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Research Article

Rational generalized Stieltjes functions

IVAN KOVALYOV*

ABSTRACT. The rational meromorphic functions on $\mathbb{C}\backslash\mathbb{R}$ are studied. We consider the some classes of one, as the generalized Nevanlinna $\mathbf{N}_\kappa$ and generalized Stieltjes $\mathbf{N}_\kappa^k$ classes. By Euclidean algorithm, we can find indices κ and k, i.e. determine which class the function belongs to $\mathbf{N}_\kappa^k$.

Keywords: Rational function, generalized Nevanlinna function, generalized Stieltjes function.

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1. INTRODUCTION

Recall a generalized Nevalinna class $\mathbf{N}_\kappa$ and a generalized Stieltjes class $\mathbf{N}_\kappa^k$.

Definition 1.1. *A function f meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if for every set $z_i \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($j = 1, \dots, n$) the form*

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \xi_i \overline{\xi}_j$$

has at most κ and for some choice of z_i ($i = 1, \dots, n$) it has exactly κ negative squares. For $f \in \mathbf{N}_\kappa$, let us write $\kappa_-(f) = \kappa$. In particular, if $\kappa = 0$ then the class $\mathbf{N}_0$ coincides with the class $\mathbf{N}$ of Nevanlinna functions. A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_\kappa^+$ (see [8, 9]) if $zf \in \mathbf{N}_k$ and to the class $\mathbf{N}_\kappa^k$ ($k \in \mathbb{N}$) if $zf \in \mathbf{N}_\kappa^k$ (see [3], [4]). In particular, if $k = 0$, then $\mathbf{N}_\kappa^0 := \mathbf{N}_\kappa^+$. The function $f \in \mathbf{N}_\kappa^{-k}$, if $f \in \mathbf{N}_\kappa$ and $\frac{1}{z} f \in \mathbf{N}_k$ (see [5]).

Recall some properties of the generalized Nevanlinna functions and generalized Stieltjes functions.

Proposition 1.1. ([8]) *Let $f \in \mathbf{N}_\kappa$, $f_1 \in \mathbf{N}_{\kappa_1}$, $f_2 \in \mathbf{N}_{\kappa_2}$. Then*

(1) $-f^{-1} \in \mathbf{N}_\kappa$.
(2) $f_1 + f_2 \in \mathbf{N}_{\kappa'}$, where $\kappa' \le \kappa_1 + \kappa_2$.
(3) If, in addition, $f_1(iy) = o(y)$ as $y \to \infty$ and f_2 is a polynomial, then

$$f_1 + f_2 \in \mathbf{N}_{\kappa_1 + \kappa_2}. \tag{1.1}$$

(4) Every real polynomial $P(t) = p_\nu t^\nu + p_{\nu-1} t^{\nu-1} + \ldots + p_1 t + p_0$ of degree ν belongs to a class $\mathbf{N}_\kappa$, where the index $\kappa = \kappa_-(P)$ can be evaluated by (see [8, Lemma 3.5])

$$\kappa_-(P) = \begin{cases} \left[\frac{\nu+1}{2}\right], & \text{if } p_\nu < 0; \text{ and } \nu \text{ is odd}; \\ \left[\frac{\nu}{2}\right], & \text{otherwise}. \end{cases} \tag{1.2}$$

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*Corresponding author: Ivan Kovalyov; i.m.kovalyov@gmail.com
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Proposition 1.2. *([2]) Let $f \in \mathbf{N}_\kappa^k$. Then the following equivalences hold:*

(1) $f \in \mathbf{N}_\kappa^k \iff -\frac{1}{f} \in \mathbf{N}_\kappa^{-k}$;
(2) $f \in \mathbf{N}_\kappa^k \iff zf(z) \in \mathbf{N}_k^{-\kappa}$.

Lemma 1.1 ([7, Lemma 3.2])**.** *Let $P(z)$ be a polynomial of the degree ν and let $\alpha \in \mathbb{R}$. Then:*

(1) if $zP(z) \in \mathbf{N}_\kappa$, then

$$(z-\alpha)P(z) \in \mathbf{N}_\kappa; \tag{1.3}$$

(2) if $P(z) \in \mathbf{N}_\kappa$, then

$$\frac{(z-\alpha)}{z}P(z) \in \mathbf{N}_{\kappa'}, \quad \text{where} \quad \kappa' = \kappa + \kappa_-\left(-\frac{\alpha P(0)}{z}\right); \tag{1.4}$$

(3) if $((z-\alpha)P(z) - g(z)) \in \mathbf{N}_\kappa^k$, then

$$(-\alpha P(0) - g(z)) \in \mathbf{N}_{\kappa-\kappa_1}^{(k-k_1)} \quad \text{and} \quad (\alpha P(0) + g(z))^{-1} \in \mathbf{N}_{\kappa-\kappa_1}^{-(k-k_1)}, \tag{1.5}$$

where $\kappa_1 = \kappa_-(zP(z))$ and $k_1 = \kappa_-(P(z))$.

The indefinite Hamburger moment in the generalized Nevanlinna class $\mathbf{N}_\kappa$ was studied in [10]. The indefinite Stieltjes moment problem in the generalized Stieltjes class $\mathbf{N}_\kappa^k$ was studied in [11], [1], [2], [6] and [7]. One is based on the Schur algorithm, i.e. the description of the solutions are found in terms of the continued fractions. In the present paper, the rational generalized Stieltjes functions are investigated. The goal is to determine class $\mathbf{N}_\kappa^k$, such that the some rational generalized Stieltjes function f belongs to one (i.e. find the indices κ and k).

