P Z)\varphi_*(\nabla \ln \lambda)), \varphi_* \omega_P W) \\
& = g_N(\mathcal{V}\nabla_X B_P Z + \mathcal{A}_X C_P Z, \phi_P U) + \frac{1}{\lambda^2} g_M(-(\nabla \varphi_*)(X, B_P Z) + \nabla_X^\varphi C_P Z, \varphi_* \omega_P W) \\
& - g_N(X(\ln \lambda) C_P Z + C_P Z(\ln \lambda) X - g_N(X, C_P Z)\nabla \ln \lambda, \omega_P W).
\end{aligned}$$

Since φ is horizontally homothetic, we obtain the result. □

5. Totally geodesic and harmonicity

We consider the conditions for ahssR maps to be totally geodesic and harmonic.

Theorem 5.1. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Then given $P \in \{I, J, K\}$ with $0 < \theta_P < \frac{\pi}{2}$, the assertions are equivalent.*

(a) The map φ is harmonic.

(b) $-\varphi_* trace|_{\ker \varphi_*} \nabla_{(\ )}(\ ) + trace|_{(\ker \varphi_*)^\perp} \Big(2g_N((\ ), \nabla \ln \lambda)\varphi_*(\ ) + \nabla_{(\ )}^{\varphi \perp} \varphi_*(\ )\Big) - (\dim \mathcal{D}_2^P + \dim \mu^P)\varphi_*(\nabla \ln \lambda) = 0.$

Proof. Since

$$\begin{aligned}
TN & = \ker \varphi_* \oplus (\ker \varphi_*)^\perp \\
& = \mathcal{D}_1^P \oplus \mathcal{D}_2^P \oplus \omega_P \mathcal{D}_2^P \oplus \mu^P,
\end{aligned}$$

we choose a local orthonormal frame $\{V_1, \cdots, V_s, PV_1, \cdots, PV_s, W_1, \cdots, W_t, \sec \theta_P \phi_P W_1, \cdots, \sec \theta_P \phi_P W_t, \csc \theta_P \omega_P W_1, \cdots, \csc \theta_P \omega_P W_t, \csc \theta_P \sec \theta_P \omega_P \phi_P W_1, \cdots, \csc \theta_P \sec \theta_P \omega_P \phi_P W_t, Z_1, \cdots, Z_r, PZ_1, \cdots, PZ_r\}$ of TN such that $\{V_1, \cdots, V_s, PV_1, \cdots, PV_s\} \subset \Gamma(\mathcal{D}_1^P)$, $\{W_1, \cdots, W_t, \sec \theta_P \phi_P W_1, \cdots, \sec \theta_P \phi_P W_t\} \subset \Gamma(\mathcal{D}_2^P)$, $\{\csc \theta_P \omega_P W_1, \cdots, \csc \theta_P \omega_P W_t, \csc \theta_P \sec \theta_P \omega_P \phi_P W_1, \cdots, \csc \theta_P \sec \theta_P \omega_P \phi_P W_t\} \subset \Gamma(\omega_P\ \mathcal{D}_2^P)$, $\{Z_1, \cdots, Z_r, PZ_1, \cdots, PZ_r\} \subset \Gamma(\mu^P)$.

