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NEW SUMMABILITY METHODS VIA $\widetilde{\phi}$ FUNCTIONS

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Abstract. In 1971, the definition of Orlicz $\widetilde{\phi}$ functions was introduced by Lindenstrauss and Tzafriri and moreover in 2006, the notion of double lacunary sequences was presented by Savaş and Patterson. The primary focus of this article is to introduce the double statistically $\widetilde{\phi}$-convergence and double lacunary statistically $\widetilde{\phi}$-convergence which are generalizations of the double statistically convergence [19] and double lacunary statistically convergence [24]. Additionally, some essential inclusion theorems are examined.

1. Introduction and Background

In 1951, Fast [6] and Steinhaus [26] independently put forward the idea of statistical convergence. Some fundamental characteristics of statistical convergence were established by Schoenberg [25] in 1959, and by Fridy [7] and in 1985 in the case of single sequences, and by Mursaleen and Edely [18] in 2003 in the case of multiple sequences. Ever since, numerous studies of single sequences have been devoted to this subject, such as ([4], [5], [16], [27]).

We recall that the concept of convergence for double sequences was presented by Pringsheim [20] as follows:

Definition 1. *[20] A double sequence $y = (y_{r,s})$ has Pringsheim limit ϖ (denoted by $P - \lim y = \varpi$) if for every $\varepsilon > 0$ there exists $\overline{N}_\varepsilon \in \mathbb{N}$ such that $|y_{r,s} - \varpi| < \varepsilon$, whenever $r, s > \overline{N}_\varepsilon$.*

In the recent past, the concept of statistical convergence for double sequences was studied by Mursaleen and Edely [18] as follows: A double sequence $y = (y_{r,s})$

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of real numbers is double statistically convergent to ϖ if for each $\varepsilon > 0$,

$$P - \lim_{z,w\to\infty} \frac{1}{zw} \left|\{(r,s) \in \mathbb{N}\times\mathbb{N} : r \leq z,\ s \leq w : |y_{r,s} - \varpi| \geq \varepsilon\}\right| = 0.$$

In such a case, we can symbolize with $st_2 - \lim y = \varpi$, and S_2 will represent the class of all statistically convergent double sequences. There are several papers dealing with double statistical convergence (see [2], [12], [19], [22], [23]). Also, in [24] the concept of lacunary statistically convergence for double sequence was introduced as follows:

Definition 2. *The double sequence $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ is named double lacunary if there exist two increasing sequences of integers such that*

$$r_0 = 0, \quad \gamma_\xi = r_\xi - r_{\xi-1} \to \infty \quad as \quad \xi \to \infty,$$

and

$$s_0 = 0, \quad \overline{\gamma}_\eta = s_\eta - s_{\eta-1} \to \infty \quad as \quad \eta \to \infty.$$

Also, $r_{\xi,\eta} = r_\xi s_\eta$, $\gamma_{\xi,\eta} = \gamma_\xi \overline{\gamma}_\eta$, $\zeta_\xi = \frac{r_\xi}{r_{\xi-1}}$, $\overline{\zeta}_\eta = \frac{s_\eta}{s_{\eta-1}}$, $\zeta_{\xi,\eta} = \zeta_\xi \overline{\zeta}_\eta$, and $\Phi_{\xi,\eta}$ is determined by

$$J_{\xi,\eta} = \left\{(r,s) : r_{\xi-1} < r \leq r_\xi \ and\ s_{\eta-1} < s \leq s_\eta\right\}.$$

We now consider the concept of double lacunary statistically convergence as follows:

Definition 3. *[24] Let $\Phi_{\xi,\eta}$ be a double lacunary sequence. The double sequence y is $S_{\theta_{\xi,\eta}} - P-$convergent to ϖ if for every $\varepsilon > 0$,*

$$P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left|\{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\}\right| = 0,$$

where the vertical bars denote the cardinality of the enclosed set.

Additionally, Lindenstrauss and Tzafriri [15] presented the following definitions.

Definition 4. *[15] A function $\widetilde{\phi}(\tau) : [0,\infty) \to [0,\infty)$ is called an Orlicz function if $\widetilde{\phi}(\tau)$ is continuous, non-decreasing and convex with $\widetilde{\phi}(0) = 0$, $\widetilde{\phi}(\tau) > 0$ for $\tau > 0$, and $\widetilde{\phi}(\tau) \to \infty$ as $\tau \to \infty$.*

Definition 5. *[15] An Orlicz function $\widetilde{\phi}(\tau)$ is said to satisfy the $\triangle_2$ condition for all values of τ, if there exists a constant $\overline{M} > 0$ such that*

$$\widetilde{\phi}(2\tau) \leq \overline{M}\,.\widetilde{\phi}(\tau), \qquad (\tau \geq 0)\,.$$

Krasnoselskii and Rutitsky [14] also showed that $\triangle_2$ condition is equivalent to the condition

$$\widetilde{\phi}\,(l\tau)) \leq \overline{M}\,(l)\ .\widetilde{\phi}(\tau), \qquad (\tau \geq 0)\,,$$

for all values of $l, \tau > 1$. Recently, some papers have been dealing with Orlicz function (see [1], [3], [8], [9], [10], [11], [13], [17], [21]).