2. Finding the index

2.1. **Euclidean algorithm.** Let us recall an Euclidean algorithm. Let P_0 and Q_0 be the polynomials, such that $\deg(P_0) = n_0$ and $\deg(Q_0) = m_0$, where $n_0, m_0 \in \mathbb{Z}_+$ and let $m_0 \le n_0$. By Euclidean algorithm, we obtain

$$\begin{aligned}
P_0(z) &= Q_0(z)a_0(z) + r_1(z),\\
Q_0(z) &= r_1(z)a_1(z) + r_2(z),\\
r_1(z) &= r_2(z)a_2(z) + r_3(z),\\
&\vdots\\
r_{n-2}(z) &= r_{n-1}(z)a_{n-1}(z) + r_n(z),\\
r_{n-1}(z) &= r_n(z)a_n(z),
\end{aligned} \tag{2.6}$$

where r_i are polynomials. Consequently, the ratio $\dfrac{P_0(z)}{Q_0(z)}$ can be represented as a continued fraction

$$\frac{P_0(z)}{Q_0(z)} = a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cdots + \cfrac{1}{a_n(z)}}}. \tag{2.7}$$

2.2. Rational generalized Nevanlinna function and its index κ.

Theorem 2.1. *Let P_0 and Q_0 be the polynomials, such that* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $m_0 < n_0$. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$. *Then f belongs to the class* $\mathbf{N}_\kappa$ *and the index κ is calculated by*

$$\kappa = \sum_{j=0}^{n} \kappa_-((-1)^{j+1}a_j(z)). \tag{2.8}$$

Proof. Assume, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ is meromorphic function on $\mathbb{C}\backslash\mathbb{R}$, where the P_0 and Q_0 are the polynomials of the power $\deg(P_0) = n_0$ and $\deg(Q_0) = m_0$, respectively. By Definition 1.1, $f \in \mathbf{N}_\kappa$.

Calculating index κ. Due to (2.7), we can rewrite f as follows

$$f(z) = \frac{1}{\frac{P_0(z)}{Q_0(z)}} = -\cfrac{1}{-a_0(z) - \cfrac{1}{a_1(z) - \cfrac{1}{-a_2(z) - \cdots - \cfrac{1}{(-1)^{n+1}a_n(z)}}}}. \tag{2.9}$$

By Proposition 1.1 (see (1.2))

$$\kappa_j = \kappa_-((-1)^{j+1}a_j(z)), \quad j = \overline{0, n},$$

i.e. $(-1)^{j+1}a_j(z) \in \mathbf{N}_{\kappa_j}$. Moreover, by Proposition 1.1 (see items (1) and (3)), we obtain

$$\begin{aligned} -\frac{1}{(-1)^{j+1}a_j(z)} &\in \mathbf{N}_{\kappa_j} \quad \text{for all } j = \overline{0, n}, \\ (-1)^n a_{n-1}(z) - \frac{1}{(-1)^{n+1}a_n(z)} &\in \mathbf{N}_{\kappa_n + \kappa_{n-1}}. \end{aligned} \tag{2.10}$$

Let us construct a recursive sequence as

$$\begin{aligned} f_n(z) :=& (-1)^n a_{n-1}(z) - \frac{1}{(-1)^{n+1}a_n(z)}, \\ f_{n-1}(z) :=& (-1)^{n-1} a_{n-2}(z) - \frac{1}{f_n(z)}, \\ &\vdots \\ f_{n-2}(z) :=& (-1)^{n-2} a_{n-3}(z) - \frac{1}{f_{n-1}(z)}, \\ f_1(z) :=& -a_0(z) - \frac{1}{f_2(z)}. \end{aligned} \tag{2.11}$$

Hence (see Proposition 1.1)

$$f_n \in \mathbf{N}_{\kappa_n + \kappa_{n-1}}, \; f_{n-1} \in \mathbf{N}_{\kappa_n + \kappa_{n-1} + \kappa_{n-2}}, \; \ldots, \; f_1 \in \mathbf{N}_{\kappa_n + \kappa_{n-1} + \ldots + \kappa_0}. \tag{2.12}$$

By the recursive sequence, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ can be rewritten as

$$f(z) = -\frac{1}{f_1(z)}. \tag{2.13}$$

Therefore $f \in \mathbf{N}_\kappa$, where the index $\kappa = \sum_{j=0}^{n} \kappa_-((-1)^{j+1}a_j(z))$. This completes the proof. $\square$

Corollary 2.1. *Let P_0 and Q_0 be the polynomials, such that* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $m_0 \leq n_0$. *Let* $f(z) = \dfrac{P_0(z)}{Q_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$. *Then f belongs to the class $\mathbf{N}_\kappa$ and the index κ is calculated by*

$$\kappa = \sum_{j=0}^{n} \kappa_-((-1)^j a_j(z)). \tag{2.14}$$

Proof. Let the rational function $f(z) = \dfrac{P_0(z)}{Q_0(z)}$ is meromorphic function on $\mathbb{C}\backslash\mathbb{R}$, where the numerator P_0 and denominator Q_0 are the polynomials of the power $\deg(P_0) = n_0$ and $\deg(Q_0) = m_0$, respectively. Hence, f belongs to the generalized Nevanlinna class $\mathbf{N}_\kappa$ (see Definition 1.1).

Let us find the index κ. By the representation (2.7), we obtain

$$f(z) = \frac{P_0(z)}{Q_0(z)} = a_0(z) - \cfrac{1}{-a_1(z) - \cfrac{1}{a_2(z) - \cdots - \cfrac{1}{(-1)^n a_n(z)}}}. \tag{2.15}$$

By Theorem 2.1 (see (2.10)-(2.13)), $f \in \mathbf{N}_\kappa$ and the index κ is calculated by (2.14). This completes the proof. $\square$

3. Rational generalized Stieltjes function and its indices κ, k

First of all, we study the simple case of the rational functions, which belong to the generalized Stieltjes classes $\mathbf{N}_\kappa^{\pm k}$ and find the formulas for the indices κ and k.

