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The de Rham Cohomology Group of a Hemi-Slant Submanifold in Metallic Riemannian Manifolds

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Abstract

In this study, we answer the question under what conditions a hemi-slant submanifold of locally decomposable metallic Riemannian manifolds admits a well defined canonical de Rham cohomology class. Firstly, we give the integrability and minimality conditions of the distributions arose from its definition. Later, we find some necessary conditions depending on the above-named concepts of the associated distributions for such a type of submanifold to define a de Rham cohomology class. Furthermore, we analyzed the non-triviality of this cohomology class. In the end, we construct two examples which enable better expressing the main results.

Keywords: *De Rham cohomology group, metallic Riemannian manifold, hemi-slant submanifold, Riemannian submanifold.*
AMS Subject Classification (2020): *14F40; 53C15; 53C25; 53C40.*

1. Introduction

The concept of a slant submanifold was first introduced by B. Y. Chen [9] as an extension of both holomorphic and totally real submanifolds in an almost Hermitian manifold. Then, the same author also collected the important results concerning slant submanifolds in his book [10]. After that, slant submanifolds were defined and studied in different ambient manifolds, such as almost contact metric manifolds [26], Sasakian manifolds [6], almost product Riemannian manifolds [31]. Also, slant submanifolds were generalized as semi-slant submanifolds by N. Papaghiuc [28] in the almost Hermitian setting. On the other hand, J. L. Cabrerizo *et a*l. introduced and examined a large class of the aforesaid submanifolds, namely, bi-slant submanifolds [5]. Moreover, another significant class of bi-slant submanifolds are anti-slant submanifolds, which were defined by A. Carriazo [7]. However, since the expression of anti-slant means that it has no slant factors, anti-slant submanifolds have also appeared in different two names in the literature, namely pseudo-slant submanifolds [25] or hemi-slant submanifolds [32].

Recently, metallic Riemannian manifolds [23] were defined by C. E. Hreţcanu and M. C. Crâşmăreanu as a generalization of golden Riemannian manifolds [11, 21, 22]. In [3, 16, 17, 18, 20], C. E. Hreţcanu and A. M. Blaga investigated invariant, anti-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds of metallic Riemannian submanifolds in terms of the characterization, the integrability of the associated distributions, totally mixed geodesicity and the parallelism of induced canonical structures. In [19], the authors also obtained some results concerning the existence and non-existence of semi-invariant, semi-slant and hemi-slant warped product submanifolds in metallic Riemannian manifolds.

On the other hand, there exist several papers regarding the idea to analyzing submanifolds on different environments by using de Rham cohomology groups in the literature: In [8], B. Y. Chen constructed a canonical de Rham cohomology group for any closed CR submanifold in a Kaehler manifold with the help of the distributions involved in its definition in order to prove that if this cohomology group is even dimensional and trivial, the holomorphic distribution is not integrable or its orthogonal complement in the tangent bundle

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is not minimal. In a similar way, the de Rham cohomology of CR submanifolds was also considered in some well known ambient manifolds, such as a nearly Kaehler manifold [12] and a quasi-Kaehler manifold [13]. S. Ianuş, S. Marchiafava and G. E. Vîlcu [24] demonstrated that there is a canonical de Rham cohomology group for a closed paraquaternionic CR submanifold of paraquaternionic Kähler manifolds and discussed necessary conditions for such a cohomology group to be non-trivial. In [33], F. Şahin found some necessary conditions for any hemi-slant submanifold of a Kaehler manifold to define a non-trivial de Rham cohomology group. For a semi-invariant submanifold in locally product Riemannian manifolds, G. Pitis [29] obtained a close relation between its de Rham cohomology group and associated distributions; moreover, the author gave some examples which substantiate his claims. In addition, the de Rham cohomology of semi-invariant submanifolds was investigated by M. Gök [14] in the locally decomposable metallic Riemannian setting.

Motivated by the above works, we study the de Rham cohomology group of a hemi-slant submanifold in locally decomposable metallic Riemannian manifolds.

The paper is prepared in the following way: Section 1 is introduction. Section 2 gives a background to clarify the main results of the paper. In section 3, we mainly focus necessary conditions for the existence and non-triviality of de Rham cohomology classes for a hemi-slant submanifold in locally decomposable metallic Riemannian manifolds. Finally, we establish two concrete examples to illustrate the results obtained in the paper.