The main goal of this paper is to extend some sequence spaces defined by Orlicz functions from ordinary (i.e., single) sequences to double sequences. To accomplish this goal we present a new notion of double lacunary statistically $\widetilde{\phi}-$convergence, as more generalizations of the double statistically convergence [18] and double lacunary statistically convergence [24]. Also, we examine some inclusions relations.

2. Main Result

We now present some definitions, which will be needed in the further section.

Definition 6. *Let $\widetilde{\phi}$ be an Orlicz function. A sequence $y = (y_{r,s})$ is said to be $\widetilde{\phi}-$double convergent to ϖ if $P - \lim_{r,s} \widetilde{\phi}(y_{r,s} - \varpi) = 0$. In such a situation, ϖ is called the $\widetilde{\phi}-$limit of $(y_{r,s})$ and symbolized by $\widetilde{\phi} - lim\, y = \varpi$.*

Note 1. *If $\widetilde{\phi}(y) = |y|$, then $\widetilde{\phi}-$double convergent notions coincide with usual double convergence. Also, it is simple to control, if $y = (y_{r,s})$ is $\widetilde{\phi}-$double convergent to ϖ, then any of its subsequence is $\widetilde{\phi}-$double convergent to ϖ.*

Definition 7. *Let $\widetilde{\phi}$ be an Orlicz function. A sequence $y = (y_{r,s})$ is said to be double statistically $\widetilde{\phi}-$convergent to ϖ if for each $\varepsilon > 0$,*

$$P - \lim_{z,w} \frac{1}{zw} \left| \left\{ r \leq z, s \leq w : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon \right\} \right| = 0.$$

ϖ is called the double statistical $\widetilde{\phi}-$ limit of the sequence $(y_{r,s})$ and we symbolize $S_2 - \widetilde{\phi}\ lim\, y = \varpi$ or $y_{r,s} \xrightarrow{P} \varpi(S_2 - \widetilde{\phi})$. We will denote the class of all double statistically $\widetilde{\phi}-$convergent sequences by $S_2 - \widetilde{\phi}$.

Note 2. *If $\widetilde{\phi}(y) = |y|$, then $S_2 - \widetilde{\phi}$ convergence coincides with double statistically convergence.*

Definition 8. *Let $\widetilde{\phi}$ be an Orlicz function. Let us define new sequence spaces $|\sigma_{1,1}|_{\widetilde{\phi}}$ and $N_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ as follows:*

$$|\sigma_{1,1}|_{\widetilde{\phi}} = \left\{ y = (y_{r,s}) : for\ some\ \varpi,\ P - \lim_{z,w} \left(\frac{1}{zw} \sum_{r,s=1,1}^{z,w} \widetilde{\phi}(y_{r,s} - \varpi) \right) = 0 \right\}$$

and

$$N_{\Phi_{\xi,\eta}} - \widetilde{\phi} = \left\{ y = (y_{r,s}) : for\ some\ \varpi,\ P - \lim_{\xi,\eta} \left(\frac{1}{\gamma_{\xi,\eta}} \sum_{(r,s) \in J_{\xi,\eta}} \widetilde{\phi}(y_{r,s} - \varpi) \right) = 0 \right\}.$$

Note 3. *If $\widetilde{\phi}(y) = |y|$, then the spaces $|\sigma_{1,1}|_{\widetilde{\phi}}$ and $N_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ coincide with $|\sigma_{1,1}|$ and $N_{\Phi_{\xi,\eta}}$, respectively, which were considered in [24].*

Definition 9. *Let $\widetilde{\phi}$ be an Orlicz function, and $\Phi_{\xi,\eta}$ be a double lacunary sequence. A sequence $y = (y_{r,s})$ is said to be double lacunary statistically $\widetilde{\phi}-$convergent to ϖ if for each $\varepsilon > 0$,*

$$P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left|\left\{(r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $S_{\Phi_{\xi,\eta}} - \widetilde{\phi} \lim y = \varpi$ or $y_{r,s} \xrightarrow{P} \varpi(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$. We will denote the class of all double lacunary statistically $\widetilde{\phi}-$convergent sequences by $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$.

Note 4. *If $\widetilde{\phi}(y) = |y|$, then $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergence coincides with $S_{\Phi_{\xi,\eta}}-$ convergence, which was studied by Savas and Patterson [24].*

