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ON INTERNAL CATEGORIES IN HIGHER DIMENSIONAL GROUPS

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Abstract. In this paper, we demonstrated the equivalencies of internal categories of cat-1 groups and cat-2 groups structures using some related equivalent categories.

1. Introduction

Establishing an equivalence involves demonstrating strong similarities between the mathematical structures concerned. Homotopical algebra performs the same thing as homological algebra in that it extends notions from classical geometric homology theory to algebraic contexts.

The concept of crossed modules over groups was first introduced by Whitehead in the late 1940s, appeared during his research on the algebraic structure of the second relative homotopy groups. It has been extensively used in a variety of branches of mathematics. Crossed modules of groups model connected homotopy 2-types. Loday defined the notion of a 'cat-n group' and showed that this notion corresponds to the crossed module and the crossed square for n=1,2, respectively. Crossed squares are two-dimensional analogous of crossed modules as well as cat-1 groups and crossed modules are known as two-dimensional generalizations of a group.

The main goal of this study is to construct the equivalence between the category of internal categories in the category of cat-1 groups and the category of cat-2 groups. Since the category of crossed modules and cat-1 groups have pullbacks, (see [5] and [1]), we can talk about internal categories in their categories. To begin with, we note from our previous paper, [6], that the equivalence between the category of cat-1 groups and those of crossed modules is also preserved in their internal categories. Later, by using the equivalence of internal categories of crossed modules and the category of crossed square, and considering the equivalence of the category of crossed square and that of cat-2 groups, we get the equivalence concerned.

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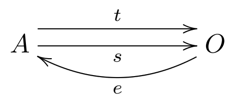
2. Preliminaries

Let $\mathcal{C}$ be a category with pullbacks. Then an internal category C in $\mathcal{C}$ consists of objects A and O in $\mathcal{C}$ together with morphism s, t, e



such that

$$se = te = 1_O$$

and a composition morphism

$$m : A_t \times_s A \longrightarrow A$$

which are also the morphisms in $\mathcal{C}$, $A_t \times_s A = \{(f,g)|t(f) = s(g)\}$ and the following equations are satisfied:

- $t(m(f,g)) = t(g), s(m(f,g)) = s(f)$,
- $m(1_A \times m) = m(m \times 1_A)$ and
- $m(es, 1_A) = m(1_A, et) = 1_A$.

Thus an internal category in $\mathcal{C}$ is defined to be a six-tuple (A, O, s, t, e, m),[3, 4].

Definition 2.1. A cat-1 group is a group G together with a normal subgroup R of G and the homomorphisms $s, t : G \longrightarrow R$ satisfy the following conditions

- $s|_N = t|_N = id_N$
- $[Kers, Kert] = 1$.

A cat-1 group is denoted by (G, R, s, t), [2].

Definition 2.2. A cat-2 group is a group G together with the subgroups R_0, R_1 of G and the homomorphisms $s_0, t_0 : G \longrightarrow R_0$ and $s_1, t_1 : G \longrightarrow R_1$ such that

$$\begin{aligned} s_0|_{R_0} &= t_0|_{R_0} = id_{R_0} \\ s_1|_{R_1} &= t_1|_{R_1} = id_{R_1} \\ [Kers_i, Kert_i] &= 1, 1 \leq i, j \leq 2 \\ s_i s_j &= s_j s_i, t_i t_j = t_j t_i, t_i s_j = s_j t_i, i \neq j. \end{aligned}$$

$(G, R_0, R_1, s_0, s_1, t_0, t_1)$ structure is called the cat-2 group, [2].

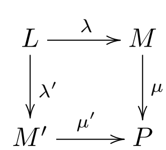
Definition 2.3. A crossed module consists of group R and S with the action of R on S denoted by ${}^r s$ for $r \in R$ and $s \in S$, and a $\partial : S \longrightarrow R$ satisfying the following two conditions:

CM1) ∂ is R-equivariant, so $\partial({}^r s) = r\partial(s)r^{-1}$

CM2) Peiffer rule; ${}^{\partial(s)}s_1 = ss_1s^{-1}$

where $r \in R$ and $s, s_1 \in S$, [8].