2. Preliminaries

A *metallic structure* $\widetilde{J}$ on a differentiable manifold $\widetilde{M}$ is an endomorphism of the tangent bundle $T\widetilde{M}$ yielding the equation

$$\widetilde{J}^2 = p\widetilde{J} + qI, \tag{2.1}$$

where p, q are non-zero natural numbers and I is the identity endomorphism on $\widetilde{M}$. In this situation, $\left(\widetilde{M}, \widetilde{J}\right)$ is called a *metallic manifold*. Furthermore, if a metallic manifold $\left(\widetilde{M}, \widetilde{J}\right)$ admits a $\widetilde{J}$-compatible Riemannian metric, i.e.,

$$\widetilde{g}\left(\widetilde{J}X, Y\right) = \widetilde{g}\left(X, \widetilde{J}Y\right) \tag{2.2}$$

for any vector fields $X, Y \in \Gamma(TM)$, then the pair $\left(\widetilde{g}, \widetilde{J}\right)$ and the triple $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ are said to be a *metallic Riemannian structure* and a *metallic Riemannian manifold*, respectively [23]. In particular, if the metallic structure is parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$, i.e.,

$$\left(\widetilde{\nabla}_X \widetilde{J}\right) Y = 0 \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(TM)$, then the triple $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ is called a *locally decomposable metallic Riemannian manifold*.

Let M be any m-dimensional isometrically immersed submanifold of an $\widetilde{m}$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. The ambient tangent bundle $T\widetilde{M}$ is given by the direct sum

$$T\widetilde{M} = TM \oplus TM^{\perp},$$

where TM and $TM^{\perp}$ are tangent and normal bundles of M. In what follows, we denote by the same symbol $\widetilde{g}$ the Riemannian metric induced on M throughout the paper.

The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h\left(X, Y\right) \tag{2.4}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.5}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, where ∇ is the induced connection on M, h is the second fundamental form, A_V is the shape operator with respect to V and $\nabla^{\perp}$ is the normal connection. In addition, the second fundamental form h and the shape operator A are related by

$$\widetilde{g}\left(h\left(X, Y\right), V\right) = \widetilde{g}\left(A_V X, Y\right) \tag{2.6}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$ [1].

For any vector field $X \in \Gamma(TM)$ and $U \in \Gamma\left(TM^{\perp}\right)$, we can split the vector fields $\widetilde{J}X$ and $\widetilde{J}U$ into tangential and normal components as follows:

$$\widetilde{J}X = TX + NX \tag{2.7}$$

and

$$\widetilde{J}U = tU + nU \tag{2.8}$$

where $TX, tU \in \Gamma(TM)$ and $NX, nU \in \Gamma(TM^{\perp})$. Thus, the following relations are valid [18]:

$$pT + qI = T^2 + tN, \tag{2.9}$$

$$pN = NT + nN, \tag{2.10}$$

$$pt = Tt + tn \tag{2.11}$$

and

$$pn + qI = n^2 + Nt. \tag{2.12}$$

Moreover, the operators T and n are $\widetilde{g}$-symmetric [3], i.e.,

$$\widetilde{g}\left(TX, Y\right) = \widetilde{g}\left(X, TY\right) \tag{2.13}$$

and

$$\widetilde{g}\left(nU, V\right) = \widetilde{g}\left(U, nV\right) \tag{2.14}$$

for any vector fields $X, Y \in \Gamma\left(TM\right)$ and $U, V \in \Gamma\left(TM^{\perp}\right)$.

An isometrically immersed submanifold M of a metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ is said to be *hemi-slant* [18] if there are two orthogonal differentiable distributions D^{θ} and $D^{\perp}$ on M such that

(a) $TM = D^{\theta} \oplus D^{\perp}$,

(b) D^{θ} is a slant distribution with the slant angle $\theta \in \left[0, \frac{\pi}{2}\right]$,

(c) $D^{\perp}$ is an anti-invariant distribution.

Furthermore, a hemi-slant submanifold M of a metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ is called *proper* if $\dim D^{\theta} \cdot \dim D^{\perp} \neq 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$.

Let M be a proper hemi-slant submanifold of a metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. In this case, it is clear that

$$TD^{\theta} = D^{\theta}, TD^{\perp} = \{0\}, ND^{\perp} = \widetilde{J}D^{\perp} \text{ and } ND^{\theta} \perp ND^{\perp}.$$

Also,

$$TM^{\perp} = ND^{\theta} \oplus ND^{\perp} \oplus \mu,$$

where μ is the orthogonal complement of $ND^{\theta} \oplus ND^{\perp}$ in the normal bundle $TM^{\perp}$.

We consider a differentiable manifold M. The quotient space

$$H^k_{dR}\left(M\right) = \ker\left(d : \Omega^k\left(M\right) \to \Omega^{k+1}\left(M\right)\right) / \operatorname{Im}\left(d : \Omega^{k-1}\left(M\right) \to \Omega^k\left(M\right)\right)$$

is called the *k-th de Rham cohomology group* of M, which is also an abelian group, where k is a positive integer, Ω^k is the vector space of all k-forms on M and d is the exterior derivative. Also, its dimension is said to be the *k-th Betti number*, denoted by b_k [27].

