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FREE RESOLUTIONS FOR THE TANGENT CONES OF SOME HOMOGENEOUS PSEUDO SYMMETRIC MONOMIAL CURVES

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Abstract. In this article, we study minimal graded free resolutions of Cohen -Macaulay tangent cones of some monomial curves associated to 4-generated pseudo symmetric numerical semigroups. We explicitly give the matrices in these minimal free resolutions.

1. Introduction

Minimal graded free resolutions are very nice objects to study the modules over finitely generated graded algebras . It carries out the information about the Hilbert series, the Castelnuovo-Mumford regularity and many other geometrical invariants of the module, which makes these resolutions very important for algebrebraic geometry and commutative algebra. Construction of an explicit minimal free resolution of a finitely generated algebra is a difficult problem in general. This problem has been studied by many mathematicans, in particular for the homogeneous coordinate ring of an affine monomial curve in [1, 5, 6, 11–13, 15].

"Describing the Betti numbers and the minimal resolution of the tangent cone of S when S is a 4-generated semigroup which is (almost) symmetric or nearly Gorenstein " was an open problem (See [16], Problem 9.9). Symmetric numerical semigroup case is studied by Mete and Zengin in [11] and in [12]. They computed the Betti numbers by explicitly computing the minimal graded free resolution. Pseudo symmetric semigroup case is studied in [15] by showing that being homogeneous and being homogeneous type are equivalent for 4 generated pseudo symmetric monomial curves with Cohen-Macaulay tangent cones by computing the Betti sequences for nonhomogeneous case. Though in the homogeneous case the Betti sequence is

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already known as $(1,5,6,2)$, an explicit computation of minimal graded free resolutions were not given. In this paper, we focus on 4 generated pseudo symmetric semigroups with Cohen-Macaulay tangent cones that are homogeneous (and hence homogeneous type) and calculate the explicit minimal graded free resolutions when n_1 is the smallest among n_1, n_2, n_3, n_4.

2. Preliminaries

Let n_1, n_2, n_3, n_4 be positive integers with $\gcd(n_1, ..., n_k) = 1$. Consider the numerical semigroup $S =< n_1, n_2, ..., n_k >= \{\sum_{i=1}^{k} u_i n_i | u_i \in \mathbb{N}\}$. Let $A = K[X_1, X_2, ..., X_k]$ be the coordinate ring over the field K and $K[S]$ be the semigroup ring $K[t^{n_1}, t^{n_2}, ..., t^{n_k}]$ of S. If we denote the kernel of the surjection

$$\begin{aligned} \phi_0 \colon A &\to K[S] \\ X_i &\mapsto t^{n_i} \end{aligned}$$

by I_S, then $K[S] \simeq A/I_S$. If we denote the affine curve with parametrization

$$X_1 = t^{n_1}, \quad X_2 = t^{n_2}, \ \ ...., \ X_k = t^{n_k}$$

corresponding to S by C_S, then the local ring corresponding to S is $R_S = K[[t^{n_1}, ..., t^{n_k}]]$. The Hilbert function of the local ring R_S is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(R_S) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. It is known that

$$gr_{\mathfrak{m}}(R_S) \cong K[S]/I_{S*}$$

where $I_{S*} =< f_* | f \in I_S >$ is the defining ideal of the tangent cone with f_* denoting the initial form of f.

s being an element of the semi-group S, the apery set of S with respect to s is defined to be $AP(S, s) = \{x \in S \mid x - s \notin S\}$ and the set of lengths of s in S is $L(s) = \{\sum_{i=1}^{k} u_i | s = \sum_{i=1}^{k} u_i n_i, u_i \geq 0\}$. A subset $T \subset S$ is said to be homogeneous if either it is empty or $L(s)$ is a singleton for all $0 \neq s \in T$. n_i being the smallest among $n_1, n_2, ..., n_k$, the numerical semigroup S is said to be homogeneous if the apery set $AP(S, n_i)$ is homogeneous. It has been shown in [9] that $AP(S, n_i)$ is homogeneous if and only if there is a minimal set of generators G of I_S such that X_i belongs to the support of all nonhomogeneous elements of E.

A semigroup S is said to be of homogeneous type if the Betti numbers of the semigroup ring $K[S]$ and the Betti numbers of the associated graded ring (tangent cone) coincide, [8]. It is known that if a semigroup is of homogeneous type then the corresponding tangent cone is Cohen-Macaulay. Furthermore, if the semigroup S is homogeneous and the tangent cone is Cohen-Macaulay then S is also of homogeneous type. Converse is not true in general: there are numerical semigroups which are of homogeneous type but not homogeneous. Some counter examples are given in embedding dimension 4 , see [9].