Definition 2.4. A crossed square is a commutative diagram of groups



with an action of P on L, M, M' and with a function $h : M \times M' \to L$ holding the following conditions:

- the homomorphisms $\lambda, \lambda', \mu, \mu'$ and $\alpha = \mu\lambda = \mu'\lambda'$ are crossed modules and the morphisms: $\lambda \to \alpha, \alpha \to \mu, \lambda' \to \alpha$ and $\alpha \to \mu'$ are crossed module morphisms.
- $\lambda h(m, m') = m^{\mu'(m')} m^{-1}$ and $\lambda' h(m, m') = {}^{\mu(m)} m' m'^{-1}$,
- $h(\lambda(l), m') = l^{m'} l^{-1}$ and $h(m, \lambda'(l)) = {}^{m} l l^{-1}$,
- $h(m_1 m_2, m') = {}^{m_1} h(m_2, m') h(m_1, m')$ and $h(m, m_1' m_2') = h(m, m_1')^{m_1'} h(m, m_2')$,
- $h({}^{p}m, {}^{p}m') = {}^{p} h(m, m')$,
- ${}^{m}({}^{m'}l) h(m, m') = h(m, m')^{m'}({}^{m}l)$,
  for all $m, m_1, m_2 \in M, m_1', m_2' \in M', p \in P$, and $l \in L$, [7].

Proposition 1. The category of cat-1 groups is categorically equivalent to the category of crossed modules, [2].

Proposition 2. The category of cat-2 groups is categorically equivalent to the category of crossed squares, [2].

3. Internal Categories within the Category of Cat-1 groups

Definition 3.1. Let G_0, G_1 and R_0, R_1 be groups. $R_0 \trianglelefteq G_0$ and $R_1 \trianglelefteq G_1$ the properties are hold. Let (G_1, R_1, s, t) and (G_0, R_0, s', t') be two cat-1 groups and $(G_1, G_0, R_1, R_0, \overline{s}, \overline{t}, \overline{s'}, \overline{t'}, m_1, m_0)$ be internal of cat-1 groups illustrated with

$$G_1 \underset{\overline{s}}{\overset{\overline{t}}{\rightrightarrows}} G_0$$

and

$$R_1 \underset{\overline{s'}}{\overset{\overline{t'}}{\rightrightarrows}} R_0.$$

Therefore, the internal of cat -1 groups is obtained

$$\begin{array}{ccc} G_1 & \underset{\overline{s}}{\overset{\overline{t}}{\rightrightarrows}} & G_0 \\ t \downarrow\downarrow s & & t' \downarrow\downarrow s' \\ R_1 & \underset{\overline{s'}}{\overset{\overline{t'}}{\rightrightarrows}} & R_0 \end{array}$$

with composition (m_1, m_0) where

$$\begin{aligned} m_1 &: G_1 \times G_1 \longrightarrow G_1 \\ m_0 &: R_1 \times R_1 \longrightarrow R_1 \end{aligned}$$

So, $(G_0, G_1, R_0, R_1, s_0, s_1, t_0, t_1, m_0, m_1)$ is called an internal of cat-1 group structure.

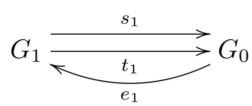
Proposition 3. The category **Cat(XMod)** of internal categories within the category of crossed modules is categorically equivalent to the category of crossed squares, [3].

Proposition 4. The category of internal categories within the category of cat-1 groups is natural equivalent to the category of internal categories within the category of crossed modules over groups, [6].

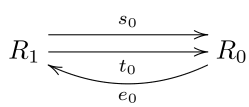
Theorem 3.2. *The category* ***Cat(Cat-1)*** *of internal categories within the category of Cat-1 groups (or categorical group) is categorically equivalent to the category of Cat-2 groups.*

Proof. Let (G_1, R_1, s, t) and (G_0, R_0, s', t') be cat-1 groups, and $(G_1, G_0, R_1, R_0, s_1, t_1, s_0, t_0, m_0, m_1)$ be an object in **Cat(Cat-1)**.

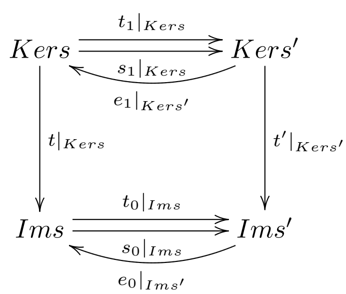
The compositions $m_1 : G_1 \times G_1 \to G_1$ and $m_0 : R_1 \times R_1 \to R_1$ are the homomorphisms of cat-1 groups.