On the other hand, Hodge Theorem [30, Theorem 8.12] is a useful method for finding harmonic representatives of $H^k_{dR}\left(M\right)$'s if M is a compact orientable Riemannian manifold without boundary. It identifies $H^k_{dR}\left(M\right)$ with the space of all harmonic k-forms on M, denoted by $\mathcal{H}^k_{\Delta}\left(M\right)$.

Let (R, S) be a pair of complementary orthogonal distributions on a Riemannian manifold M endowed with the Levi-Civita connection ∇. In this case, the vector field $\nabla_X Y$ can be decomposed into two parts as follows:

$$\nabla_X Y = \left(\nabla_X Y\right)^R + \left(\nabla_X Y\right)^S$$

for any vector fields $X, Y \in \Gamma(TM)$, where $(\nabla_X Y)^R \in \Gamma(R)$ and $(\nabla_X Y)^S \in \Gamma(S)$. We set $\dim R = r$ and $\dim S = s$. Let $\mathfrak{B}_R = \{E_1, \ldots, E_r\}$ and $\mathfrak{B}_S = \{E_{r+1}, \ldots, E_{r+s}\}$ be the local orthonormal frames for the distributions R and S, respectively. The distribution R is named *minimal* [8] if its mean curvature vector given by $H_R = \frac{1}{r}\sum_{i=1}^{r}(\nabla_{E_i}E_i)^S$ is identically zero. Besides, the distribution R is called *nearly autoparallel* [2] or *geodesically invariant* [4] if

$$\nabla_X X \in \Gamma(R),$$

or equivalently

$$(\nabla_X Y + \nabla_Y X) \in \Gamma(R)$$

for any vector fields $X, Y \in \Gamma(R)$. Same definitions can be applied to the distribution S. In fact, minimality of a distribution on a Riemannian manifold is equivalent to its *nearly autoparallelness or geodesically invariance.*

Let α and β be two forms such that

$$\alpha = \alpha^1 \Lambda \cdots \Lambda \alpha^r \text{ and } \beta = \beta^{r+1} \Lambda \cdots \Lambda \beta^{r+s},$$

where $\alpha^1, \ldots, \alpha^r, \beta^{r+1}, \ldots, \beta^{r+s}$ are 1-forms on M determined by

$$\alpha^i(Z) = 0,\ \alpha^i(E_j) = \delta_{ij},\ 1 \le i, j \le r \tag{2.15}$$

and

$$\beta^{r+A}(X) = 0,\ \beta^{r+A}(E_{r+B}) = \delta_{AB},\ 1 \le A, B \le s \tag{2.16}$$

for any vector fields $X \in \Gamma(R)$ and $Z \in \Gamma(S)$. From (2.15) and (2.16), we get

$$\alpha(E_1, \ldots, E_r) = \det\left[\alpha^i(E_j)\right] = 1 \tag{2.17}$$

and

$$\beta(E_{r+1}, \ldots, E_{r+s}) = \det\left[\beta^{r+A}(E_{r+B})\right] = 1. \tag{2.18}$$

Thus, it follows that the distributions R and S are orientable with respect to the ordered local orthonormal frames $\mathfrak{B}_R$ and $\mathfrak{B}_S$, respectively. For the same reason, it is seen that α (resp., β) is a globally well defined r-form (resp., a globally well defined s-form). Moreover, it is notable that $\alpha\Lambda\beta$ is a globally well defined m-form and M is orientable with respect to the local orthonormal frame $\mathfrak{B}_R \cup \mathfrak{B}_S$ of the tangent bundle TM.

Before starting, we give two following propositions [14]:

Proposition 2.1. *Let M be any submanifold of a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ such that $TM = R \oplus S$, where R and S are orthogonal distributions on M. If the distribution S is integrable and the distribution R is minimal, then the r-form $\alpha = \alpha^1 \Lambda \cdots \Lambda \alpha^r$ is closed.*

Proposition 2.2. *Let M be any submanifold of a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ such that $TM = R \oplus S$, where R and S are orthogonal distributions on M. If the distribution R is integrable and the distribution S is minimal, then the s-form $\beta = \beta^1 \Lambda \cdots \Lambda \beta^s$ is closed.*

Remark 2.1. For semi-invariant submanifolds of locally product Riemannian manifolds, Propositions 2.1 and 2.2 were proved by G. Pitis in [29].