In [10] the generators of I_S corresponding to a 4-generated pseudo symmetric numerical semigroup are given as $< f_1, f_2, f_3, f_4, f_5 >$ where

$$\begin{aligned} f_1 &= X_1^{\alpha_1} - X_3X_4^{\alpha_4-1}, \qquad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2, \\ f_4 &= X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, \quad f_5 = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} - X_2X_4^{\alpha_4-1} \end{aligned}$$

where here $\alpha_i > 1$, $1 \le i \le 4$, and $0 < \alpha_{21} < \alpha_1$, such that $n_1 = \alpha_2\alpha_3(\alpha_4 - 1) + 1$, $n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3$, $n_3 = \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1$, $n_4 = \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2$.

Barucci, Fröberg and Şahin in [1] showed that the Betti sequence of $K[S]$ is $(1, 5, 6, 2)$ for 4 generated pseudo symmetric monomial curves but I_{S_*} or the Betti numbers of the tangent cone were not known. In [14], we described the Cohen–Macaulay property of the tangent cone in terms of Komeda's parametrization for 4-generated pseudo symmetric monomial curves.

3. Free Resolutions

When n_1 is the smallest among $\{n_1, n_2, n_3, n_4\}$, since the semigroup is always homogeneous, it is known that the Betti sequence is $(1, 5, 6, 2)$. It is also known from [14] that the tangent cone is Cohen-Macaulay iff $\alpha_4 \le \alpha_2 + \alpha_3 \le \alpha_{21} + \alpha_3 - 1 \le \alpha_1$. To compute these homogeneous summands, we will use:

Lemma 1 ([14], page 16)**.** *When n_1 is the smallest and the tangent cone is Cohen-Macaulay, $\{f_1, f_2, f_3, f_4, f_5\}$ forms a standard basis for I_S.*

Since the homogeneous summands change when there are equalities, there are 8 different possibilities for the tangent cone that should be considered:

(1) $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$
(2) $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$
(3) $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
(4) $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
(5) $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$
(6) $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$
(7) $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$
(8) $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$

However, case 8 is irredundant as can be seen from the next proposition.

Proposition 1. *If $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$ then $n_1 = n_2$*

Proof. $n_1 = \alpha_2\alpha_3(\alpha_4 - 1) + 1 = (\alpha_{21} + 1)\alpha_3(\alpha_{21} + \alpha_3 - 1) + 1 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_3(\alpha_{21} + \alpha_3 - 1) + 1$.

On the other hand,

$n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3) + (\alpha_3 - 1)(\alpha_3 - 1) + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_{21}\alpha_3 + (\alpha_3 - 1)^2 + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_3(\alpha_{21} + \alpha_3 - 1) + 1 = n_1$ □

There is a general form of the minimal graded free resolution of the tangent cone in possibilities (1) and (3), (2) and (4), (5) and (7). We will list these and the minimal graded free resolution in case (6) respectively.

The content of the rest of the paper will be as follows: for each of these four possibilities, we will give the generators of I_{S*} as a corollary of lemma 3.4 of [14] and give our main theorems to compute the minimal graded free resolutions. To find the generators of I_{S*}, since we only take the homogeneous summands of the elements in G in Lemma 1 in respective cases, we will not write the proofs of corollaries. To prove the given complexes in our theorems are exact, we will use Buchsbaum-Eisenbud criterion, see [2] for the details. θ being a matrix, we will denote the minor obtained from θ by erasing its ith row , jth column with $[\theta]_{r_i,c_j}$.

3.1. **If** $\alpha_4 < \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 < \alpha_1$**.**

Corollary 1. *I_{S*} is generated by $G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3}, X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}\}$ where $X = X_2^{\alpha_2}$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$ and $X = f_2$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$.*

Theorem 1. *If S is a 4-generated pseudo symmetric semigroup, then minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} & X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -X_2 & 0 & -X_3^{\alpha_3-1} & 0 & X_4 & 0 \\ 0 & -X_3^{\alpha_3} & 0 & 0 & 0 & -X_4^{\alpha_4-1} \\ 0 & X & X_4^{\alpha_4-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_2 & -X_3 & Y \\ X_3 & 0 & 0 & X_4 & 0 & X_2^{\alpha_2-1} \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ 0 & X_4^{\alpha_4-1} \\ 0 & -X \\ -X_3 & 0 \\ X_2 & Z \\ 0 & -X_3^{\alpha_3} \end{bmatrix}$$

with $(X,Y,Z) = (X_2^{\alpha_2}, 0, 0)$ if $\alpha_2 \neq \alpha_{21}+1$ and $(X,Y,Z) = (f_2, -X_1^{\alpha_{21}}, X_1^{\alpha_{21}}X_3^{\alpha_3-1})$ if $\alpha_2 = \alpha_{21}+1$.