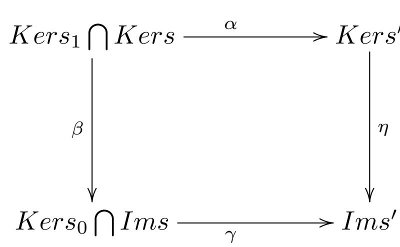
and



are the internal categories of the category of groups. By using the equivalence of the category of cat-1 groups and that of crossed modules, we get an internal of crossed modules as illustrated by the following diagram.



By considering the equivalence of the internal category of crossed modules and crossed squares, we obtain the crossed square with the following diagram



where $Kers_1 \bigcap Kers = Kers_1|_{Kers}, Kers_0 \bigcap Ims = Kers_0|_{Ims}, \alpha = t_1|_{Kers_1 \bigcap Kers}$, $\beta = t|_{Kers_1 \bigcap Kers}, \gamma = t_0|_{Kers_0 \bigcap Ims}, \eta = t'|_{Kers'}$ and h-map $h(g_0, r_0) = r_0 e_1(g_0)$ ${r_0}^{-1} e_1({g_0}^{-1})$ for $r_0 \in R_0, g_0 \in G_0$.

There exist groups $A \rtimes B$ and $C \rtimes D$ for $A = Kers_1 \bigcap Kers, B = Kers_0 \bigcap Ims$, $C = Kers'$ and $D = Ims'$. Also we get that $(A \rtimes B) \rtimes (C \rtimes D)$ is a semi-direct group with an action on $A \rtimes B$ of $C \rtimes D$

$$^{(c,d)}(a,b) = (^{c}(^{d}a)h(c,^{d}b),^{d}b)$$

and the group operation

$$(a,b,c,d) \triangleleft (a',b',c',d') = ((a,b) *^{(c,d)} (a',b'),(c,d) * (c',d'))$$

for $a \in A, b \in B, c \in C, d \in D$ where operation '$*$' is semi-direct product.

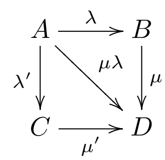
If we take $G = (A \rtimes B) \rtimes (C \rtimes D)$, $R_1 = C \rtimes D$ and $R_2 = B \rtimes D$, $(G, R_1, \overline{s}, \overline{t})$ come out a cat-1 group as illustrated below

$$A \rtimes B \rtimes C \rtimes D \overset{\overline{s}}{\underset{\overline{t}}{\rightrightarrows}} C \rtimes D$$

where $\overline{s}$ is the projection of R_1 and $\overline{t}(a,b,c,d) = (t_1|_A(a)^{t_0|_B(b)}c, t_0|_B(b)d)$.

We can change the role of B and C, that is we can use an action of $B \rtimes D$ on $A \rtimes C$ such that G is canonically isomorphic to $(A \rtimes C) \rtimes (B \rtimes D)$. Similarly, there is a cat-1 group $(G, R_2, \overline{\overline{s}}, \overline{\overline{t}})$ with $R_2 = B \rtimes D$. Finally, there exists a cat-2 group.

Conversely, let $(G, R_1, R_2, \overline{s}, \overline{t}, \overline{s'}, \overline{t'})$ be a cat-2 group. Define $A = Ker\overline{s} \bigcap Ker\overline{s'}$, $B = R_1 \bigcap Ker\overline{s'}$, $C = R_2 \bigcap Ker\overline{s}$, $D = R_1 \bigcap R_2$ and the morphisms $\lambda = \overline{t}|_A, \lambda' = \overline{t'}|_A, \mu = \overline{t'}|_B, \mu' = \overline{t}|_C$.



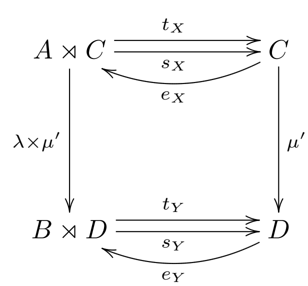
The above diagram is commutative with $\mu\lambda(a) = \overline{t}\overline{t'}|_A(a) = \overline{t}\overline{t'}(a) = \mu'\lambda'(a)$. All morphisms which are indicated in the diagram are crossed modules. Also defining h-map by $h(b,c) = [b,c]$ for $b \in B$, $c \in C$ and all the actions by conjugation, the axioms of crossed square are verified.