Then we get

$$\begin{aligned}
& \tau(\varphi) \\
& = \operatorname{trace}(\nabla \varphi_*) \\
& = \sum_{i=1}^{s} ((\nabla \varphi_*)(V_i, V_i) + (\nabla \varphi_*)(PV_i, PV_i)) + \sum_{j=1}^{t} ((\nabla \varphi_*)(W_j, W_j) \\
& + (\nabla \varphi_*)(\sec \theta_P \phi_P W_j, \sec \theta_P \phi_P W_j) + (\nabla \varphi_*)(\csc \theta_P \omega_P W_j, \csc \theta_P \omega_P W_j) \\
& + (\nabla \varphi_*)(\csc \theta_P \sec \theta_P \omega_P \phi_P W_j, \csc \theta_P \sec \theta_P \omega_P \phi_P W_j)) \\
& + \sum_{k=1}^{r} ((\nabla \varphi_*)(Z_k, Z_k) + (\nabla \varphi_*)(PZ_k, PZ_k)).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{j=1}^{t}(\nabla\varphi_*)(\csc\theta_P\omega_P W_j, \csc\theta_P\omega_P W_j)\\
&= \csc^2\theta_P\sum_{j=1}^{t}(\nabla\varphi_*)(\omega_P W_j, \omega_P W_j)\\
&= \csc^2\theta_P\sum_{j=1}^{t}(2\omega_P W_j(\ln\lambda)\varphi_*\omega_P W_j - g_N(\omega_P W_j, \omega_P W_j)\varphi_*(\nabla\ln\lambda)\\
&+ \nabla^{\varphi\perp}_{\omega_P W_j}\varphi_*\omega_P W_j)\\
&= \sum_{j=1}^{t}(2\csc\theta_P\omega_P W_j(\ln\lambda)\varphi_*\csc\theta_P\omega_P W_j - \varphi_*(\nabla\ln\lambda)\\
&+ \nabla^{\varphi\perp}_{\csc\theta_P\omega_P W_j}\varphi_*\csc\theta_P\omega_P W_j),
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{t}(\nabla\varphi_*)(\csc\theta_P\sec\theta_P\omega_P\phi_P W_j, \csc\theta_P\sec\theta_P\omega_P\phi_P W_j)\\
&= \csc^2\theta_P\sec^2\theta_P\sum_{j=1}^{t}(\nabla\varphi_*)(\omega_P\phi_P W_j, \omega_P\phi_P W_j)\\
&= \csc^2\theta_P\sec^2\theta_P\sum_{j=1}^{t}(2\omega_P\phi_P W_j(\ln\lambda)\varphi_*\omega_P\phi_P W_j\\
&- g_N(\omega_P\phi_P W_j, \omega_P\phi_P W_j)\varphi_*(\nabla\ln\lambda) + \nabla^{\varphi\perp}_{\omega_P\phi_P W_j}\varphi_*\omega_P\phi_P W_j)\\
&= \sum_{j=1}^{t}(2\csc\theta_P\sec\theta_P\omega_P\phi_P W_j(\ln\lambda)\varphi_*\csc\theta_P\sec\theta_P\omega_P\phi_P W_j - \varphi_*(\nabla\ln\lambda)\\
&+ \nabla^{\varphi\perp}_{\csc\theta_P\sec\theta_P\omega_P\phi_P W_j}\varphi_*\csc\theta_P\sec\theta_P\omega_P\phi_P W_j)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{r}((\nabla\varphi_*)(Z_k, Z_k) + (\nabla\varphi_*)(PZ_k, PZ_k))\\
&= \sum_{k=1}^{r}(2Z_k(\ln\lambda)\varphi_*Z_k - g_N(Z_k, Z_k)\varphi_*(\nabla\ln\lambda) + \nabla^{\varphi\perp}_{Z_k}\varphi_*Z_k\\
&+ 2PZ_k(\ln\lambda)\varphi_*PZ_k - g_N(PZ_k, PZ_k)\varphi_*(\nabla\ln\lambda) + \nabla^{\varphi\perp}_{PZ_k}\varphi_*PZ_k)\\
&= \sum_{k=1}^{r}(2(Z_k(\ln\lambda)\varphi_*Z_k + PZ_k(\ln\lambda)\varphi_*PZ_k) - 2\varphi_*(\nabla\ln\lambda)\\
&+ \nabla^{\varphi\perp}_{Z_k}\varphi_*Z_k + \nabla^{\varphi\perp}_{PZ_k}\varphi_*PZ_k).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\tau(\varphi)\\
&= -\varphi_* \mathrm{trace}\,|_{\ker\varphi_*}\nabla_{(\ )}(\ ) + \mathrm{trace}\,|_{(\ker\varphi_*)^\perp}(2g_N((\ ),\nabla\ln\lambda)\varphi_*(\ ) + \nabla^{\varphi\perp}_{(\ )}\varphi_*(\ ))\\
&\quad - (\dim \mathcal{D}_2^P + \dim\mu^P)\varphi_*(\nabla\ln\lambda).
\end{aligned}
$$