Example 1. *Let $\widetilde{\phi}(y) = y^2$ and $\Phi_{\xi,\eta} = (2^\xi, 3^\eta)$. It is quite clear that $\widetilde{\phi}$ satisfies the $\triangle_2$ condition. Let us define the sequence $(y_{r,s})$ as follows:*

$$y_{r,s} = \begin{cases} \sqrt{rs}, & r = z^2 \text{ and } s = w^2, (z,w) \in \mathbb{N} \times \mathbb{N}; \\ \frac{1}{\sqrt{rs}}, & \text{otherwise}, \end{cases}$$

then the sequence $(y_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent to 0 despite the fact that $(y_{r,s})$ is not convergent. To confirm, we obtain the following

$$\begin{aligned} & P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left|\left\{(r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon\right\}\right| \\ = \; & P - \lim_{\xi,\eta} \frac{1}{2^{\xi-1}3^{\eta-1}} \left|\left\{(r,s) \in (2^{\xi-1}, 3^\xi] \times (3^{\eta-1}, 3^\eta] : \widetilde{\phi}(y_{r,s} - 0) \geq \varepsilon\right\}\right| \\ = \; & P - \lim_{\xi,\eta} \frac{6}{2^{\xi-1}3^{\eta-1}} \left|\left\{(r,s) \in (2^{\xi-1}, 2^\xi] \times (3^{\eta-1}, 3^\eta] : \, y_{r,s}^2 \geq \varepsilon\right\}\right| \\ \leq \; & P - \lim_{\xi,\eta} \frac{6}{2^{\xi-1}3^{\eta-1}} \left|\left\{r \leq 2^\xi, s \leq 3^\eta : y_{r,s}^2 \geq \varepsilon\right\}\right| \\ = \; & P - \lim_{\xi,\eta} \frac{6}{zw} \left|\left\{r \leq z, s \leq w : \, y_{r,s}^2 \geq \varepsilon\right\}\right| = 0. \end{aligned}$$

This demonstrates that the double sequence $(y_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent to 0 even $(y_{r,s})$ is not convergent.

Example 2. *Let $\widetilde{\phi}$ be an Orlicz function with $\widetilde{\phi}(y) = |y|$, $\Phi_{\xi,\eta}$ be any double lacunary sequence, then the sequence $(y_{r,s})$ defined by $y_{r,s} = r^2 s^2$, for every $(r,s) \in \mathbb{N} \times \mathbb{N}$ is not $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent. To verify this let us hold any $\varpi \in \mathbb{R}$. Then $\varpi \leq 0$ or $\varpi > 0$. If $\varpi \leq 0$, choose $\varepsilon = \frac{1}{2}$, then for every $(r,s) \in \mathbb{N} \times \mathbb{N}$,*

$$R(\varepsilon) = \{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\} = J_{\xi,\eta}.$$

Hence, for $\varpi \leq 0$,

$$P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left|\{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\}\right| = \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} |J_{\xi,\eta}| = 1.$$

If $\varpi > 0$*, then there exists* $(r_0, s_0) \in \mathbb{N} \times \mathbb{N}$ *in such a manner* $y_{r_0-1,s_0-1} \leq \varpi \leq y_{r_0,s_0}$*. In this circumstances, if* $\varpi < 1$*, by taking* $\varepsilon = \frac{1}{2} \min\{\varpi, 1 - \varpi\}$*, we obtain*

$$R(\varepsilon) = \{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\} = J_{\xi,\eta}.$$

Also, if $\varpi \geq 1$*, by taking* $\varepsilon = \frac{1}{2} \min\{\varpi - y_{r_0-1,s_0-1}, y_{r_0,s_0} - \varpi\}$*, we get*

$$R(\varepsilon) = \{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\} = J_{\xi,\eta}.$$

Thus, for $\varpi > 0$,

$$\begin{aligned} & P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left|\{(r,s) \in J_{\xi,\eta} : |y_{r,s} - \varpi| \geq \varepsilon\}\right| \\ = \quad & P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} |J_{\xi,\eta}| = 1. \end{aligned}$$

Definition 10. *A double sequence* $y = (y_{r,s})$ *is said to be double* $\widetilde{\phi}-$ *bounded with regard to the Orlicz function* $\widetilde{\phi}$*, if there exists* $\overline{M} > 0$ *such that* $\widetilde{\phi}(y_{r,s}) \leq \overline{M}$*, for every* $(r,s) \in \mathbb{N} \times \mathbb{N}$.

In subsequent theorem, we present the relationship between the spaces $N_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ and $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ and demonstrate that these two concepts are equivalent for double $\widetilde{\phi}-$ bounded spaces.

Theorem 1. *Let* $\Phi_{\xi,\eta} = (r_\xi, s_\eta)$ *be a double lacunary sequence, then*

(1) $y_{r,s} \xrightarrow{P} \varpi(N_{\Phi_{\xi,\eta}} - \widetilde{\phi})$ *implies* $y_{r,s} \xrightarrow{P} \varpi(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$*, and converse is not true.*

(2) *If* y *is double* $\widetilde{\phi}-$ *bounded and* $y_{r,s} \xrightarrow{P} \varpi(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$ *then* $y_{r,s} \xrightarrow{P} \varpi(N_{\Phi_{\xi,\eta}} - \widetilde{\phi})$.