An internal category of crossed modules is obtained by having regard to the above crossed square.

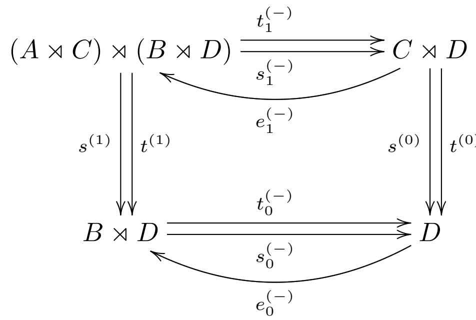
$(X_1, Y_1, \partial_1) = (A \rtimes C, B \rtimes D, \lambda \times \mu')$ is a crossed module with the action $^{(b,d)}(a,c) = ((^{b}(^{d}a))h(b,^{d}c),^{d}c)$. It is clear that $(X_0, Y_0, \partial_0) = (C, D, \mu')$ is a crossed module. By defining $s_X(a,c) = c, t_X(a,c) = \lambda(a)c, e_X(c) = (1,c), s_Y(b,d) = d, t_Y(b,d) = \mu'(b)d, e_Y(d) = (1,d)$ and the compositions

$$\begin{aligned}(a', \lambda'(a)c) \circ (a,c) &= (a'a, c)\\ (b', \mu(b)d) \circ (b,d) &= (b'b, d)\end{aligned}$$

we get the internal category of crossed modules as seen in the diagram:



Therefore, by considering the equivalence between the category of crossed module and that of cat-1 group, we get the internal category of cat-1 group as follows.



In the diagram vertical structures are the cat-1 groups with the following equations

$$\begin{aligned} s^{(1)}(a,c,b,d) &= (b,d) \\ t^{(1)}(a,c,b,d) &= (\lambda \times \mu')(a,c)(b,d) \\ s^{(0)}(c,d) &= d, t^{(0)}(c,d) = \mu'(c)d \end{aligned}$$

Since

$$\begin{aligned} s^{(0)}((c_1,d_1) * (c_0,d_0)) &= s^{(0)}(c_1^{d_1}c_0, d_1d_0) \\ &= d_1d_0 \\ &= s^{(0)}(c_1,d_1)s^{(0)}(c_0,d_0) \end{aligned}$$

$$\begin{aligned} t^{(0)}((c_1,d_1) * (c_0,d_0)) &= t^{(0)}(c_1^{d_1}c_0, d_1d_0) \\ &= \mu(c_1^{d_1}c_0)d_1d_0 \\ &= \mu(c_1)d_1\mu(c_0){d_1}^{-1}d_1d_0 \\ &= t^{(0)}(c_1,d_1)t^{(0)}(c_0,d_0) \end{aligned}$$

and

$$\begin{aligned} & t^{(1)}((a_1,c_1,b_1,d_1) \triangleleft (a_0,c_0,b_0,d_0)) = t^{(1)}((a_1,c_1) *^{(b_1,d_1)} (a_0,c_0), (b_1,d_1) * (b_0,d_0)) \\ & = t^{(1)}((a_1,c_1) * ({}^{b_1}({}^{d_1}a_0)h(b_1,{}^{d_1}c_0), {}^{d_1}c_0), (b_1^{d_1}b_0, d_1d_0)) \\ & = (\lambda \times \mu')(a_1^{c_1}({}^{b_1}({}^{d_1}a_0)h(b_1,{}^{d_1}c_0)), c_1^{d_1}c_0) * (b_1^{d_1}b_0, d_1d_0) \\ & = (\lambda(a_1^{c_1}({}^{b_1}({}^{d_1}a_0)h(b_1,{}^{d_1}c_0))), \mu'(c_1^{d_1}c_0)) * (b_1^{d_1}b_0, d_1d_0) \\ & = (\lambda(a_1)\lambda({}^{c_1}({}^{b_1}({}^{d_1}a_0)h(b_1,{}^{d_1}c_0))^{\mu'(c_1^{d_1}c_0)}(b_1^{d_1}b_0), \mu'(c_1^{d_1}c_0)d_1d_0) \\ & = (\lambda(a_1)^{\mu'(c_1)}\lambda({}^{b_1}({}^{d_1}a_0))^{\mu'(c_1)}\lambda(h(b_1,{}^{d_1}c_0))^{\mu'(c_1^{d_1}c_0)}(b_1({}^{d_1}b_0)), \mu'(c_1^{d_1}c_0)d_1d_0) \\ & = (\lambda(a_1))^{\mu'(c_1)}b_1^{\mu'(c_1)d_1}\lambda(a_0)^{\mu'(c_1)}({b_1}^{-1})^{\mu'(c_1)}\lambda h(b_1,{}^{d_1}c_0)^{\mu'(c_1^{d_1}c_0)}(b_1({}^{d_1}b_0)), \mu'(c_1^{d_1}c_0)d_1d_0) \\ & = (\lambda(a_1)^{\mu'(c_1)}b_1^{\mu'(c_1)d_1}\lambda(a_0)^{\mu'(c_1)}({b_1}^{-1})^{\mu'(c_1)}(b_1)^{\mu'({}^{d_1}c_0)}({b_1}^{-1})^{\mu'(c_1^{d_1}c_0)}(b_1({}^{d_1}c_0)), \mu'(c_1^{d_1}c_0)d_1d_0) \\ & = (\lambda(a_1)^{\mu'(c_1)}b_1^{\mu'(c_1)d_1}\lambda(a_0)^{\mu'(c_1)d_1\mu'(c_0)}b_0, \mu'(c_1)d_1\mu'(c_0)d_0) \\ & = t^{(1)}(a_1,c_1,b_1,d_1) * t^{(1)}(a_0,c_0,b_0,d_0) \end{aligned}$$