3.1. Rational function of the generalized Stieltjes class $\mathbf{N}_\kappa^k$.

Theorem 3.2. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $m_0 < n_0$. *Let f admit the representation* (2.9) *and let $a_{2i}(z)$ vanish at zero for all* $i = \overline{0, [n/2]}$ *(i.e.* $a_{2i}(0) = 0$*). Then f belongs to the class $\mathbf{N}_\kappa^k$, where the index κ is calculated by* (2.8) *and index k is found by*

$$k = \begin{cases} \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-\frac{a_{2j}(z)}{z}\right) + \sum\limits_{j=0}^{[n/2]-1} \kappa_-(za_{2j+1}(z)), & \text{if } n \text{ is even;} \\ \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-\frac{a_{2j}(z)}{z}\right) + \sum\limits_{j=0}^{[n/2]} \kappa_-(za_{2j+1}(z)), & \text{if } n \text{ is odd.} \end{cases} \tag{3.16}$$

Proof. By Definition 1.1, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ meromorphic on $\mathbb{C}\backslash\mathbb{R}$ belongs to the generalized Stiektjes class $\mathbf{N}_\kappa^k$ (i.e. $f \in \mathbf{N}_\kappa$ and $zf \in \mathbf{N}_k$) and by Theorem 2.1, the index κ is calculated by (2.8).

Let us find an index k. Assume f admits the representation (2.9) and $a_{2i}(0) = 0$ for all $i = \overline{0, [n/2]}$. Hence, we get the two cases, where n is even or odd.

First of all we consider the even case (i.e. $n = 2m$, $m \in \mathbb{Z}_+$), we obtain

$$zf(z) = \frac{Q_0(z)}{P_0(z)} = \frac{1}{\dfrac{P_0(z)}{zQ_0(z)}} \tag{3.17}$$
$$= -\frac{1|}{\left|-\dfrac{a_0(z)}{z}\right.} - \frac{1|}{|za_1(z)} - \dots - \frac{1|}{|za_{2m-1}(z)} - \frac{1|}{\left|-\dfrac{a_{2m}(z)}{z}\right.}.$$

The terms $-\dfrac{a_{2i}(z)}{z}$ are polynomials, i.e. $a_{2i}(0) = 0$ for all $i = \overline{0, [n/2]}$. By Theorem 2.1 , we get

$$k = \sum_{j=0}^{[n/2]} \kappa_-\left(-\frac{a_{2j}(z)}{z}\right) + \sum_{j=0}^{[n/2]-1} \kappa_-(za_{2j+1}(z)).$$

The next step, let n is odd (i.e. $n = 2m+1$, $m \in \mathbb{Z}_+$). Consequently

$$zf(z) = -\frac{1|}{\left|-\dfrac{a_0(z)}{z}\right.} - \frac{1|}{|za_1(z)} - \dots - \frac{1|}{\left|-\dfrac{a_{2m}(z)}{z}\right.} - \frac{1|}{|za_{2m+1}(z)}. \tag{3.18}$$

Similarly, $-\dfrac{a_{2i}(z)}{z}$ are the polynomials and the index k is

$$\sum_{j=0}^{[n/2]} \kappa_-\left(-\frac{a_{2j}(z)}{z}\right) + \sum_{j=0}^{[n/2]} \kappa_-(za_{2j+1}(z)).$$

This completes the proof. ☐

Corollary 3.2. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $n_0 \le m_0$. *Then* f *belongs to the class* $\mathbf{N}_\kappa^k$ *and admits the representation* (2.15).

Moreover, the index κ *is calculated by* (2.14). *In addition, if the all polynomials* $a_{2i+1}(z)$ *vanish at zero in the representation* (2.15), *then the index* k *is found by*

$$k = \begin{cases} \sum\limits_{j=0}^{[n/2]} \kappa_-\left(za_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]-1} \kappa_-(-\frac{a_{2j+1}(z)}{z}), & \text{if } n \text{ is even;} \\ \sum\limits_{j=0}^{[n/2]} \kappa_-\left(za_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]} \kappa_-(-\frac{a_{2j+1}(z)}{z}), & \text{if } n \text{ is odd.} \end{cases} \tag{3.19}$$

Proof. By Definition 1.1, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ belongs to the generalized Stieltjes class $\mathbf{N}_\kappa^k$ and by Corollary 2.1, f admits the representation (2.15) and the index κ is calculated by (2.14). By Theorem 3.2, the index k can be found by (3.19). This completes the proof. ☐

Corollary 3.3. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $m_0 + 1 < n_0$. *Then the rational function* $zf(z)$ *admits the following representation*

$$zf(z) = -\frac{1|}{|-\widetilde{a}_0(z)} - \frac{1|}{|\widetilde{a}_1(z)} - \dots - \frac{1|}{|(-1)^{n+1}\widetilde{a}_n(z)} \tag{3.20}$$

and f *belongs to the class* $\mathbf{N}_\kappa^k$.

Furthermore, in addition, if $\widetilde{a}_{2i+1}$ *vanish at zero for all* $i = \overline{0, [n/2]}$*, then the indices* κ *and* k *can be found by*

$$k = \sum_{j=0}^{n} \kappa_-((-1)^{j+1}\widetilde{a}_j(z)), \tag{3.21}$$

$$\kappa = \begin{cases} \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-z\widetilde{a}_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]-1} \kappa_-\left(\frac{\widetilde{a}_{2j+1}(z)}{z}\right), & \text{if } n \text{ is even;} \\ \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-z\widetilde{a}_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]} \kappa_-\left(\frac{\widetilde{a}_{2j+1}(z)}{z}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.22}$$

Proof. By Euclidean algorithm, the rational function $zf(z) = \frac{zQ_0(z)}{P_0(z)}$ admits the representation (3.20). By Theorem 3.2, the rational function f belongs to the generalized Stieltjes class $\mathbf{N}_\kappa^k$, where the indices k and κ are found by (3.21) and (3.22), respectively. This completes the proof. □

Corollary 3.4. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$*, where* $\deg(P_0) = n_0$*,* $\deg(Q_0) = m_0$ *and* $n_0 \le m_0 + 1$*. Then the rational function* $zf(z)$ *admits the following representation*

$$zf(z) = \hat{a}_0(z) - \frac{1|}{|-\hat{a}_1(z)} - \frac{1|}{|\hat{a}_2(z)} - \cdots - \frac{1|}{|(-1)^n\hat{a}_n(z)} \tag{3.23}$$

and f *belongs to the class* $\mathbf{N}_\kappa^k$.