Proof. (1) Provided that $\varepsilon > 0$ and $y_{r,s} \xrightarrow{P} \varpi(N_{\Phi_{\xi,\eta}} - \widetilde{\phi})$, then

$$\begin{aligned} \sum_{(r,s)\in J_{\xi,\eta}} \widetilde{\phi}(y_{r,s} - \varpi) \quad & \geq \quad \sum_{\substack{(r,s)\in J_{\xi,\eta} \\ \widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon}} \widetilde{\phi}(y_{r,s} - \varpi) \\ & \geq \quad \varepsilon \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon \right\} \right| \end{aligned}$$

where the first result follows. In order to demonstrate the converse part, we will consider a double sequence which is in $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ but not in $N_{\Phi_{\xi,\eta}} - \widetilde{\phi}$. For this, let $\widetilde{\phi}(y) = |y|$, proceeding as in Savas and Patterson [24], $\Phi_{\xi,\eta}$ be given and the sequence $(y_{r,s})$ is defined by

$$y_{r,s} = \begin{pmatrix} 1 & 2 & 3 & \cdots & \sqrt{\gamma_{\xi,\eta}} & 0 & \cdots \\ 2 & 2 & 3 & \cdots & \sqrt{\gamma_{\xi,\eta}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{\gamma_{\xi,\eta}} & \sqrt{\gamma_{\xi,\eta}} & \cdots & \cdots & \sqrt{\gamma_{\xi,\eta}} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \end{pmatrix}.$$

Note that $y_{r,s}$ is not double bounded. It was displayed in Savas and Patterson [24] that $y_{r,s} \overset{P}{\to} 0(S_{\Phi_{\xi,\eta}})$. However, $y_{r,s}$ is not convergent to 0 $(N_{\Phi_{\xi,\eta}})$. From Note 2.3 and 2.4 we conclude that $y_{r,s} \overset{P}{\to} 0(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$, yet $y_{r,s}$ is not convergent to 0 $(N_{\Phi_{\xi,\eta}} - \widetilde{\phi})$. Therefore, $(N_{\Phi_{\xi,\eta}} - \widetilde{\phi}) \subseteq (S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$.

(2) Let $y_{r,s} \overset{P}{\to} \varpi(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$ and y is double $\widetilde{\phi}-$ bounded, in other way $\widetilde{\phi}(y_{r,s}) \leq \overline{M}$ for every $(r,s) \in \mathbb{N} \times \mathbb{N}$. For $\varepsilon > 0$, we obtain

$$\begin{aligned} \frac{1}{\gamma_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \widetilde{\phi}(y_{r,s} - \varpi) &= \frac{1}{\gamma_{\xi,\eta}} \sum_{\substack{(r,s)\in J_{\xi,\eta} \\ \widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon}} \widetilde{\phi}(y_{r,s} - \varpi) + \frac{1}{\gamma_{\xi,\eta}} \sum_{\substack{(r,s)\in J_{\xi,\eta} \\ \widetilde{\phi}(y_{r,s}-\varpi)<\varepsilon}} \widetilde{\phi}(y_{r,s} - \varpi) \\ &\leq \frac{\overline{M} + \widetilde{\phi}(\varpi)}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon \right\} \right| + \varepsilon \end{aligned}$$

which gives us the result.

□

Note 5. *From (1) and (2) of the above theorem, we conclude that if y is double $\widetilde{\phi}-$ bounded, then $S_{\Phi_{\xi,\eta}} - \widetilde{\phi} = N_{\Phi_{\xi,\eta}} - \widetilde{\phi}$.*

In the following theorems we examine the relationship between $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ and $S_2 - \widetilde{\phi}$ under certain restrictions on $\Phi_{\xi,\eta} = (r_\xi, s_\eta)$.

Theorem 2. *For any double lacunary sequence $\Phi_{\xi,\eta}$ and any Orlicz function $\widetilde{\phi}$, $S_2 - \widetilde{\phi}\ lim\, y = \varpi$ implies $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\ lim\, y = \varpi$ if and only if $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$. Provided that $\liminf_\xi \zeta_\xi = 1$ and $\liminf_\eta \overline{\zeta}_\eta = 1$, then there exists a double bounded $S_2 - \widetilde{\phi}$ double sequence that is not $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$.*

Proof. (Sufficiency) Suppose that $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$, then there exists a $\overline{\delta} > 0$ such that $\zeta_\xi > 1 + \overline{\delta}$ and $\overline{\zeta}_\eta > 1 + \overline{\delta}$ for sufficiently large ξ and η, which implies that $\frac{\gamma_{\xi,\eta}}{r_{\xi,\eta}} > \frac{\overline{\delta}}{1+\overline{\delta}}$. If $y_{r,s} \overset{P}{\to} \varpi(S_2 - \widetilde{\phi})$, then for every $\varepsilon > 0$ and for sufficiently large ξ and η, we are granted the following

$$\begin{aligned}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon\right\}\right| &= \frac{r_{\xi,\eta}}{\gamma_{\xi,\eta}}\frac{1}{r_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon\right\}\right| \\ &\leq \frac{1+\overline{\delta}}{\overline{\delta}}\frac{1}{r_{\xi,\eta}}\left|\left\{r\leq r_{\xi},s\leq s_{\eta}:\widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon\right\}\right|.\end{aligned}$$

Thus, $y_{r,s}\xrightarrow{P}\varpi(S_{\Phi_{\xi,\eta}}-\widetilde{\phi})$.