$s^{(0)}, t^{(0)}$ and $t^{(1)}$ are group homomorphisms. It is also seen that the rest of the axioms of cat-1 group are satisfied.

In the diagram horizontal structures are the internal of groups with following equations.

$$\begin{aligned} s_1^{(-)}(a,c,b,d) &= (c,d), t_1^{(-)}(a,c,b,d) = (\lambda'(a)c, \mu(b)d), e_1^{(-)}(c,d) = (1,c,1,d) \\ s_0^{(-)}(b,d) &= d, t_0^{(-)}(b,d) = \mu(b)d, e_0^{(-)}(d) = (1,d) \end{aligned}$$

Since

$$\begin{aligned} s_1^{(-)}((a_1,c_1,b_1,d_1) \lhd (a_0,c_0,b_0,d_0)) &= s_1^{(-)}((a_1,c_1) *^{(b_1,d_1)} (a_0,c_0), (b_1,d_1) * (b_0,d_0)) \\ &= (b_1^{d_1} b_0, d_1 d_0) \\ &= s_1^{(-)}((a_1,c_1,b_1,d_1)) * s_1^{(-)}(a_0,c_0,b_0,d_0), \end{aligned}$$

$$\begin{aligned} t_1^{(-)}((a_1,c_1,b_1,d_1) \lhd (a_0,c_0,b_0,d_0)) &= t_1^{(-)}((a_1,c_1) *^{(b_1,d_1)} (a_0,c_0), (b_1,d_1) * (b_0,d_0)) \\ &= t_1^{(-)}((a_1,c_1) * ({}^{b_1}({}^{d_1}a_0)h(b_1, {}^{d_1}c_0), {}^{d_1}c_0), (b_1^{d_1}b_0, d_1d_0)) \\ &= t_1^{(-)}(a_1^{c_1}({}^{b_1}({}^{d_1}a_0)h(b_1, {}^{d_1}c_0)), c_1^{d_1}c_0), b_1^{d_1}b_0, d_1d_0) \\ &= (\lambda'(a_1^{c_1}({}^{b_1}({}^{d_1}a_0)h(b_1, {}^{d_1}c_0)))c_1^{d_1}c_0, \mu(b_1^{d_1}b_0)d_1d_0) \\ &= (\lambda'(a_1)c_1\lambda'({}^{b_1}({}^{d_1}a_0)h(b_1, {}^{d_1}c_0))c_1{}^{-1}c_1^{d_1}c_0, \mu(b_1^{d_1}b_0)d_1d_0) \\ &= (\lambda'(a_1)c_1\lambda'({}^{b_1}({}^{d_1}a_0))({}^{\mu(b_1)}({}^{d_1}c_0)({}^{d_1}c_0{}^{-1}))^{d_1}c_0, \mu(b_1^{d_1}b_0)d_1d_0) \\ &= (\lambda'(a_1)c_1^{\mu(b_1)d_1}\lambda'(a_0)^{\mu(b_1)}({}^{d_1}c_0), \mu(b_1^{d_1}b_0)d_1d_0) \\ &= (\lambda'(a_1)c_1, \mu(b_1)d_1) * (\lambda'(a_0)c_0, \mu'(b_0)d_0) \\ &= t_1^{(-)}(a_1,c_1,b_1,d_1) * t_1^{(-)}(a_0,c_0,b_0,d_0), \end{aligned}$$