3. Main Results

Let M be any hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$ such that $TM = D^\theta \oplus D^\perp$, where D^θ is the slant distribution with the slant angle θ and $D^\perp$ is the anti-invariant distribution. We put $\dim D^\theta = u$ and $\dim D^\perp = v$. Let us denote by $\mathfrak{B}_{D^\theta} = \{H_1, \ldots, H_u\}$ and $\mathfrak{B}_{D^\perp} = \{H_{u+1}, \ldots, H_{u+v}\}$ the local orthonormal frames of the distributions D^θ and $D^\perp$, respectively.

We consider two forms $\psi = \psi^1 \Lambda \cdots \Lambda \psi^u$ and $\omega = \omega^{u+1} \Lambda \cdots \Lambda \omega^{u+v}$, where $\psi^1, \ldots, \psi^u, \omega^{u+1}, \ldots, \omega^{u+v}$ are 1-forms on M satisfying the following relations:

$$\psi^i(Z) = 0,\ \psi^i(H_j) = \delta_{ij},\ 1 \le i, j \le u \tag{3.1}$$

and

$$\omega^{u+A}(X) = 0,\ \omega^{u+A}(H_{u+B}) = \delta_{AB},\ 1 \le A, B \le v \tag{3.2}$$

for any vector fields $X \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$.

Proposition 3.1. *Let M be any hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. A necessary and sufficient condition for the slant distribution D^θ to be integrable is that for any vector fields $X, Y \in \Gamma\left(D^\theta\right)$, the vector field*

$$h\left(X, TY\right) - h\left(TX, Y\right) + \nabla_Y^\perp NX - \nabla_X^\perp NY$$

is perpendicular to the distribution $\widetilde{J}D^\perp$ of the normal bundle $TM^\perp$.

Proof. By use of (2.2), (2.3), (2.4), (2.5) and (2.7), by a straightforward computation, we get

$$\widetilde{g}\left(\nabla_X Y, Z\right) = \frac{1}{q}\widetilde{g}\left(h\left(X, TY\right) + \nabla_X^\perp NY - ph\left(X, Y\right), \widetilde{J}Z\right) \tag{3.3}$$

for any vector fields $X, Y \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$. Similarly, interchanging the roles of X and Y in (3.3), we can obtain

$$\widetilde{g}\left(\nabla_Y X, Z\right) = \frac{1}{q}\widetilde{g}\left(h\left(TX, Y\right) + \nabla_Y^\perp NX - ph\left(X, Y\right), \widetilde{J}Z\right). \tag{3.4}$$

Then we infer from (3.3) and (3.4) that

$$\widetilde{g}\left(\left[X, Y\right], Z\right) = \widetilde{g}\left(h\left(X, TY\right) - h\left(TX, Y\right) + \nabla_Y^\perp NX - \nabla_X^\perp NY, \widetilde{J}Z\right)$$

for any vector fields $X, Y \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$, which proves our assertion. □

Proposition 3.2. *[18, Theorem 4.9] Let M be any hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. A necessary and sufficient condition for the anti-invariant distribution $D^\perp$ to be integrable is that*

$$A_{\widetilde{J}Z} W = 0 \tag{3.5}$$

for any vector fields $Z, W \in \Gamma\left(D^\perp\right)$.

Proposition 3.3. *Let M be any hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. A necessary and sufficient condition for the slant distribution D^θ to be minimal in M is that for any vector fields $X \in \Gamma\left(D^\theta\right)$, the vector field*

$$h\left(X, TX\right) + \nabla_X^\perp NX - ph\left(X, X\right)$$

has no components in the distribution $\widetilde{J}D^\perp$ of the normal bundle $TM^\perp$. Specially, the slant distribution D^θ is minimal in M if the following statements are correct:

- **(a)** *The hemi-slant submanifold M is D^θ-geodesic,*
- **(b)** *For any vector fields $X \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$,*

$$\widetilde{g}\left(\nabla_X^\perp NX, NZ\right) = 0.$$

Proof. We derive from (2.2), (2.3), (2.4), (2.5) and (2.7) that

$$\widetilde{g}\left(\nabla_X X, Z\right) = \frac{1}{q}\widetilde{g}\left(h\left(X, TX\right) + \nabla_X^\perp NX - ph\left(X, X\right), \widetilde{J}Z\right) \tag{3.6}$$

for any vector fields $X \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$, which completes the proof. □

Proposition 3.4. *Let M be any hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$. A necessary and sufficient condition for the anti-invariant distribution $D^\perp$ to be minimal in M is that*

$$\widetilde{g}\left(A_{\widetilde{J}Z} Z, TX\right) - \widetilde{g}\left(\nabla_Z^\perp NZ, NX\right) - p\widetilde{g}\left(A_{\widetilde{J}Z} Z, X\right) = 0$$

for any vector fields $X \in \Gamma\left(D^\theta\right)$ and $Z \in \Gamma\left(D^\perp\right)$. Especially, each leaf of the anti-invariant distribution $D^\perp$ is minimal in M if the following statements hold:

(a) *For any vector fields $Z, W \in \Gamma\left(D^{\perp}\right)$,*

$$A_{\widetilde{J}Z}W = 0,$$

(b) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(\nabla_{Z}^{\perp}NZ, NX\right) = 0.$$

Proof. Using the parallelism of the metallic structure $\widetilde{J}$, we obtain from (2.2), (2.4), (2.5) and (2.7) that

$$\widetilde{g}\left(\nabla_{Z}Z, X\right) = \frac{1}{q}\left\{-\widetilde{g}\left(A_{\widetilde{J}Z}Z, TX\right) + \widetilde{g}\left(\nabla_{Z}^{\perp}NZ, NX\right) + p\widetilde{g}\left(A_{\widetilde{J}Z}Z, X\right)\right\}$$

for any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$, which shows that our claim is true. □

Theorem 3.1. *For an arbitrary compact proper hemi-slant submanifold M without boundary of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$, the v-form ω defines a well defined canonical de Rham cohomology class denoted by $[\omega]$ in $H_{dR}^{v}(M)$ if the following statements are true:*

(a) *For any vector fields $X, Y \in \Gamma\left(D^{\theta}\right)$, the vector field*

$$h(X, TY) - h(TX, Y) + \nabla_{Y}^{\perp}NX - \nabla_{X}^{\perp}NY$$

is perpendicular to the distribution $\widetilde{J}D^{\perp}$ of the normal bundle $TM^{\perp}$,

(b) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(A_{\widetilde{J}Z}Z, TX\right) - \widetilde{g}\left(\nabla_{Z}^{\perp}NZ, NX\right) - p\widetilde{g}\left(A_{\widetilde{J}Z}Z, X\right) = 0.$$

Moreover, the canonical de Rham cohomology class $[\omega]$ is non-trivial if the following statements are satisfied:

(c) *For any vector fields $Z, W \in \Gamma\left(D^{\perp}\right)$,*

$$A_{\widetilde{J}Z}W = 0,$$

(d) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$, the vector field*

$$h(X, TX) + \nabla_{X}^{\perp}NX - ph(X, X)$$

has no components in the distribution $\widetilde{J}D^{\perp}$ of the normal bundle $TM^{\perp}$.

Proof. As is seen from Proposition 3.1, (a) shows that the slant distribution D^{θ} is integrable. Proposition 3.4 means that (b) is equivalent to the minimality of the anti-invariant distribution $D^{\perp}$. Hence, under the assumptions (a) and (b), by Proposition 2.1, we get that the v-form ω is closed. In this case, there is a canonical de Rham cohomology class $[\omega] \in H_{dR}^{v}(M)$ associated to ω.

Now, we show that such a cohomology class is non-trivial. Proposition 3.2 implies that (c) is a necessary and sufficient condition for the integrability of the anti-invariant distribution $D^{\perp}$. At the same time, in view of Proposition 3.3, it results from (d) that the slant distribution D^{θ} is minimal. Thus, if the assumptions (c) and (d) are satisfied, it follows from Proposition 2.2 that the u-form ψ is closed. Taking account of that the u-form ψ is the Hodge dual of ω, denoted by $\star\omega$, i.e., $\psi = \star\omega$, it is clear that ω is a co-closed v-form. Therefore, by reason of the fact that M is a compact and boundaryless submanifold, i.e., a closed submanifold, ω is a harmonic v-form. Also, we recall that the submanifold M is orientable with respect to $\mathfrak{B}_{D^{\theta}} \cup \mathfrak{B}_{D^{\perp}}$. Consequently, in the light of the aforementioned evaluations, Hodge Theorem completes the proof, i.e., the cohomology class $[\omega]$ is non-trivial in $H_{dR}^{v}(M)$. □

Theorem 3.2. *For an arbitrary compact proper hemi-slant submanifold M without boundary of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$, the v-form ω determines a well defined canonical de Rham cohomology class given by $[\omega]$ in $H_{dR}^{v}(M)$ if the following statements are valid:*

(a) *For any vector fields* $X, Y \in \Gamma\left(D^{\theta}\right)$*, the vector field*

$$h(X, TY) - h(TX, Y) + \nabla^{\perp}_{Y} NX - \nabla^{\perp}_{X} NY$$

is perpendicular to the distribution $\widetilde{J}D^{\perp}$ *of the normal bundle* $TM^{\perp}$*,*

(b) *For any vector fields* $Z, W \in \Gamma\left(D^{\perp}\right)$*,*

$$A_{\widetilde{J}Z} W = 0,$$

(c) *For any vector fields* $X \in \Gamma\left(D^{\theta}\right)$ *and* $Z \in \Gamma\left(D^{\perp}\right)$*,*

$$\widetilde{g}\left(\nabla^{\perp}_{Z} NZ, NX\right) = 0.$$

Also, the canonical de Rham cohomology class $[\omega]$ *is non-trivial if the following statements are verified:*