Proof. It is easy to see that $\phi_1\phi_2 = \phi_2\phi_3$ so that we have a complex. To show the complex is exact, $\text{rank}\phi_1 = 1$, $\text{rank}\phi_2 = 4$ and $\text{rank}\phi_3 = 2$ and hence $\text{rank}\phi_1 + \text{rank}\phi_2 = \text{rank}A^5$, $\text{rank}\phi_2 + \text{rank}\phi_3 = \text{rank}A^6$. Then by Buchsbaum- Eisenbud criterion, it is enough to check that $I(\phi_i)$ has a regular sequence of length i for $i = 1, 2, 3$. There is nothing to show for $i = 1$. A regular sequence of length 2 can be obtained as $-X_3^{2\alpha_3+1}$ from the minor $[\phi_2]_{r_3,c_4,c_6}$ and X_2X^2 from the minor

$[\phi_2]_{r_2,c_3,c_5}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained for ϕ_3 as $X_4^{\alpha_4}$ from the minor $[\phi_3]_{r_3,r_4,r_5,r_6}$, $X_3^{\alpha_3+1}$ from the minor $[\phi_3]_{r_1,r_2,r_3,r_5}$ and $-X_2X$ from the minor $[\phi_3]_{r_1,r_2,r_4,r_6}$. □

3.2. **If** $\alpha_4 = \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 < \alpha_1$**.**

Corollary 2. *I_{S*} is generated by*

$$G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3}, X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, Y\}$$

where $(X,Y) = (f_2, f_5)$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$,
$(X,Y) = (X_2^{\alpha_2}, -X_2X_4^{\alpha_4-1})$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$

Theorem 2. *In this case, minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} & X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} & Y \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} X_2 & 0 & X_4 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1X_3^{\alpha_3-1} & 0 & 0 & -X_4^{\alpha_4-1} & X_3^{\alpha_3} \\ -X_1Z & 0 & -X_1X_2^{\alpha_2-1} & -X_4^{\alpha_4-1} & 0 & -X \\ 0 & -X_2 & -X_3 & 0 & -Z & 0 \\ X_3 & -X_4 & 0 & 0 & -X_2^{\alpha_2-1} & 0 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ X_3 & 0 \\ -X_2 & (-X_2X_4^{\alpha_4-1} - Y)/X_1 \\ 0 & -X \\ 0 & X_3^{\alpha_3} \\ X_1 & X_4^{\alpha_4-1} \end{bmatrix}$$

where
$(X,Y,Z) = (f_2, f_5, X_1^{\alpha_{21}})$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
$(X,Y,Z) = (X_2^{\alpha_2}, -X_2X_4^{\alpha_4-1}, 0)$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ so that we have a complex. Similarly to the previous case, it is easy to see that rank$\phi_1 = 1$, rank$\phi_2 = 4$ and rank$\phi_3 = 2$ and hence rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. A regular sequence of length 2 is $X_3^{2\alpha_3+1}$ from the minor $[\phi_2]_{r_3,c_2,c_5}$, $X_2^{2\alpha_2+1}$ if $\alpha_2 - 1 < \alpha_{21} + \alpha_3$ and $-X_2f_2^2$ if $\alpha_2 - 1 = \alpha_{21} + \alpha_3$ from the minor $[\phi_2]_{r_3,c_3,c_4}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained as f_4 from the minor $[\phi_3]_{r_2,r_3,r_4,r_5}$, $X_3^{\alpha_3+1}$ from the minor $[\phi_3]_{r_1,r_3,r_4,r_6}$, $X_2^{\alpha_2+1}$ if $\alpha_2 - 1 < \alpha_{21} + \alpha_3$ and X_2f_2 if $\alpha_2 - 1 = \alpha_{21} + \alpha_3$ from the minor $[\phi_3]_{r_1,r_2,r_5,r_6}$ for $I(\phi_3)$. □

3.3. **If** $\alpha_4 < \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 = \alpha_1$**.**

Corollary 3. *I_{S*} is generated by*

$$G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2, X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}\}$$

where $X = f_2$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$ and $X_2^{\alpha_2}$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$.