$$\begin{aligned} s_0^{(-)}((b_1,d_1) * (b_0,d_0)) &= s_0^{(-)}(b_1{}^{d_1}b_0, d_1d_0) \\ &= d_1d_0 \\ &= s_0^{(-)}(b_1,d_1)s_0^{(-)}(b_0,d_0) \end{aligned}$$

and

$$\begin{aligned} t_0^{(-)}((b_1,d_1) * (b_0,d_0)) &= t_0^{(-)}(b_1{}^{d_1}b_0, d_1d_0) \\ &= \mu'(b_1{}^{d_1}b_0)d_1d_0 \\ &= \mu'(b_1)d_1\mu'(b_0)d_1{}^{-1}d_1d_0 \\ &= t_0^{(-)}(b_1,d_1)t_0^{(-)}(b_0,d_0) \end{aligned}$$

$s_1^{(-)}, t_1^{(-)}, s_0^{(-)}, t_0^{(-)}$ are the group homomorphisms.

We define the composition $m' = (m_1', m_0')$ as $m_1' : X_1' \times X_1' \to X_1'$, $m_1'((a',c',b',d'),(a,c,b,d)) = (a'a, c, b'b, d)$ and $m_0' : X_0' \times X_0' \to X_0'$, $m_0'((b,d),(b',d')) = (b'b, d)$ where $X_1' = A \rtimes C \rtimes B \rtimes D$,$X_0' = B \rtimes D$,$c' = \lambda'(a)c, d' = \mu(b)d$.

The composition m must be a group homomorphism that is the equation

$$\begin{aligned} &m(((a', \lambda'(a)c, b', \mu(b)d),(a,c,b,d)).((\overline{a'}, \lambda'\overline{a}\overline{c}, \overline{b'}, \mu(\overline{b}\overline{d}),(\overline{a},\overline{c},\overline{b},\overline{d})))) = \\ &\quad m((a', \lambda'(a)c, b', \mu(b)d),(a,c,b,d)) \lhd m((\overline{a'}, \lambda'\overline{a}\overline{c}, \overline{b'}, \mu(\overline{b}\overline{d}),(\overline{a},\overline{c},\overline{b},\overline{d}))) \end{aligned}$$

must be verified.

$$m_1'(((a',\lambda'(a)c,b',\mu(b)d),(a,c,b,d))*((\overline{a}',\lambda'(\overline{a})\overline{c},\overline{b}',\mu(\overline{b})\overline{d}),(\overline{a},\overline{c},\overline{b},\overline{d})))$$

$$= m_1'((a',\lambda'(a)c,b',\mu(b)d)*(\overline{a}',\lambda'(\overline{a})\overline{c},\overline{b}',\mu(\overline{b})\overline{d}),(a,c,b,d)*(\overline{a},\overline{c},\overline{b},\overline{d}))$$

$$= m_1'((a',\lambda'(a)c)*^{(b',\mu(b)d)}(\overline{a}',\lambda'(\overline{a})\overline{c}),(b',\mu(b)d)*(\overline{b}',\mu(\overline{b})\overline{d}),(a,c)*^{(b,d)}(\overline{a},\overline{c}),(b,d)*(\overline{b},\overline{d}))$$

$$= m_1'((a',\lambda'(a)c)*({}^{b'}({}^{\mu(b)d}\overline{a}')h(b',{}^{\mu(b)d}\lambda'(\overline{a})\overline{c}),{}^{\mu(b)d}\lambda'(\overline{a})\overline{c}),b'^{(\mu(b)d)}\overline{b}',\mu(b)d\mu(\overline{b})\overline{d}),$$

$$(a,c)*({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c}),{}^{d}\overline{c}),b^{d}\overline{b},d\overline{d})$$