(Necessity) Assume that $\liminf_{\xi}\zeta_{\xi}=1$ or $\liminf_{\eta}\overline{\zeta}_{\eta}=1$ and consider a sequence which is $S_2-\widetilde{\phi}$ convergent, but not $S_{\Phi_{\xi,\eta}}-\widetilde{\phi}$ convergent. For this, let $\widetilde{\phi}(y)=|y|$ and let us select a double subsequence $(r_{\xi_{\bar{i}},\eta_{\bar{j}}}=r_{\xi_{\bar{i}}}s_{\eta_{\bar{j}}})$ of the double lacunary sequence $\Phi_{\xi,\eta}$ such that $\frac{r_{\xi_{\bar{i}}}}{r_{\xi_{\bar{i}-1}}}<1+\frac{1}{\bar{i}}$, $\frac{s_{\eta_{\bar{j}}}}{s_{\eta_{\bar{j}-1}}}<1+\frac{1}{\bar{j}}$, $\frac{r_{\xi_{\bar{i}}-1}}{r_{\xi_{\bar{i}-1}}}>\bar{i}$, and $\frac{s_{\eta_{\bar{j}}-1}}{s_{\eta_{\bar{j}-1}}}>\bar{j}$ where $\xi_{\bar{i}}\geq\xi_{\bar{i}-1}+2$ and $\eta_{\bar{j}}\geq\eta_{\bar{j}-1}+2$. Also, let us define $y=(y_{\bar{i},\bar{j}})$ by

$$y_{\bar{i},\bar{j}}=\begin{cases}1, & \text{if } (r,s)\in J_{\xi_{\bar{i}},\eta_{\bar{j}}},\ \bar{i},\bar{j}=1,2,3...\\ 0, & \text{otherwise.}\end{cases}$$

Then for any real ϖ, we are granted the following

$$\frac{1}{\gamma_{\xi_{\bar{i}},\eta_{\bar{j}}}}\sum_{J_{\xi_{\bar{i}},\eta_{\bar{j}}}}\left|y_{\bar{i},\bar{j}}-\varpi\right|=|1-\varpi|\ \text{ for } \bar{i},\bar{j}=1,2,3,...,$$

and

$$\frac{1}{\gamma_{\xi_{\bar{i}},\eta_{\bar{j}}}}\sum_{J_{\xi,\eta}}\left|y_{\bar{i},\bar{j}}-\varpi\right|=|\varpi|\,,\text{for } \xi\neq\xi_{\bar{i}} \text{ and } \eta\neq\eta_{\bar{j}}.$$

That means

$$P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon\right\}\right|\neq 0.$$

Therefore, y is not $S_{\Phi_{\xi,\eta}}-\widetilde{\phi}$ convergent to ϖ. However, y is $S_2-\widetilde{\phi}-$convergent, since if $\bar{t}$ and $\overline{v}$ are any quite enough large integers we can identify the unique $\bar{i}$ and $\bar{j}$ for which $r_{\xi_{\bar{i}}-1}<\bar{t}\leq r_{\xi_{\bar{i}+1}-1}$ and $s_{\eta_{\bar{j}}-1}<\overline{v}\leq s_{\eta_{\bar{j}+1}-1}$, and we write the following

$$\begin{aligned}\frac{1}{\bar{t}\overline{v}}\sum_{\bar{i},\bar{j}=1,1}^{\bar{t}\overline{v}}\widetilde{\phi}\left(y_{\bar{i},\bar{j}}\right) &= \frac{1}{\bar{t}\overline{v}}\sum_{\bar{i},\bar{j}=1,1}^{\bar{t}\overline{v}}\left|y_{\bar{i},\bar{j}}\right| \\ &\leq \left(\frac{r_{\xi_{\bar{i}}-1}+\gamma_{\xi_{\bar{i}}}}{r_{\xi_{\bar{i}}-1}}\right)\cdot\left(\frac{s_{\eta_{\bar{j}}-1}+\overline{\gamma}_{\eta_{\bar{j}}}}{s_{\eta_{\bar{j}}-1}}\right)\end{aligned}$$