(d) *The hemi-slant submanifold* M *is* D^{θ}*-geodesic,*

(e) *For any vector fields* $X \in \Gamma\left(D^{\theta}\right)$ *and* $Z \in \Gamma\left(D^{\perp}\right)$*,*

$$\widetilde{g}\left(\nabla^{\perp}_{X} NX, NZ\right) = 0.$$

Proof. Taking account of Propositions 3.3 and 3.4, the proof is an immediate consequence of Theorem 3.1. □

Theorem 3.3. *For an arbitrary compact totally geodesic proper hemi-slant submanifold* M *without boundary of a locally decomposable metallic Riemannian manifold* $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$*, the* v*-th Betti number* b_v *is not zero if the following statements hold:*

(a) *For any vector fields* $X \in \Gamma\left(D^{\theta}\right)$ *and* $Z, W \in \Gamma\left(D^{\perp}\right)$*,*

$$\widetilde{g}\left(\nabla^{\perp}_{Z} NW, NX\right) = 0,$$

(b) *For any vector fields* $X, Y \in \Gamma\left(D^{\theta}\right)$ *and* $Z \in \Gamma\left(D^{\perp}\right)$*,*

$$\widetilde{g}\left(\nabla^{\perp}_{X} NY, NZ\right) = 0.$$

Proof. Since M is totally geodesic, if (a) and (b) are satisfied, then we can see that all the conditions of Theorem 3.1 are hold automatically. Thus, the proof has been demonstrated. □

Theorem 3.4. *For an arbitrary compact proper hemi-slant submanifold* M *without boundary of a locally decomposable metallic Riemannian manifold* $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$*, there exists a well defined de Rham cohomology class* $[\psi]$ *determined by the* u*-form* ψ *in* $H^{u}_{dR}(M)$ *if the following statements are correct:*

(a) *For any vector fields* $Z, W \in \Gamma\left(D^{\perp}\right)$*,*

$$A_{\widetilde{J}Z} W = 0,$$

(b) *For any vector fields* $X \in \Gamma\left(D^{\theta}\right)$*, the vector field*

$$h(X, TX) + \nabla^{\perp}_{X} NX - ph(X, X)$$

has no components in the distribution $\widetilde{J}D^{\perp}$ *of the normal bundle* $TM^{\perp}$*.*

Furthermore, the canonical de Rham cohomology class $[\psi]$ *is non-trivial if the following two conditions are satisfied:*

(c) *For any vector fields* $X, Y \in \Gamma\left(D^{\theta}\right)$*, the vector field*

$$h(X, TY) - h(TX, Y) + \nabla^{\perp}_{Y} NX - \nabla^{\perp}_{X} NY$$

is perpendicular to the distribution $\widetilde{J}D^{\perp}$ *of the normal bundle* $TM^{\perp}$*,*

(d) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(A_{\widetilde{J}Z}Z, TX\right) - \widetilde{g}\left(\nabla_{Z}^{\perp}NZ, NX\right) - p\widetilde{g}\left(A_{\widetilde{J}Z}Z, X\right) = 0.$$

Proof. The proof can be demonstrated in a manner similar to that of Theorem 3.1. □

Theorem 3.5. *For an arbitrary compact proper hemi-slant submanifold M without boundary of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$, the u-form ψ represents a well defined de Rham cohomology class denoted by $[\psi]$ in $H_{dR}^{u}(M)$ if the following statements are verified:*

(a) *The hemi-slant submanifold M is D^{θ}-geodesic,*

(b) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(\nabla_{X}^{\perp}NX, NZ\right) = 0,$$

(c) *For any vector fields $Z, W \in \Gamma\left(D^{\perp}\right)$,*

$$A_{\widetilde{J}Z}W = 0.$$

In addition, the canonical de Rham cohomology class $[\psi]$ is non-trivial if the following two conditions are correct:

(d) *For any vector fields $X, Y \in \Gamma\left(D^{\theta}\right)$, the vector field*

$$h(X, TY) - h(TX, Y) + \nabla_{Y}^{\perp}NX - \nabla_{X}^{\perp}NY$$

has no components in the distribution $\widetilde{J}D^{\perp}$ of the normal bundle $TM^{\perp}$,

(e) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(\nabla_{Z}^{\perp}NZ, NX\right) = 0.$$

Proof. With the help of Propositions 3.3 and 3.4, the proof follows directly from Theorem 3.4. □

Theorem 3.6. *For an arbitrary compact totally geodesic proper hemi-slant submanifold without boundary of a locally decomposable metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{J}\right)$, the u-th Betti number b_u is not zero if the following statements are true:*

(a) *For any vector fields $X \in \Gamma\left(D^{\theta}\right)$ and $Z, W \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(\nabla_{Z}^{\perp}NW, NX\right) = 0,$$

(b) *For any vector fields $X, Y \in \Gamma\left(D^{\theta}\right)$ and $Z \in \Gamma\left(D^{\perp}\right)$,*

$$\widetilde{g}\left(\nabla_{X}^{\perp}NY, NZ\right) = 0.$$

Proof. The proof is the same as that of Theorem 3.3. □

Finally, we give two examples.