Furthermore, if $\hat{a}_{2i}$ *vanish at zero for all* $i = \overline{1, [n/2]}$*, then the indices* κ *and* k *can be found by*

$$k = \sum_{j=0}^{n} \kappa_-((-1)^j\hat{a}_j(z)), \tag{3.24}$$

$$\kappa = \begin{cases} \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-z\hat{a}_{2j+1}(z)\right) + \sum\limits_{j=0}^{[n/2]-1} \kappa_-(\frac{\hat{a}_{2j}(z)}{z}), & \text{if } n \text{ is even;} \\ \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-z\hat{a}_{2j+1}(z)\right) + \sum\limits_{j=0}^{[n/2]} \kappa_-(\frac{\hat{a}_{2j}(z)}{z}), & \text{if } n \text{ is odd.} \end{cases} \tag{3.25}$$

3.2. Rational function of the generalized Stieltjes class $\mathbf{N}_\kappa^{-k}$.

Theorem 3.3. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$*, where* $\deg(P_0) = n_0$*,* $\deg(Q_0) = m_0$ *and* $m_0 \le n_0$*. Let* f *admits the representation* (2.9) *and let the all odd polynomials* $a_{2i+1}(z)$ *vanish at zero (i.e.* $a_{2i+1}(0) = 0$*). Then* f *belongs to the class* $\mathbf{N}_\kappa^{-k}$*, where the index* κ *is calculated by* (2.8) *and index* k *is found by*

$$k = \begin{cases} \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-za_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]-1} \kappa_-\left(\frac{a_{2j+1}(z)}{z}\right), & \text{if } n \text{ is even;} \\ \sum\limits_{j=0}^{[n/2]} \kappa_-\left(-za_{2j}(z)\right) + \sum\limits_{j=0}^{[n/2]} \kappa_-\left(\frac{a_{2j+1}(z)}{z}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.26}$$

Proof. By Definition 1.1, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ meromorphic on $\mathbb{C}\backslash\mathbb{R}$ belongs to the generalized Stieltjes class $\mathbf{N}_\kappa^{-k}$ (i.e. $f \in \mathbf{N}_\kappa$ and $\frac{f}{z} \in \mathbf{N}_k$) and by Theorem 2.1, the index κ is calculated by (2.8).

Suppose f admits representation (2.9) and the all odd polynomials $a_{2i}(0)$ vanish at zero (i.e. $a_{2i+1}(0)=0$).

If n is even (i.e. $n=2m$, $m\in\mathbb{Z}_+$), then

$$\begin{aligned}\frac{f(z)}{z}&=\frac{Q_0(z)}{zP_0(z)}\\&=\frac{1}{\frac{zP_0(z)}{Q_0(z)}}\\&=-\frac{1|}{|-za_0(z)}-\frac{1|}{\left|\frac{a_1(z)}{z}\right.}-\frac{1|}{|-za_2(z)}-\dots-\frac{1|}{\left|\frac{a_{2m-1}(z)}{z}\right.}-\frac{1|}{|-za_{2m}(z)}.\end{aligned}\tag{3.27}$$

Due to the all odd polynomials $a_{2i+1}(0)=0$, $\frac{a_{2j+1}}{z}$ are polynomials and by Theorem 2.1, we obtain

$$k=\sum_{j=0}^{[n/2]}\kappa_-\left(-za_{2j}(z)\right)+\sum_{j=0}^{[n/2]-1}\kappa_-\left(\frac{a_{2j+1}(z)}{z}\right).$$

If n is odd (i.e. $n=2m+1$, $m\in\mathbb{Z}_+$), then

$$\frac{f(z)}{z}=-\frac{1|}{|-za_0(z)}-\frac{1|}{\left|\frac{a_1(z)}{z}\right.}-\dots-\frac{1|}{\left|\frac{a_{2m-1}(z)}{z}\right.}-\frac{1|}{|-za_{2m}(z)}-\frac{1|}{\left|\frac{a_{2m+1}(z)}{z}\right.}.$$

Obviously, $a_{2i+1}(0)=0$, $\frac{a_{2j+1}}{z}$ are polynomials and by Theorem 2.1, we find index k as follow

$$k=\sum_{j=0}^{[n/2]}\kappa_-\left(-za_{2j}(z)\right)+\sum_{j=0}^{[n/2]}\kappa_-\left(\frac{a_{2j+1}(z)}{z}\right).$$

This completes the proof. □

Corollary 3.5. *Let the rational function* $f(z)=\dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0)=n_0$, $\deg(Q_0)=m_0$ *and* $n_0\le m_0$. *Then* f *belongs to the class* $\mathbf{N}_\kappa^{-k}$ *and admits the representation* (2.15).