$$= m_1'(a'^{(\lambda'(a)c)}({}^{b'}({}^{(\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})),(\lambda'(a)c)\,({}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})),b'^{(\mu(b)d)}\overline{b}',\mu(b)d\mu(\overline{b})\overline{d}),$$

$$a^{c}({}^{b}({}^{d}\overline{a})h(b,d\cdot\overline{c})),c\left({}^{d}\overline{c}\right),b^{d}\overline{b},d\overline{d})$$

$$= (a'^{(\lambda'(a)c)}({}^{b'}({}^{(\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})a^{c}({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c})),c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= (a'^{(\lambda'(a)c)}({}^{b'}({}^{(\mu(b)d)}\overline{a}'))^{(\lambda'(a)c)}h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})a^{c}({}^{b}({}^{d}\overline{a})h(b,d\cdot\overline{c})),c\left({}^{d}\overline{c}\right),$$

$$\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}\left[{}^{b'}({}^{(\mu(b)d)}\overline{a}')\right]a^{-1}a\left[{}^{c}\left(h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})\right)\right]a^{-1}a^{c}({}^{b}({}^{d}\overline{a})h(b,d\cdot\overline{c})),c\left({}^{d}\overline{c}\right),$$

$$\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d}) = a'a^{c}\left[{}^{(\mu(b'))}({}^{(\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))\right],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}\left[({}^{(\mu(b')\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))\right],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}\left[({}^{(\mu(b')\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})^{(\mu(b)d)}\overline{c})({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))\right],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}\left[({}^{(\mu(b')\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})^{(\mu(b)d)}\overline{c})({}^{\mu(b)}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))\right],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}[({}^{(\mu(b')\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a}))^{{}^{(\mu(b)d)}\lambda'(\overline{a})}h(b',{}^{(\mu(b)d)}\overline{c})({}^{\mu(b)}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)$$

$$\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}({}^{(\mu(b')\mu(b)d)}\overline{a}')h(b',\lambda'({}^{(\mu(b)d)}\overline{a}))^{\lambda'({}^{(\mu(b)d)}\overline{a})}h(b',{}^{(\mu(b)d)}\overline{c})({}^{(\mu(b)d)}\overline{a})h(b,{}^{d}\overline{c}))],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)$$

$$\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}({}^{(\mu(b')\mu(b)d)}\overline{a}')({}^{b'}({}^{(\mu(b)d)}\overline{a}))({}^{(\mu(b)d)}\overline{a}^{-1})^{\lambda'({}^{(\mu(b)d)}\overline{a})}h(b',{}^{(\mu(b)d)}\overline{c})({}^{(\mu(b)d)}\overline{a})h(b,{}^{d}\overline{c}))],c\left({}^{d}\overline{c}\right),$$

$$\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}({}^{(\mu(b')\mu(b)d)}\overline{a}')({}^{b'}({}^{(\mu(b)d)}\overline{a}))h(b',{}^{(\mu(b)d)}\overline{c})h(b,{}^{d}\overline{c}))],c\left({}^{d}\overline{c}\right),\left(b'^{(\mu(b)d)}\overline{b}'\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}({}^{(\mu(b')\mu(b)d)}\overline{a}')({}^{\mu(b')}({}^{(\mu(b)d)}\overline{a}))h(b',{}^{(\mu(b)d)}\overline{c})h(b,{}^{d}\overline{c}))],c\left({}^{d}\overline{c}\right),\left(b'\left(b\left({}^{d}\overline{b}'\right)b^{-1}\right)\right)\left(b^{d}\overline{b}\right),d\overline{d})$$