Example 3.1. Let J be a tensor field of type $(1,1)$ on the 5-dimensional Euclidean space $\left(\mathbb{R}^5, \langle,\rangle\right)$ defined by

$$J(x_1, x_2, x_3, x_4, x_5) = \left(\frac{p}{2}x_1 + \frac{\sqrt{\lambda}}{2}x_3, \frac{p}{2}x_2 + \frac{\sqrt{\lambda}}{2}x_4, \frac{p}{2}x_3 + \frac{\sqrt{\lambda}}{2}x_1, \frac{p}{2}x_4 + \frac{\sqrt{\lambda}}{2}x_2, 0\right)$$

where $(x_1, x_2, x_3, x_4, x_5)$ is the local coordinates of $\mathbb{R}^5$, $\lambda = p^2 + 4q$ and $p, q \in \mathbb{N}^{+}$.

We define a submanifold M by the immersion $i : M \longrightarrow \mathbb{R}^5$ such that

$$i(u, v, w) = (u\cos t, v\sin t, v, 0, w),$$

where $u, v > 0$ and $t \in \left(0, \frac{\pi}{2}\right)$. In this case, a local orthonormal frame of the tangent bundle TM can be chosen as follows:

$$E_1 = \frac{\partial}{\partial x_1},$$

$$E_2 = \frac{1}{\sqrt{1+\sin^2 t}}\left(\sin t \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right)$$

and

$$E_3 = \frac{\partial}{\partial x_5}.$$

Putting $\omega^1 = dx_5$, then we get

$$J^2 = pJ + q\left(I - \omega^1 \otimes E_3\right).$$

Also, it is worthy to note that J is a framed $f_{a,b}(3,2,1)$-structure of rank 2 [15] with $a = p$ and $b = q$. We set

$$D^\theta = \left\{X \in \Gamma(TM) : \omega^1(X) = 0\right\}$$

and denote by $D^\perp$ its orthogonal complement in the tangent bundle TM. Then it follows that $D^\theta = Span\{E_1, E_2\}$ and $D^\perp = Span\{E_3\}$.

Let us consider a tensor field $\widetilde{J}$ of type $(1,1)$ on the product manifold $M \times \mathbb{R}$ such that

$$\widetilde{J}X = JX \text{ for any vector field } X \in \Gamma\left(D^\theta\right), \text{ that is, } \widetilde{J}E_1 = JE_1, \widetilde{J}E_2 = JE_2,$$

$$\widetilde{J}E_3 = (p - \sigma_{p,q})\frac{\partial}{\partial l} \text{ and } \widetilde{J}\left(\frac{\partial}{\partial l}\right) = (p - \sigma_{p,q})E_3,$$

where $\sigma_{p,q}$ is the (p,q)-metallic number and l is the parameter on $\mathbb{R}$. Hence, we derive that $\left(M \times \mathbb{R}, \langle,\rangle, \widetilde{J}\right)$ is a locally decomposable metallic Riemannian manifold.

Taking account of the definitions of the distributions D^θ and $D^\perp$, by a straightforward computation, it follows that D^θ is a slant distribution with the Wirtinger angle $\theta = \arccos\left(\frac{p}{\sqrt{p^2+\lambda}}\right)$ and $\widetilde{J}D^\perp = TM^\perp$. Thus, M is a 3-dimensional hemi-slant submanifold of the ambient manifold $\left(M \times \mathbb{R}, \langle,\rangle, \widetilde{J}\right)$.

Now, let us define a 2-form ψ and a 1-form ω by $\psi = \psi^1 \Lambda \psi^2$ and $\omega = \omega^1$, respectively, where

$$\psi^1 = dx_1$$

and

$$\psi^2 = \frac{1}{\sqrt{1+\sin^2 t}}\left(\sin t dx_2 + dx_3\right).$$

It is clear that $d\omega = \delta\omega = d\psi = \delta\psi = 0$. Hence, we find

$$\Delta\omega = \Delta\psi = 0,$$

which means that the forms ψ and ω are harmonic. Therefore, by Hodge Theorem, the cohomology groups $H^1_{dR}(M)$ and $H^2_{dR}(M)$ are non-trivial, in other words, $b_1 \neq 0$ and $b_2 \neq 0$.