Moreover, the index κ *is calculated by* (2.14). *In addition, if the all polynomials* $a_{2i}(z)$ *vanish at zero in the representation* (2.15), *then the index* k *is culculated by*

$$k=\begin{cases}\sum\limits_{j=0}^{[n/2]-1}\kappa_-\left(-za_{2j+1}(z)\right)+\sum\limits_{j=0}^{[n/2]}\kappa_-(\frac{a_{2j}(z)}{z}), & \text{if } n \text{ is even};\\ \sum\limits_{j=0}^{[n/2]}\kappa_-\left(-za_{2j+1}(z)\right)+\sum\limits_{j=0}^{[n/2]}\kappa_-(\frac{a_{2j}(z)}{z}), & \text{if } n \text{ is odd}.\end{cases}\tag{3.28}$$

Proof. By Definition 1.1, the rational function $f(z)=\dfrac{Q_0(z)}{P_0(z)}$ belongs to $\mathbf{N}_\kappa^{-k}$ and by Corollary 2.1, f admits representation (2.15) and the index κ can be calculated by (2.14). By Theorem 3.3, the index k can be found by (3.19). This completes the proof. □

Corollary 3.6. *Let the rational function* $f(z)=\dfrac{Q_0(z)}{P_0(z)}$ *be meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0)=n_0$, $\deg(Q_0)=m_0$ *and* $m_0<n_0+1$. *Then the rational function* $zf(z)$ *admits the following representation*

$$\frac{f(z)}{z}=\frac{Q_0(z)}{zP_0(z)}=-\frac{1|}{|-\widetilde{a}_0(z)}-\frac{1|}{|\widetilde{a}_1(z)}-\dots-\frac{1|}{|(-1)^{n+1}\widetilde{a}_n(z)}\tag{3.29}$$

and f belongs to the class $\mathbf{N}_\kappa^{-k}$.

Furthermore, if $\widetilde{a}_{2i}$ vanish at zero for all $i=\overline{0,[n/2]}$, then the indices κ and k can be found by

$$k=\sum_{j=0}^{n}\kappa_-((-1)^{j+1}\widetilde{a}_j(z)), \tag{3.30}$$

$$\kappa=\begin{cases}\sum\limits_{j=0}^{[n/2]}\kappa_-\left(-\frac{\widetilde{a}_{2j}(z)}{z}\right)+\sum\limits_{j=0}^{[n/2]-1}\kappa_-(z\widetilde{a}_{2j+1}(z)), & \text{if } n \text{ is even;}\\ \sum\limits_{j=0}^{[n/2]}\kappa_-\left(-\frac{\widetilde{a}_{2j}(z)}{z}\right)+\sum\limits_{j=0}^{[n/2]}\kappa_-(z\widetilde{a}_{2j+1}(z)), & \text{if } n \text{ is odd.}\end{cases} \tag{3.31}$$

Proof. By Euclidean algorithm, the rational function $\frac{f(z)}{z}=\frac{Q_0(z)}{zP_0(z)}$ admits the representation (3.29). By Theorem 3.3, the rational function f belongs to the generalized Stieltjes class $\mathbf{N}_\kappa^{-k}$, the indices k and κ are found by (3.30) and (3.31), respectively. This completes the proof. □

Corollary 3.7. *Let the rational function $f(z)=\dfrac{Q_0(z)}{P_0(z)}$ be the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, where $\deg(P_0)=n_0$, $\deg(Q_0)=m_0$ and $n_0+1\le m_0$. Then the rational function $zf(z)$ admits the following representation*

$$\frac{f(z)}{z}=\hat{a}_0(z)-\frac{1|}{|-\hat{a}_1(z)}-\frac{1|}{|\hat{a}_2(z)}-\cdots-\frac{1|}{|(-1)^n\hat{a}_n(z)} \tag{3.32}$$

and f belongs to the class $\mathbf{N}_\kappa^{-k}$.

Furthermore, if $\hat{a}_{2i+1}$ vanish at zero, then the indices κ and k can be found by

$$k=\sum_{j=0}^{n}\kappa_-((-1)^j\hat{a}_j(z)), \tag{3.33}$$

$$\kappa=\begin{cases}\sum\limits_{j=0}^{[n/2]}\kappa_-\left(-\frac{\hat{a}_{2j+1}(z)}{z}\right)+\sum\limits_{j=0}^{[n/2]-1}\kappa_-(z\hat{a}_{2j}(z)), & \text{if } n \text{ is even;}\\ \sum\limits_{j=0}^{[n/2]}\kappa_-\left(-\frac{\hat{a}_{2j+1}(z)}{z}\right)+\sum\limits_{j=0}^{[n/2]}\kappa_-(z\hat{a}_{2j}(z)), & \text{if } n \text{ is odd.}\end{cases} \tag{3.34}$$

4. General cases

4.1. General case in the class $\mathbf{N}_\kappa^k$.

Proposition 4.3. *Let the rational function $f(z)=\dfrac{Q_0(z)}{P_0(z)}$ be the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, where $\deg(P_0)=n_0$, $\deg(Q_0)=m_0$ and $m_0<n_0$, let f admits representation (2.9). Then f belongs to the class $\mathbf{N}_\kappa^k$, such that*

$$\kappa=\sum_{j=0}^{n}\kappa_j \text{ and } k\le\sum_{i=0}^{n}k_i+\sum_{i=0}^{[n/2]}k_i^0, \tag{4.35}$$

where the indices κ_i, k_i and k_i^0 can be found by

(4.36)
$$\begin{aligned} &\kappa_i = \kappa_-((-1)^{i+1}a_i(z)), \quad k_{2i} = \kappa_-\left(-\frac{a_{2i}(z)-a_{2i}(0)}{z}\right), \\ &k_{2i+1} = \kappa_-(za_{2i+1}(z)), \quad k_i^0 = \begin{cases} 1, & \text{if } a_{2i}(0) < 0; \\ 0, & \text{if } a_{2i}(0) > 0. \end{cases} \end{aligned}$$

Proof. (i) The first case. Let $n = 2m+1$, $m \in \mathbb{Z}_+$, then the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ can be rewritten by formula (2.9) as follows

(4.37)
$$f(z) = -\frac{1|}{\left|-a_0(z) - \dfrac{1}{a_1(z)}\right.} - \frac{1|}{\left|-a_2(z) - \dfrac{1}{a_3(z)}\right.} - \cdots - \frac{1|}{\left|-a_{2m}(z) - \dfrac{1}{a_{2m+1}(z)}\right.}.$$