$$\begin{aligned}
&\le \frac{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}}{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}} + \frac{r_{\xi_{\bar{i}}-1}\overline{\gamma}_{s_{\bar{j}}}}{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}} + \frac{\gamma_{\xi_{\bar{i}}} s_{\eta_{\bar{j}}-1}}{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}} + \frac{\gamma_{\xi_{\bar{i}}}\overline{\gamma}_{\eta_{\bar{j}}}}{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}} \\
&\le 1 + \frac{\overline{\gamma}_{\eta_{\bar{j}}}}{s_{\eta_{\bar{j}}-1}} + \frac{\gamma_{\xi_{\bar{i}}}}{r_{\xi_{\bar{i}}-1}} + \frac{\gamma_{\xi_{\bar{i}}}\overline{\gamma}_{\eta_{\bar{j}}}}{r_{\xi_{\bar{i}}-1,\eta_{\bar{j}}-1}} \\
&\le 1 + \frac{\left(s_{\eta_{\bar{j}}} - s_{\eta_{\bar{j}}-1}\right)}{s_{\eta_{\bar{j}}-1}} + \frac{\left(r_{\xi_{\bar{i}}} - r_{\xi_{\bar{i}}-1}\right)}{r_{\xi_{\bar{i}}-1}} + \left(\frac{r_{\xi_{\bar{i}}} - r_{\xi_{\bar{i}}-1}}{r_{\xi_{\bar{i}}-1}}\right)\left(\frac{s_{\eta_{\bar{j}}} - s_{\eta_{\bar{j}}-1}}{s_{\eta_{\bar{j}}-1}}\right) \\
&\le 1 + \frac{s_{\eta_{\bar{j}}}}{s_{\eta_{\bar{j}}-1}} - 1 + \frac{r_{\xi_{\bar{i}}}}{r_{\xi_{\bar{i}}-1}} - 1 + \left(\frac{r_{\xi_{\bar{i}}}}{r_{\xi_{\bar{i}}-1}} - 1\right)\left(\frac{s_{\eta_{\bar{j}}}}{s_{\eta_{\bar{j}}-1}} - 1\right) \\
&\le \left(1 + \frac{1}{\bar{i}}\right) + \left(1 + \frac{1}{\bar{j}}\right) + \frac{1}{\bar{i}}\frac{1}{\bar{j}} - 1 \\
&\le 1 + \frac{1}{\bar{i}} + \frac{1}{\bar{j}} + \frac{1}{\bar{i}}\frac{1}{\bar{j}}
\end{aligned}$$

as $\bar{t} \to \infty$ and $\overline{v} \to \infty$, it follows that $\bar{i}, \bar{j} \to \infty$. Thus, from Note 2, y is $S_2 - \widetilde{\phi}$ convergent. □

The following example demonstrates that there exists a $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent sequence which has a double subsequence that is not $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$–convergent.

Example 3. *Let $\Phi_{\xi,\eta} = (2^{\xi}, 3^{\eta})$ be the double lacunary sequence, $\widetilde{\phi}(y) = |y|$ be an Orlicz function and $(y_{r,s})$ is defined by*

$$y_{r,s} = \begin{cases} rs, & r = z^2 \text{ and } s = w^2, (z,w) \in \mathbb{N}\times\mathbb{N} \\ \frac{1}{rs}, & \text{otherwise} \end{cases}$$

then the sequence $(y_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent to 0. However, $(y_{r,s})$ has a double subsequence, which is not $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent.

Theorem 3. *For any lacunary sequence $\Phi_{\xi,\eta}$ and any Orlicz function $\widetilde{\phi}$, $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\ lim\, y = \varpi$ implies $S_2 - \widetilde{\phi}\ lim\, y = \varpi$ if and only if $\lim\sup_{\xi} \zeta_{\xi} < \infty$ and $\lim\sup_{\eta} \overline{\zeta}_{\eta} < \infty$. If $\lim\sup_{\xi} \zeta_{\xi} = \infty$ and $\lim\sup_{\eta} \overline{\zeta}_{\eta} = \infty$ then there exists a double bounded $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ summable sequence that is not $S_2 - \widetilde{\phi}$.*

Proof. If $\lim\sup_{\xi} \zeta_{\xi} < \infty$ and $\lim\sup_{\eta} \overline{\zeta}_{\eta} < \infty$, then there is an $\widehat{H} > 0$ such that $\zeta_{\xi} < \widehat{H}$ and $\overline{\zeta}_{\eta} < \widehat{H}$ for all ξ and η. It is assumed that $y_{r,s} \xrightarrow{P} \varpi(S_{\Phi_{\xi,\eta}} - \widetilde{\phi})$, and let

$$N_{\xi,\eta} = \left|\left\{(r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi) \ge \varepsilon\right\}\right|.$$

By the definition of $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent and given any $\varepsilon' > 0$, there is an $\xi_0, \eta_0 \in \mathbb{N}$ such that $\frac{N_{\xi,\eta}}{\gamma_{\xi,\eta}} < \varepsilon'$ for all $\xi > \xi_0$ and $\eta > \eta_0$. Now let

$$M = max\left\{N_{\xi,\eta} : 1 \leq \xi \leq \xi_0 \ \& \ 1 \leq \eta \leq \eta_0\right\}$$

and let z and w be any integers satisfying $r_{\xi-1} < z \leq r_\xi$ and $s_{\eta-1} < w \leq s_\eta$; then we can write