Therefore, the result follows. □

Corollary 5.2. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). If all the fibers are minimal and the map φ is horizontally homothetic, then the map φ is harmonic.*

Theorem 5.3. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahssR map with an ahssR basis (I, J, K). Suppose that $X(\ln\lambda)\varphi_* Z + Z(\ln\lambda)\varphi_* X - g_N(X, Z)\varphi_*(\nabla\ln\lambda) + \nabla^{\varphi\perp}_X \varphi_* Z = 0$ for $X, Z \in \Gamma((\ker\varphi_*)^\perp)$. Then the assertions are equivalent.*

(a) The map φ is totally geodesic.

(b)

(i) $\varphi_*(\omega_I(\widehat{\nabla}_V\phi_I W + \mathcal{T}_V\omega_I W) + C_I(\mathcal{T}_V\phi_I W + \mathcal{H}\nabla_V\omega_I W)) = 0$ for $V, W \in \Gamma(\ker\varphi_*)$.

(ii) $\varphi_*(\omega_I(\widehat{\nabla}_W B_I Z + \mathcal{T}_W C_I Z) + C_I(\mathcal{T}_W B_I Z + \mathcal{H}\nabla_W C_I Z)) = 0$ for $W \in \Gamma(\ker\varphi_*)$ and $Z \in \Gamma((\ker\varphi_*)^\perp)$.

(c)

(i) $\varphi_*(\omega_J(\widehat{\nabla}_V\phi_J W + \mathcal{T}_V\omega_J W) + C_J(\mathcal{T}_V\phi_J W + \mathcal{H}\nabla_V\omega_J W)) = 0$ for $V, W \in \Gamma(\ker\varphi_*)$.

(ii) $\varphi_*(\omega_J(\widehat{\nabla}_W B_J Z + \mathcal{T}_W C_J Z) + C_J(\mathcal{T}_W B_J Z + \mathcal{H}\nabla_W C_J Z)) = 0$ for $W \in \Gamma(\ker\varphi_*)$ and $Z \in \Gamma((\ker\varphi_*)^\perp)$.

(d)

(i) $\varphi_*(\omega_K(\widehat{\nabla}_V\phi_K W + \mathcal{T}_V\omega_K W) + C_K(\mathcal{T}_V\phi_K W + \mathcal{H}\nabla_V\omega_K W)) = 0$ for $V, W \in \Gamma(\ker\varphi_*)$.

(ii) $\varphi_*(\omega_K(\widehat{\nabla}_W B_K Z + \mathcal{T}_W C_K Z) + C_K(\mathcal{T}_W B_K Z + \mathcal{H}\nabla_W C_K Z)) = 0$ for $W \in \Gamma(\ker\varphi_*)$ and $Z \in \Gamma((\ker\varphi_*)^\perp)$.

Proof. Given $V, W \in \Gamma(\ker\varphi_*)$ and $P \in \{I, J, K\}$, we have

$$
\begin{aligned}
&(\nabla\varphi_*)(V, W)\\
&= \varphi_* P\nabla_V PW\\
&= \varphi_* P(\widehat{\nabla}_V\phi_P W + \mathcal{T}_V\phi_P W + \mathcal{T}_V\omega_P W + \mathcal{H}\nabla_V\omega_P W)\\
&= \varphi_*(\omega_P(\widehat{\nabla}_V\phi_P W + \mathcal{T}_V\omega_P W) + C_P(\mathcal{T}_V\phi_P W + \mathcal{H}\nabla_V\omega_P W)).
\end{aligned}
$$