Setting

$$f_m(z) := -\frac{1}{-a_{2m}(z) - \dfrac{1}{a_{2m+1}(z)}},$$

then zf_m takes the following form

$$\begin{aligned} zf_m(z) &= -\frac{z}{-a_{2m}(z) - \dfrac{1}{a_{2m+1}(z)}} \\ &= -\frac{1}{-\dfrac{a_{2m}(z)-a_{2m}(0)}{z} - \dfrac{a_{2m}(0)}{z} - \dfrac{1}{za_{2m+1}(z)}}. \end{aligned}$$

By Proposition 1.1 and Proposition 1.2, $f_m \in \mathbf{N}_{\widetilde{\kappa}_m}^{\widetilde{k}_m}$, where

$$\widetilde{\kappa}_m = \kappa_-(-a_{2m}) + \kappa_-(a_{2m+1}) \text{ and } \widetilde{k}_m \le k_{2m} + k_{2m+1} + k_m^0,$$

where

(4.38)
$$\begin{aligned} &k_{2m} := \kappa_-\left(-\frac{a_{2m}(z)-a_{2m}(0)}{z}\right), \; k_{2m+1} = \kappa_-(za_{2m+1}), \\ &k_m^0 := \kappa_-\left(-\frac{a_{2m}(0)}{z}\right) = \begin{cases} 1, & \text{if } a_{2m}(0) < 0; \\ 0, & \text{if } a_{2m}(0) > 0. \end{cases} \end{aligned}$$

The next step. Let us define the function f_{m-1} by

$$f_{m-1}(z) = -\frac{1}{-a_{2m-2}(z) - \dfrac{1}{a_{2m-1}(z) + f_m(z)}}.$$

Consequently, zf_{m-1} takes the following form

$$zf_{m-1}(z) = -\frac{1}{-\dfrac{a_{2m-2}(z)-a_{2m-2}(0)}{z} - \dfrac{a_{2m-2}(0)}{z} - \dfrac{1}{za_{2m-11}(z) + zf_m(z)}}.$$

Hence $f_{m-1} \in \mathbf{N}_{\widetilde{\kappa}_{m-1}}^{\widetilde{k}_{m-1}}$ (see Propositions 1.1 and 1.2), where the indices $\widetilde{\kappa}_{m-1}$ and $\widetilde{k}_{m-1}$ are

$$\widetilde{\kappa}_{m-1} = \widetilde{\kappa}_m + \kappa_-(-a_{2m-2}) + \kappa_-(a_{2m-1}) \text{ and } \widetilde{k}_{m-1} \le k_{2m-2} + k_{2m-1} + k_{m-1}^0 + \widetilde{k}_m,$$

where

$$
\begin{aligned}
&k_{2m} := \kappa_-\left(-\frac{a_{2m}(z)-a_{2m}(0)}{z}\right),\ k_{2m+1} = \kappa_-(za_{2m+1}),\\
&k_m^0 := \kappa_-\left(-\frac{a_{2m}(0)}{z}\right) = \begin{cases} 1, & \text{if } a_{2m}(0) < 0;\\ 0, & \text{if } a_{2m}(0) > 0.\end{cases}
\end{aligned}
\tag{4.39}
$$

Step-by-step, we obtain that $f \in \mathbf{N}_\kappa^k$ and (4.35)–(4.36) hold.

(ii) The second case. Let $n = 2m+2$, $m \in \mathbb{Z}_+ \cup \{-1\}$, then the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ can be rewritten by

$$
f(z) = -\frac{1|}{\left|-a_0(z) - \dfrac{1}{a_1(z)}\right.} - \cdots - \frac{1|}{\left|-a_{2m}(z) - \dfrac{1}{a_{2m+1}(z)}\right.} - \frac{1|}{|-a_{2m+2}(z)}.
\tag{4.40}
$$

Let us set the function f_{m+1} by

$$
f_{m+1}(z) = -\frac{1}{-a_{2m+2}(z)}.
$$

Hence, the function zf_{m+1} takes the form

$$
zf_{m+1}(z) = -\frac{z}{-a_{2m+2(z)}} = -\frac{1}{-\dfrac{a_{2m+2}(z)-a_{2m+2}(0)}{z} - \dfrac{a_{2m+2}(0)}{z}}.
$$

By Proposition 1.1 and Proposition 1.2, $f_{m+1} \in \mathbf{N}_{\widetilde{\kappa}_{m+1}}^{\widetilde{k}_{m+1}}$, where the indices $\widetilde{\kappa}_{m+1}$ and $\widetilde{k}_{m+1}$ are defined by

$$
\begin{aligned}
&\widetilde{\kappa}_{m+1} = \kappa_-(-a_{2m+2}),\\
&\widetilde{k}_{m+1} \le \kappa_-\left(-\frac{a_{2m+2}(z)-a_{2m+2}(0)}{z}\right) + \kappa_-\left(-\frac{a_{2m+2}(0)}{z}\right).
\end{aligned}
$$

By the first case (i), we obtain $f \in \mathbf{N}_\kappa^k$, where the indices κ and k satisfy the formulas (4.35)–(4.36). This completes the proof. □

Corollary 4.8. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be the meromorphic on* $\mathbb{C}\backslash\mathbb{R}$, *where* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and* $n_0 \le m_0$. *Then* f *admits representation*

$$
f(z) = a_{-1}(z) - \frac{1|}{|-a_0(z)} - \frac{1|}{|a_1(z)} - \cdots - \frac{1|}{|(-1)^{n+1}a_n(z)}.
\tag{4.41}
$$

Furthermore, f *belongs to the class* $\mathbf{N}_\kappa^k$, *such that*

$$
\kappa = \sum_{j=-1}^{n} \kappa_j \text{ and } k \le \sum_{i=-1}^{n} k_i + \sum_{i=-1}^{[n/2]} k_i^0,
\tag{4.42}
$$

where the indices κ_i, k_i *and* k_i^0 *can be found by*

$$
\begin{aligned}
&k_{2i+1} = \kappa_-(za_{2i+1}(z)), \quad \kappa_i = \kappa_-((-1)^{i+1}a_i(z)),\\
&k_i^0 = \begin{cases} 1, & \text{if } a_{2i}(0) < 0;\\ 0, & \text{if } a_{2i}(0) > 0.\end{cases} \quad k_{2i} = \kappa_-\left(-\frac{a_{2i}(z)-a_{2i}(0)}{z}\right).
\end{aligned}
\tag{4.43}
$$

Proof. Assume the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ be the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, where $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ and $n_0 \le m_0$. By Euclidean algorithm, the function f admits representation (4.41).