$$\begin{aligned}
& \frac{1}{zw}\left|\left\{r \leq z,\ s \leq w : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon\right\}\right| \\
\leq\ & \frac{1}{r_{\xi-1}s_{\eta-1}}\left|\left\{r \leq r_\xi, s \leq s_\eta : \widetilde{\phi}(y_{r,s} - \varpi) \geq \varepsilon\right\}\right| \\
=\ & \frac{1}{r_{\xi-1}s_{\eta-1}}\left\{N_{1,1} + N_{2,2} + \cdots + N_{\xi_0+1,\eta_0+1} + \cdots + N_{\xi,\eta}\right\} \\
\leq\ & \frac{M^2}{r_{\xi-1}s_{\eta-1}}\xi_0\eta_0 \\
& + \frac{1}{r_{\xi-1}s_{\eta-1}}\left\{\gamma_{\xi_0+1,\eta_0+1}\frac{N_{\xi_0+1,\eta_0+1}}{\gamma_{\xi_0+1,\eta_0+1}} + \cdots + \gamma_{\xi,\eta}\frac{N_{\xi,\eta}}{\gamma_{\xi,\eta}}\right\} \\
\leq\ & \frac{\xi_0\eta_0 M^2}{r_{\xi-1}s_{\eta-1}} + \\
& \frac{1}{r_{\xi-1}s_{\eta-1}}\Big(\sup_{\xi>\xi_0\&\ \eta>\eta_0}\frac{N_{\xi,\eta}}{\gamma_{\xi,\eta}}\Big)\left\{\gamma_{\xi_0+1,\eta_0+1} + \cdots\cdots + \gamma_{\xi,\eta}\right\} \\
\leq\ & \frac{\xi_0 M}{r_{\xi-1}}\frac{\eta_0 M}{s_{\eta-1}} + \varepsilon'\frac{r_\xi - r_{\xi_0}}{r_{\xi-1}}\frac{s_\eta - s_{\eta_0}}{s_{\eta-1}} \\
\leq\ & \frac{\xi_0 M}{r_{\xi-1}}\frac{\eta_0 M}{s_{\eta-1}} + \varepsilon'\zeta_\xi\overline{\zeta}_\eta \leq \frac{\xi_0 M}{r_{\xi-1}}\frac{\eta_0 M}{s_{\eta-1}} + \varepsilon'\widehat{H}^2.
\end{aligned}$$

For converse, suppose that $\lim\sup_\xi \zeta_\xi = \infty$ and $\lim\sup_\eta \overline{\zeta}_\eta = \infty$ and consider a sequence which is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent, but not $S_2 - \widetilde{\phi}$ convergent. For this, let $\widetilde{\phi}(y) = |y|$ and select a double subsequence $(r_{\xi_{\overline{i}},\eta_{\overline{j}}})$ of the double lacunary sequence $\Phi_{\xi,\eta} = (r_\xi, s_\eta)$ such that $\zeta_{\xi_{\overline{i}}} > \overline{i}$ and $\overline{\zeta}_{\eta_{\overline{j}}} > \overline{j}$ define a double bounded sequence $y = (y_{z,w})$ by

$$y_{z,w} = \begin{cases} 1, & r_{\xi_{\overline{i}}-1} < z \leq 2r_{\xi_{\overline{i}}-1} \ \& \ s_{\eta_{\overline{j}}-1} < w \leq 2s_{\eta_{\overline{j}}-1} \text{ for } \overline{i},\ \overline{j} = 1,2,3... \\ 0, & otherwise. \end{cases}$$

Hence, $y \in N_{\Phi_{\xi,\eta}}$ but $y \notin |\sigma_{1,1}|$. From [24], y is $S_{\Phi_{\xi,\eta}}$-convergent. The above Note 2.4 implies that y is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent, but y is not S_2-convergent. Consequently, by above Note 2 implies y is not $S_2 - \widetilde{\phi}$ convergent.

□

By combining the above two theorems let us present following theorem.

Theorem 4. *Let $\Phi_{\xi,\eta}$ be any double lacunary sequence; then $S_2 - \widetilde{\phi} = S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ if and only if*

$$1 < \lim_{\xi} inf\, \zeta_{\xi} \leq \lim_{\xi} sup\, \zeta_{\xi} < \infty$$

and

$$1 < \lim_{\eta} inf\, \overline{\zeta}_{\eta} \leq \lim_{\eta} sup\, \overline{\zeta}_{\eta} < \infty.$$

Theorem 5. *Let $\Phi_{\xi,\eta}$ be any double lacunary sequence and $\widetilde{\phi}$ be an Orlicz function. If the sequence $(y_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent, then $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ limit of $(y_{r,s})$ is unique.*

Proof. Let $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, y_{r,s} = \varpi_0$ and $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, y_{r,s} = \kappa_0$. Then

$$P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi_0) \geq \varepsilon \right\} \right| = 0$$

and

$$P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \kappa_0) \geq \varepsilon \right\} \right| = 0$$

i.e.

$$\begin{aligned} & P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \varpi_0) < \varepsilon \right\} \right| \\ = \quad & 1 = P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(y_{r,s} - \kappa_0) < \varepsilon \right\} \right|. \end{aligned}$$

Let us consider such $(r,s) \in J_{\xi,\eta}$ for which both of $\widetilde{\phi}(y_{r,s} - \varpi_0) < \varepsilon$ and $\widetilde{\phi}(y_{r,s} - \kappa_0) < \varepsilon$ are true. For such $(r,s) \in J_{\xi,\eta}$ we have

$$\begin{aligned} \widetilde{\phi}\left(\frac{1}{2}(\varpi_0 - \kappa_0)\right) &= \widetilde{\phi}\left(\frac{1}{2}(\varpi_0 - y_{r,s} + y_{r,s} - \kappa_0)\right) \\ &\leq \frac{1}{2}\widetilde{\phi}(y_{r,s} - \varpi_0) + \frac{1}{2}\widetilde{\phi}(y_{r,s} - \kappa_0) = \varepsilon. \end{aligned}$$

□

For the consequence we presume the Orlicz function which fulfils $\triangle_2$ condition, unless otherwise claimed.