Theorem 3. *In this case, minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2 & X_4^{\alpha_4} & X_2X_4^{\alpha_4-1}\end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -X_3^{\alpha_3-1} & 0 & X_2 & -X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_3 & X_4^{\alpha_4-1} \\ X_4^{\alpha_4-1} & 0 & 0 & 0 & X & 0 \\ 0 & X_2 & 0 & X_3 & 0 & Y \\ X_1^{\alpha_1-\alpha_{21}-1} & -X_4 & -X_3 & 0 & 0 & -X_2^{\alpha_2-1} \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 0 & X \\ -X_3 & Z \\ X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ X_2 & YX_3^{\alpha_3-1} \\ 0 & -X_4^{\alpha_4-1} \\ 0 & -f_3 \end{bmatrix}$$

where (X, Y, Z) equals to $(f_2, X_1^{\alpha_{21}}, -X_1^{\alpha_1-1})$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$ and $(X_2^{\alpha_2}, 0, 0)$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ is obvious and rank$\phi_1 = 1$, rank$\phi_2 = 4$ and rank$\phi_3 = 2$ and hence rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. $I(\phi_2)$ has a regular sequence of length 2 as $X_4^{2\alpha_4}$ from the minor $[\phi_2]_{r_4,r_3,c_5}$, X_2X^2 from the minor $[\phi_2]_{r_2,c_1,c_4}$. A regular sequence of length 3 can be obtained as X_3f_3 from the minor $[\phi_3]_{r_1,r_3,r_4,r_5}$, $-X_4^{\alpha_4}$ from the minor $[\phi_3]_{r_1,r_2,r_4,r_6}$, $-X_2X$ from the minor $[\phi_3]_{r_2,r_3,r_5,r_6}$ for $I(\phi_3)$. □

Finally, minimal graded free resolution of the tangent cone in (6) is:

3.4. **If** $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$**.**

Corollary 4. *In this case I_{S_*} is generated by*

$$G_* = \{X_1^{\alpha_1-\alpha_{21}-1}X_2 - X_3^{\alpha_3}, X_2^{\alpha_2}, X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} - X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}, X_3X_4^{\alpha_4-1}\}$$

Theorem 4. *If $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$, then minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1-\alpha_{21}-1}X_2 - X_3^{\alpha_3} & X_2^{\alpha_2} & X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} - X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} & X_3X_4^{\alpha_4-1} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 0 & 0 & X_1X_2^{\alpha_2-1} & -X_4^{\alpha_4-1} & -X_2^{\alpha_2} & 0 \\ 0 & -X_1X_3^{\alpha_3-1} & -X_1^{\alpha_1-\alpha_{21}} & 0 & -f_3 & -X_4^{\alpha_4-1} \\ 0 & X_2 & X_3 & 0 & 0 & 0 \\ -X_3 & X_4 & 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 & X_2^{\alpha_2-1} \\ X_2 & 0 & X_4 & -X_3^{\alpha_3-1} & 0 & 0 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} -X_4 & -X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ -X_3 & 0 \\ X_2 & 0 \\ 0 & -X_2^{\alpha_2} \\ X_1 & X_4^{\alpha_4-1} \\ 0 & -f_3 \end{bmatrix}$$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ and rank$\phi_1 = 1$, rank$\phi_2 = 4$, rank$\phi_3 = 2$. Thus, rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. $I(\phi_2)$ has a regular sequence of length 2 as $X_3^2X_4^{\alpha_4-1}f_3$ from the minor $[\phi_2]_{r_5,c_2,c_6}$, $-X_2^{2\alpha_2+1}$ from the minor $[\phi_2]_{r_2,c_3,c_4}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained as f_4 from the minor $[\phi_3]_{r_2,r_3,r_4,r_6}$, X_3f_3 from the minor $[\phi_3]_{r_1,r_3,r_4,r_5}$, $X_2^{\alpha_2+1}$ from the minor $[\phi_3]_{r_1,r_2,r_5,r_6}$ for $I(\phi_3)$. □

4. Conclusion

Since we investigated 4-generated pseudo symmetric semigroups that are homogeneous with Cohen-Macaulay tangent cones when n_1 is the smallest, and since these semigroups are of homogeneous type automatically, in addition to the known Betti sequence $(1, 5, 6, 2)$ of the tangent cone, which Barucci, Fröberg and Şahin obtained in [1], using the standard basis found in [14], we computed the generators of I_{S_*} and we have given a complete characterization to the minimal graded-free resolution of the tangent cone in all possible situations.

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