By Proposition 1.1, $a_{-1} \in \mathbf{N}_{\kappa_{-1}}^{k_{-1}}$, where indices κ_{-1} and k_{-1} are defined by (4.43).

By Proposition 4.3, $(f - a_{-1}) \in \mathbf{N}_{\widetilde{\kappa}}^{\widetilde{k}}$, where the indices $\widetilde{\kappa}$ and $\widetilde{k}$ are defined by formulas (4.35)–(4.36). Therefore, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ belongs to the class $\mathbf{N}_{\kappa}^{k}$ and the formulas (4.42)–(4.43) hold. This completes the proof. $\square$

Theorem 4.4. *Let $\tau \in \mathbf{N}_{\kappa^*}^{k^*}$ and let $f(z) = \dfrac{Q_0(z)}{P_0(z)} + \tau(z)$, where the P_0 and Q_0 are polynomials, such that* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and $m_0 < n_0$. Then $f \in \mathbf{N}_{\kappa}^{k}$, where*

$$\kappa \le \kappa^* + \sum_{j=0}^{n} \kappa_j \text{ and } k \le k^* + \sum_{i=0}^{n} k_i + \sum_{i=0}^{[n/2]} k_i^0, \tag{4.44}$$

where the indices κ_i, k_i and k_i^0 can be found by (4.43).

Proof. This proof is based on Proposition 4.3 and Proposition 1.1. $\square$

4.2. General case in the class $\mathbf{N}_{\kappa}^{-k}$.

Proposition 4.4. *Let the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ be the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, where* $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ *and $m_0 < n_0$ and let f admits representation* (2.9). *Then f belongs to the class $\mathbf{N}_{\kappa}^{-k}$, such that*

$$\kappa = \sum_{j=0}^{n} \kappa_j \text{ and } k \le \sum_{i=0}^{n} k_i + \sum_{i=0}^{[n/2]} k_i^0, \tag{4.45}$$

where the indices κ_i, k_i and k_i^0 can be found by

$$\begin{aligned} &\kappa_i = \kappa_-((-1)^{i+1} a_i(z)), \quad k_{2i+1} = \kappa_-\left(\frac{a_{2i+1}(z) - a_{2i+1}(0)}{z}\right), \\ &k_{2i} = \kappa_-(-z a_{2i}(z)), \quad k_i^0 = \begin{cases} 1, & \text{if } a_{2i+1}(0) > 0; \\ 0, & \text{if } a_{2i+1}(0) < 0. \end{cases} \end{aligned} \tag{4.46}$$

Proof. By Euclidean algorithm, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ admits the representation (2.9) and by Theorem (2.1), $f \in \mathbf{N}_{\kappa}$, where the index κ are calculated by

$$\kappa = \sum_{j=0}^{n} \kappa_j = \sum_{j=0}^{n} \kappa_-((-1)^{j+1} a_j(z)).$$

By Defenition (1.1), the function f is the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, then $f \in \mathbf{N}_{\kappa}^{-k}$. Find index k.

(i) The first case. Let $n = 2m + 1$ in (2.9), then

$$\frac{f(z)}{z} = -\frac{1\,|}{|\,-za_0(z) - \dfrac{1}{\frac{a_1(z)}{z}}} - \cdots - \frac{1\,|}{|\,-za_{2m}(z) - \dfrac{1}{\frac{a_{2m+1}(z)}{z}}}.$$

Setting

$$\phi_m(z) = -\cfrac{1}{-za_{2m}(z) - \cfrac{1}{\cfrac{a_{2m+1}(z)}{z}}} = -\cfrac{1}{-za_{2m}(z) - \cfrac{1}{\cfrac{a_{2m+1}(z)-a_{2m+1}(0)}{z} + \cfrac{a_{2m+1}(z)}{z}}},$$

by Proposition 1.1, $\phi_m \in \mathbf{N}_{\widetilde{k}_m}$, where the index $\widetilde{k}_m$ is defined by

$$\widetilde{k}_m \le k_{2m} + k_{2m+1} + k_m^0,$$

where the indices k_{2m}, k_{2m+1} and k_m^0 can be calculated by

$$k_{2m} = \kappa_-(-za_{2m}(z)), \quad k_{2m+1} = \kappa_-\left(\frac{a_{2m+1}(z)-a_{2m+1}(0)}{z}\right),$$

$$k_m^0 = \begin{cases} 1, & \text{if } a_{2m+1}(0) > 0; \\ 0, & \text{if } a_{2m+1}(0) < 0. \end{cases}$$

So, let ϕ_{m-1} is defined by

$$\phi_{m-1}(z) = -\cfrac{1}{-za_{2m-2}(z) - \cfrac{1}{\cfrac{a_{2m-11}(z)}{z} + \phi_m(z)}} = -\cfrac{1}{-za_{2m-2}(z) - \cfrac{1}{\cfrac{a_{2m-1}(z)-a_{2m-1}(0)}{z} + \cfrac{a_{2m-1}(z)}{z} + \phi_m(z)}}.$$

Due to Proposition 1.1, $\phi_{m-1} \in \mathbf{N}_{\widetilde{k}_{m-1}}$, where the index $\widetilde{k}_{m-1}$ is

$$\widetilde{k}_m \le k_{2m-2} + k_{2m-11}k_{2m} + k_{2m+1} + k_{m-1}^0 + k_m^0,$$

where the indices k_{2m-2}, k_{2m-1} and k_m^0 are defined by

$$k_{2m-2} = \kappa_-(-za_{2m-2}(z)), \quad k_{2m-1} = \kappa_-\left(\frac{a_{2m-1}(z)-a_{2m-1}(0)}{z}\right),$$

$$k_m^0 = \begin{cases} 1, & \text{if } a_{2m-1}(0) > 0; \\ 0, & \text{if } a_{2m-1}(0) < 0. \end{cases}$$

By induction, we obtain the sequence $\phi_m, \phi_{m-1}, ..., \phi_1$, where $\phi_1(z) = \frac{f(z)}{z}$ and $\phi_1 \in \mathbf{N}_k$ and k is defined by (4.45)–(4.46). Therefore, the function $f \in \mathbf{N}_\kappa^{-k}$, where the indices κ and k are generated by (4.45)–(4.46).