$$= a'a^{c}({}^{(\mu(b'b)d)}\overline{a}')({}^{(\mu(b'b)d)}\overline{a})\left[h(b',{}^{(\mu(b)d)}\overline{c})\right]h(b,{}^{d}\overline{c}),c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}({}^{(\mu(b'b)d)})\overline{a}')^{(\mu(b'b)d)}\overline{a}\left[h({}^{\mu(b)}\left({}^{b^{-1}}b'\right),{}^{\mu(b)}\left({}^{d}\overline{c}\right))\right]h(b,{}^{d}\overline{c}),c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}({}^{(\mu(b'b)d)})\overline{a}')({}^{(\mu(b'b)d)}\overline{a})^{\mu(b)}\left[h(\left({}^{b^{-1}}b'\right),{}^{d}\overline{c})\right]h(b,{}^{d}\overline{c}),c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}({}^{(\mu(b'b)d)})\overline{a}')^{(\mu(b'b)d)}\overline{a})^{b}\left[h(\left({}^{b^{-1}}b'\right),{}^{d}\overline{c})\right]h(b,{}^{d}\overline{c}),c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}({}^{(\mu(b'b)d)}\overline{a}')^{(\mu(b'b)d)}\overline{a})h(b\left(b^{-1}b'b\right),{}^{d}\overline{c}),c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}\left[({}^{(\mu(b'b)d)}\overline{a}')^{(\mu(b'b)d)}\overline{a})h(b'b,{}^{d}\overline{c})\right],c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}\left[{}^{(\mu(b'b)d)}(\overline{a}'\overline{a}))h(b'b,{}^{d}\overline{c})\right],c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= a'a^{c}\left[{}^{(b'b)}({}^{d}(\overline{a}'\overline{a}))h(b'b,{}^{d}\overline{c})\right],c\left({}^{d}\overline{c}\right)),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$$= (a'a,c)*({}^{((b'b)}({}^{d}(\overline{a}'\overline{a}))h(b'b,{}^{d}\overline{c}),{}^{d}\overline{c}),\left(b'b^{d}\left(\overline{b}'\overline{b}\right),d\overline{d}\right)$$

$= \left((a'a,c) *^{(b'b,d)} (\overline{a}'\overline{a},\overline{c}), (b'b,d) * \left(\overline{b}'\overline{b},\overline{d}\right)\right)$
$= (a'a,c,b'b,d) * \left(\overline{a}'\overline{a},\overline{c},\overline{b}'\overline{b},\overline{d}\right)$
$= m_1'((a',\lambda'((a)c,b',\mu(b)d),(a,c,b,d)) * m_1'((\overline{a}',\lambda'(\overline{a})\overline{c},\overline{b}',\mu(\overline{b})\overline{d}),(\overline{a},\overline{c},\overline{b},\overline{d}))$

We note that $(a'^{(\lambda'(a)c)}({}^{b'}({}^{(\mu(b)d)}\overline{a}')h(b',{}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})),(\lambda'(a)c)\,({}^{(\mu(b)d)}\lambda'(\overline{a})\overline{c})),$ $b'^{(\mu(b)d)}\overline{b}',\mu(b)d\mu(\overline{b})\overline{d})$ and $(a^{c}({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c})),c\left({}^{d}\overline{c}\right),b^{d}\overline{b},d\overline{d})$ are composable since

$$\begin{aligned}
\lambda'(a^{cb}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))(c^{d}\overline{c}) &= \lambda'(a)c\lambda'({}^{b}({}^{d}\overline{a})h(b,{}^{d}\overline{c}))c^{-1}(c^{d}\overline{c})\\
&= \lambda'(a)c\lambda'({}^{b}({}^{d}\overline{a}))\lambda'h(b,{}^{d}\overline{c})^{d}\overline{c}\\
&= \lambda'(a)c^{\mu(b)d}\lambda'(\overline{a})^{\mu(b)d}\overline{c}({}^{d}\overline{c}^{-1})^{d}\overline{c}\\
&= \lambda'(a)c^{\mu(b)d}\lambda'(\overline{a})^{\mu(b)d}\overline{c}
\end{aligned}$$

and

$$\begin{aligned}
\mu(b^{d}\overline{b})d\overline{d} &= \mu(b)\mu({}^{d}\overline{b})d\overline{d}\\
&= \mu(b)d\mu(\overline{b})d^{-1}d\overline{d}\\
&= \mu(b)d\mu(\overline{b})\overline{d}.
\end{aligned}$$

It is clear that m_0' is a group homomorphism. Also the other required conditions of the internal category of cat-1 group are satisfied.

Thus we have determined the internal category of cat-1 group. □

4. Conclusion

In the work of Loday, [2], it is seen that the category of crossed modules is equivalent to that of cat-1 groups. On the other hand, in our previous paper, [6], we showed that this equivalence is also preserved in their inner categories. Also, according to Şahan and Mohammed, [3], [4], it is known that there is a natural equivalence between the category of crossed squares and the category of internal categories within the category of crossed modules. When we regard all these results with the equivalence of the category of cat-2 groups and that of crossed squares which takes place again in [2], we conclude that the category of internal categories in the category of cat-1 groups and the category of cat-2 groups are equivalent.

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