Theorem 6. *If $(y_{r,s})$ and $(z_{r,s})$ are $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent and α is any real constant, then*

(1) $(y_{r,s} + z_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent and $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, (y_{r,s} + z_{r,s}) = S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, y_{r,s} + S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, z_{r,s}$,

(2) $(\alpha y_{r,s})$ is $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}$ convergent and $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\, lim\, (\alpha y_{r,s}) = \alpha.\, S_{\theta_{\xi,\eta}} - \widetilde{\phi}\, lim\, y_{r,s}$.

Proof. Since $\widetilde{\phi}$ fullfils the $\triangle_2-$ condition, then there exists $\overline{M}>0$ such that $\widetilde{\phi}(2y)\leq \overline{M}.\widetilde{\phi}(y)$, for every $y\in\mathbb{R}$.

(1) Let $S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\,lim\,y_{r,s}=L$ and $S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\ lim\,z_{r,s}=\varpi$
i.e.

$$\begin{aligned} & P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)\geq\varepsilon\right\}\right| \\ = \ & 0=P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(z_{r,s}-\kappa)\geq\varepsilon\right\}\right| \end{aligned}$$

i.e.

$$\begin{aligned} & P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)<\varepsilon\right\}\right| \\ = \ & 1=P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(z_{r,s}-\kappa)<\varepsilon\right\}\right|. \end{aligned}$$

Let us consider such $(r,s)\in J_{\xi,\eta}$ for which both of $\widetilde{\phi}(y_{r,s}-\varpi)<\frac{\varepsilon}{2\overline{M}}$ and $\widetilde{\phi}(z_{r,s}-\kappa)<\frac{\varepsilon}{2\overline{M}}$ are true. Then for such $(r,s)\in J_{\xi,\eta}$, we are granted the following

$$\begin{aligned} \widetilde{\phi}((y_{r,s}+z_{r,s})-(\varpi+\kappa)) &= \widetilde{\phi}((y_{r,s}-\varpi)+(z_{r,s}-\kappa))\leq\widetilde{\phi}(2(y_{r,s}-\varpi)+2(z_{r,s}-\kappa)) \\ &\leq \overline{M}(\widetilde{\phi}(y_{r,s}-\varpi)+\widetilde{\phi}(z_{r,s}-\kappa))=\overline{M}.\left(\frac{\varepsilon}{2\overline{M}}+\frac{\varepsilon}{2\overline{M}}\right)=\varepsilon. \end{aligned}$$

Therefore,

$$P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}+z_{r,s}-\varpi-\kappa)<\varepsilon\right\}\right|=1$$

i.e.

$$P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}+z_{r,s}-\varpi-\kappa)\geq\varepsilon\right\}\right|=0$$

that means

$S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\ lim\,(y_{r,s}+z_{r,s})=\varpi+\kappa=S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\ \ lim\,y_{r,s}+S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\ \ lim\,z_{r,s}.$

(2) Let $p\in\mathbb{N}$ such that $|\alpha|\leq 2^p$ and $S_{\Phi_{\xi,\eta}}-\widetilde{\phi}\ lim\,y_{r,s}=\varpi$, then

$$P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(y_{r,s}-\varpi)<\varepsilon\right\}\right|=1.$$

Let us consider such $(r,s)\in J_{\xi,\eta}$ for which $\widetilde{\phi}(y_{r,s}-\varpi)<\frac{\varepsilon}{2^p}$, then

$$\begin{aligned} \widetilde{\phi}(\alpha(y_{r,s}-\varpi)) &= \widetilde{\phi}(|\alpha|\,(y_{r,s}-\varpi))\leq\widetilde{\phi}(2^p(y_{r,s}-\varpi)) \\ &\leq 2^p\widetilde{\phi}(y_{r,s}-\varpi)\leq 2^p.\frac{\varepsilon}{2^p}=\varepsilon. \end{aligned}$$

Hence,

$$P-\lim_{\xi,\eta}\frac{1}{\gamma_{\xi,\eta}}\left|\left\{(r,s)\in J_{\xi,\eta}:\widetilde{\phi}(\alpha(y_{r,s}-\varpi))<\varepsilon\right\}\right|=1,$$

i.e.

$$P - \underset{\xi,\eta}{lim} \frac{1}{\gamma_{\xi,\eta}} \left| \left\{ (r,s) \in J_{\xi,\eta} : \widetilde{\phi}(\alpha(y_{r,s} - \varpi)) \geq \varepsilon \right\} \right| = 0.$$

Thus, $S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\ lim\,(\alpha y_{r,s}) = \alpha.y = \alpha.\, S_{\Phi_{\xi,\eta}} - \widetilde{\phi}\ lim\, y_{r,s}$. This concludes the proof of the theorem. □

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