(ii) The second case. Let $n = 2m+2$ in (2.9), then

$$\frac{f(z)}{z} = -\frac{1|}{\left| -za_0(z) - \cfrac{1}{\cfrac{a_1(z)}{z}}\right.} - \cdots - \frac{1|}{\left| -za_{2m}(z) - \cfrac{1}{\cfrac{a_{2m+1}(z)}{z}}\right.} - \frac{1|}{|-za_{2m+2}(z)}.$$

Let us set

$$\phi_{m+1}(z) = -\frac{1}{-za_{2m+2}(z)}.$$

By Proposition 1.1, $\phi_{m+1} \in \mathbf{N}_{k_{2m+2}}$, where $k_{2m+2} = \kappa_-(-za_{2m+2}(z))$. The next step, we apply the first case (i) and obtain $f \in \mathbf{N}_{\kappa}^{-k}$, where the indices κ and k satisfy (4.45)–(4.46). This completes the proof. □

Corollary 4.9. *Let the rational function* $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ *be the meromorphic on* $\mathbb{C}\backslash\mathbb{R}$*, where* $\deg(P_0) = n_0$*,* $\deg(Q_0) = m_0$ *and* $n_0 \le m_0$*. Then f admits representation*

$$f(z) = a_{-1}(z) - \frac{1|}{|-a_0(z)} - \frac{1|}{|a_1(z)} - \dots - \frac{1|}{|(-1)^{n+1}a_n(z)}. \tag{4.47}$$

Furthermore, f belongs to the class $\mathbf{N}_{\kappa}^{-k}$*, such that*

$$\kappa = \sum_{j=-1}^{n} \kappa_j \text{ and } k \le \sum_{i=-1}^{n} k_i + \sum_{i=-1}^{[n/2]} k_i^0, \tag{4.48}$$

where the indices κ_i*,* k_i *and* k_i^0 *can be found by*

$$\begin{aligned} &k_{2i} = \kappa_-(-za_{2i}(z)),\ \kappa_i = \kappa_-((-1)^{i+1}a_i(z)),\ k_i^0 = \begin{cases} 1, a_{2i+1}(0) > 0; \\ 0, a_{2i+1}(0) < 0, \end{cases} \\ &k_{-1} = \kappa_{-1}\left(\frac{a_{-1}(z) - a_{-1}(0)}{z}\right), \quad k_{2i+1} = \kappa_-\left(\frac{a_{2i+1}(z) - a_{2i+1}(0)}{z}\right). \end{aligned} \tag{4.49}$$

Proof. Suppose the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ is the meromorphic on $\mathbb{C}\backslash\mathbb{R}$, where $\deg(P_0) = n_0$, $\deg(Q_0) = m_0$ and $n_0 \le m_0$. By Euclidean algorithm, the function f admits representation (4.41).

We can rewrite the ratio $\frac{a_{-1}(z)}{z}$ as

$$\frac{a_{-1}(z)}{z} = \frac{a_{-1}(z) - a_{-1}(0)}{z} + \frac{a_{-1}(0)}{z}.$$

By Proposition 1.1 and Proposition 1.2, $a_{-1} \in \mathbf{N}_{\kappa_{-1}}^{-\widetilde{k}_{-1}}$, where $\widetilde{k}_{-1} \le k_{-1} + k_{-1}^0$ and κ_{-1}, k_{-1}, k_{-1}^0 are defined by (4.49).

By Proposition 4.3, $(f - a_{-1}) \in \mathbf{N}_{\widetilde{\kappa}}^{\widetilde{k}}$, where the indices $\widetilde{\kappa}$ and $\widetilde{k}$ are defined by formulas (4.45)–(4.46). Therefore, the rational function $f(z) = \dfrac{Q_0(z)}{P_0(z)}$ belongs to the class $\mathbf{N}_{\kappa}^{-k}$ and the formulas (4.48)–(4.49) hold. This completes the proof. □

Theorem 4.5. *Let* $\tau \in \mathbf{N}_{\kappa^*}^{-k^*}$ *and let* $f(z) = \dfrac{Q_0(z)}{P_0(z)} + \tau(z)$*, where the* P_0 *and* Q_0 *are polynomials, such that* $\deg(P_0) = n_0$*,* $\deg(Q_0) = m_0$ *and* $m_0 < n_0$*. Then* $f \in \mathbf{N}_{\kappa}^{-k}$*, where*

$$\kappa \le \kappa^* + \sum_{j=0}^{n} \kappa_j \text{ and } k \le k^* + \sum_{i=0}^{n} k_i + \sum_{i=0}^{[n/2]} k_i^0, \tag{4.50}$$

where the indices κ_i*,* k_i *and* k_i^0 *can be found by* (4.46)

Proof. This proof is based on Proposition 4.4 and Proposition 1.1. □

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IVAN KOVALYOV
UNIVERSITÄT OSNABRÜCK
DEPARTMENT OF MATHEMATICS
SNEUER GRABEN/SCHLOSS, 49074 OSNABRÜCK, GERMANY
ORCID: 0000-0001-8464-3377
E-mail address: `i.m.kovalyov@gmail.com`