Example 3.2. We consider a metallic Riemannian structure on the 8-dimensional Euclidean space $\left(\mathbb{R}^8, \langle,\rangle\right)$ given by

$$\widetilde{J}\left(\frac{\partial}{\partial x_\iota}\right) = \sigma_{p,q}\frac{\partial}{\partial x_\iota}, \ \iota = 1,3,5,7$$

and

$$\widetilde{J}\left(\frac{\partial}{\partial x_\kappa}\right) = (p - \sigma_{p,q})\frac{\partial}{\partial x_\kappa}, \ \kappa = 2,4,6,8,$$

where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is the local coordinates of $\mathbb{R}^8$ and $\sigma_{p,q}$ is the (p,q)-metallic number. Also, $\left(\mathbb{R}^8, \langle,\rangle, \widetilde{J}\right)$ is a locally decomposable metallic Riemannian manifold.

We consider a submanifold defined by the immerson $i : M \longrightarrow \mathbb{R}^8$ such that

$$i\left(u,v,w,t\right)=\left(u,\frac{\sigma_{p,q}}{\sqrt{q}}u,\frac{\sqrt{q}}{\sigma_{p,q}}v,v,w-t,w,w+t,t\right).$$

In this case, the tangent bundle TM is spanned by the following vector fields:

$$E_1=\sqrt{\frac{q}{\sigma_{p,q}^2+q}}\left(\frac{\partial}{\partial x_1}+\frac{\sigma_{p,q}}{\sqrt{q}}\frac{\partial}{\partial x_2}\right),$$

$$E_2=\frac{\sigma_{p,q}}{\sqrt{\sigma_{p,q}^2+q}}\left(\frac{\sqrt{q}}{\sigma_{p,q}}\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right),$$

$$E_3=\frac{1}{\sqrt{3}}\left(\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_6}+\frac{\partial}{\partial x_7}\right)$$

and

$$E_4=\frac{1}{\sqrt{3}}\left(-\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_7}+\frac{\partial}{\partial x_8}\right).$$

Choosing $D^\theta = Span\left\{E_3, E_4\right\}$ and $D^\perp = Span\left\{E_1, E_2\right\}$, then it can be easily seen that D^θ is a slant distribution with the Wirtinger angle $\theta = \arccos\left(\frac{p+\sigma_{p,q}}{\sqrt{3\left(2\sigma_{p,q}^2+(p-\sigma_{p,q})^2\right)}}\right)$ and $\widetilde{J}D^\perp \subset TM^\perp$. Hence, M is a 4-dimensional hemi-slant submanifold of the ambient manifold $\left(\mathbb{R}^8, \langle,\rangle, \widetilde{J}\right)$.

Now, we consider two 2-forms ψ and ω given by $\psi = \psi^1\Lambda\psi^2$ and $\omega = \omega^1\Lambda\omega^2$, respectively, where

$$\psi^1=\sqrt{\frac{q}{\sigma_{p,q}^2+q}}\left(dx_1+\frac{\sigma_{p,q}}{\sqrt{q}}dx_2\right),$$

$$\psi^2=\frac{\sigma_{p,q}}{\sqrt{\sigma_{p,q}^2+q}}\left(\frac{\sqrt{q}}{\sigma_{p,q}}dx_3+dx_4\right),$$

$$\omega^1=\frac{1}{\sqrt{3}}\left(dx_5+dx_6+dx_7\right)$$

and

$$\omega^2=\frac{1}{\sqrt{3}}\left(-dx_5+dx_7+dx_8\right).$$

In this situation, it is easy to verify that $d\omega = \delta\omega = d\psi = \delta\psi = 0$. Thus, we obtain

$$\Delta\omega = \Delta\psi = 0,$$

which says the forms ψ and ω are harmonic. Consequently, from Hodge Theorem, the cohomology group $H^2_{dR}(M)$ is non-trivial, i.e., the $2nd$ Betti number is not zero.

4. Conclusions and Future Work

In this paper, we have investigated the geometry of hemi-slant submanifolds in metallic Riemannian manifolds with the help of de Rham cohomology groups. In this sense, the paper establishes a connection between the associated distributions of a hemi-slant submanifold of locally decomposable metallic Riemannian manifolds and its the de Rham cohomology group. Concrete examples confirm the validity of the theoretical results of the paper. The results obtained in the paper can be extended to bi-slant submanifolds of metallic Riemannian manifolds. Also, it is well known that the de Rham cohomology is a generalization of Maxwell's theory of the electromagnetic field, so we hope that the current work will be useful in physics as well as in differential geometry.

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