Given $W \in \Gamma(\ker\varphi_*)$, $Z \in \Gamma((\ker\varphi_*)^\perp)$ and $P \in \{I, J, K\}$, we get

$$
\begin{aligned}
&(\nabla\varphi_*)(W, Z)\\
&= \varphi_* P\nabla_W PZ\\
&= \varphi_* P(\mathcal{T}_W B_P Z + \widehat{\nabla}_W B_P Z + \mathcal{T}_W C_P Z + \mathcal{H}\nabla_W C_P Z)\\
&= \varphi_*(\omega_P(\widehat{\nabla}_W B_P Z + \mathcal{T}_W C_P Z) + C_P(\mathcal{T}_W B_P Z + \mathcal{H}\nabla_W C_P Z)).
\end{aligned}
$$

Given $X, Z \in \Gamma((\ker\varphi_*)^\perp)$, we obtain

$$
\begin{aligned}
&(\nabla\varphi_*)(X, Z)\\
&= X(\ln\lambda)\varphi_* Z + Z(\ln\lambda)\varphi_* X - g_N(X, Z)\varphi_*(\nabla\ln\lambda) + \nabla^{\varphi\perp}_X\varphi_* Z.
\end{aligned}
$$

Hence, we have $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

Therefore, we obtain the result. □

Corollary 5.4. *Let $\varphi : (N, I, J, K, g_N) \mapsto (M, g_M)$ be an ahss submersion with an ahss basis (I, J, K). Suppose that the map φ is horizontally homothetic. Then the assertions are equivalent.*

(a) The map φ is totally geodesic.

(b)

(i) $\varphi_*(\omega_I(\widehat{\nabla}_V \phi_I W + \mathcal{T}_V \omega_I W) + C_I(\mathcal{T}_V \phi_I W + \mathcal{H}\nabla_V \omega_I W)) = 0$ *for* $V, W \in \Gamma(\ker \varphi_*)$.

(ii) $\varphi_*(\omega_I(\widehat{\nabla}_W B_I Z + \mathcal{T}_W C_I Z) + C_I(\mathcal{T}_W B_I Z + \mathcal{H}\nabla_W C_I Z)) = 0$ *for* $W \in \Gamma(\ker \varphi_*)$ *and* $Z \in \Gamma((\ker \varphi_*)^\perp)$.

(c)

(i) $\varphi_*(\omega_J(\widehat{\nabla}_V \phi_J W + \mathcal{T}_V \omega_J W) + C_J(\mathcal{T}_V \phi_J W + \mathcal{H}\nabla_V \omega_J W)) = 0$ *for* $V, W \in \Gamma(\ker \varphi_*)$.

(ii) $\varphi_*(\omega_J(\widehat{\nabla}_W B_J Z + \mathcal{T}_W C_J Z) + C_J(\mathcal{T}_W B_J Z + \mathcal{H}\nabla_W C_J Z)) = 0$ *for* $W \in \Gamma(\ker \varphi_*)$ *and* $Z \in \Gamma((\ker \varphi_*)^\perp)$.

(d)

(i) $\varphi_*(\omega_K(\widehat{\nabla}_V \phi_K W + \mathcal{T}_V \omega_K W) + C_K(\mathcal{T}_V \phi_K W + \mathcal{H}\nabla_V \omega_K W)) = 0$ *for* $V, W \in \Gamma(\ker \varphi_*)$.

(ii) $\varphi_*(\omega_K(\widehat{\nabla}_W B_K Z + \mathcal{T}_W C_K Z) + C_K(\mathcal{T}_W B_K Z + \mathcal{H}\nabla_W C_K Z)) = 0$ *for* $W \in \Gamma(\ker \varphi_*)$ *and* $Z \in \Gamma((\ker \varphi_*)^\perp)$.

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