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Araştırma Makalesi- Research Article

Traveling Wave Solutions of the Oskolkov Equation Arising in Incompressible Viscoelastic Kelvin–Voigt Fluid

Sıkıştırılamaz Visko Elastik Kelvin-Voigt Sıvısında Ortaya Çıkan Oskolkov Denkleminin Gezici Dalga Çözümleri

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ABSTRACT

In this manuscript, exact solutions of the Oskolkov equation, which describes the dynamics of incompressible viscoelastic Kelvin-Voigt fluid, are presented. The $(1/G')$ -expansion method is used to search for these solutions. The dynamics of the obtained exact solutions are analyzed with the help of appropriate parameters and presented with graphics. The applied method is efficient and reliable to search for fundamental nonlinear waves that enrich the various dynamical models seen in engineering fields. It is concluded that the analytical method used in the study of the Oskolkov equation is reliable, valid and useful tool for created traveling wave solutions.

Keywords- $(1/G')$ ***-Expansion Method, Oskolkov Equation, Traveling Wave Solution, Exact Solution***

ÖZ

Bu çalışmada, sıkıştırılamaz visko-elastik Kelvin-Voigt akışkanının dinamiklerini tanımlayan Oskolkov denkleminin tam çözümleri sunulmuştur. Bu çözümleri aramak için $(1/G')$ -açılım yöntemi kullanılmaktadır. Elde edilen tam çözümlerinin dinamikleri uygun parametreler yardımıyla analiz edilmiş ve grafiklerle sunulmuştur. Uygulanan yöntem, mühendislik alanlarında görülen çeşitli dinamik modelleri zenginleştiren temel doğrusal olmayan dalgaları aramak için etkili ve güvenilirdir. Oskolkov denkleminin çalşmasında kullanılan analitik metodun gezici dalga çözümlerini ortaya koymakta güvenilir, geçerli ve faydalı bir araç olduğu sonucu elde edilir.

Anahtar Kelimeler- $(1/G')$ ***-Açılım Yöntemi, Oskolkov Denklemi, Gezici Dalga Çözümü, Tam Çözüm***

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I. INTRODUCTION

Nonlinear partial differential equations (NPEs) are mathematical models of many physical phenomena we encounter in the universe. The physical phenomena discussed are modeled in mathematical terms. These mathematical models describing a physical event were in a simple form when they first appeared. Therefore, the mathematical solutions of these problems and the interpretation of the obtained solutions are easy. It is known that the closer a mathematical model is to reality, the more meaningful it is in the academic universe. To bring the simple form of a model closer to reality, many new terms and variables need to be added to the model. Then, over time, the problem becomes difficult to solve due to the reasons arising from the non-linear terms of this problem.

Recently, there has been much concentration on NPEs in fields as diverse as fluid mechanics, signal processing, mathematical physics, chemical physics, plasma physics, optics, solid state physics, and geochemistry [1, 2]. There are many methods for generating analytical solutions of NPEs have been employed successfully, such as tanh method [3], first integral method [4], new Kudryashov method [5], modified Kudryashov method [6], improved tanh method [7], sub-equation method [8], modified $(1/G')$-expansion method [9], auxiliary equation method [10], ansatz method [11], ϕ^6-model expansion method [12], Hirota bilinear method [13] and so on [14-21].

Consider the (1+1)-Dimensional Oskolkov equation of the form [2],

$$u_t - \beta u_{xxt} - \alpha u_{xx} + uu_x = 0, \tag{1}$$

here β, α are constants and u is a function of *x* and *t.*

The Oskolkov equation appears in various studies such as different types of traveling wave solutions of the Oskolkov equation have been investigated using the modified (G'/G)-expansion method [1], Ghanbari has been presented exact solutions for two Oskolkov-type equations [22], exact solutions have been attained for the Oskolkov equation [23], Thabet et al. have been presented exact solutions for Oskolkov equations with exponential rational function method [24], Gozukızıl and Akcagıl have been created the exact solutions for the Oskolkov equation via tanh-coth method [25].

There is no solution of this model with this method in the literature. The purpose of this study to generate traveling wave solutions for the Oskolkov equation of by using $(1/G')$-expansion method. This method produces a hyperbolic type solution of the equation.

The outline of this article is as follows; in section 2, material and method for NPEs is introduced. In section 3, we attain traveling wave solutions for the Oskolkov equation using the expansion method. In section 4, we present the conclusions and discussions. In section 5, the differences of the traveling wave solutions generated in this study are explained after reviewing the earlier works in the literature. We also provide some significant conclusions about our work.

II.MATERIAL AND METHOD

A. Description of the Method

For NPEs, consider $(1/G')$-expansion method [26]

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{2}$$

Let $u = u(\xi) = u(x,t), \quad \xi = x - ct, \quad c \neq 0,$ here c is constant and the velocity of the wave. Following nonlinear ordinary differential equation (nODE) for $u(\xi)$, we can transform it

$$W(u, u', u'', \ldots) = 0. \tag{3}$$

The solution of equation (3) is assumed that with the form

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$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \left(\frac{1}{G'}\right)^i, \tag{4}$$

in here a_i, $i = (1,...,n)$ are scalars and $G = G(\xi)$ enables following second-order linear ordinary differential equation (IODE)

$$G'' + \lambda G' + \mu = 0, \quad \lambda \; and \, \mu \in \mathbb{R}. \tag{5}$$

Solution defined as expansion method is as follows

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + A\cosh[\xi\lambda] - A\sinh[\xi\lambda]} = \frac{\lambda}{Ae^{-\xi\lambda}\lambda - \mu}. \tag{6}$$

The desired derivatives of equation (4) are calculated and inserted into equation (3), attaining a polynomial with $(1/G')$. By setting coefficients of the polynomial equal to zero, a system of algebraic equations is produced. Package program is used to solve the equation. After, the default equation (3) is replaced in the solution function. Solutions to equation (2) are eventually discovered.

III. SOLUTIONS OF THE OSKOLKOV EQUATION

Consider equation (1) and using $u = u(x,t) = u(\xi)$, $\xi = x - ct$, equation (1) is transformed to an ODE

$$-cU' + \beta cU''' - \alpha U'' + UU' = 0. \tag{7}$$

The balancing term is a constant obtained between the highest order linear term and the highest order nonlinear term in any nODE [27]. So, balancing between highest order linear term U''' with highest nonlinear term UU' in equation (7), we obtain $n = 2$ and considering in equation (4), following situation is attain

$$u(\xi) = a_0 + a_1\left(\frac{1}{G'}\right) + a_2\left(\frac{1}{G'}\right)^2, \; a_1 \neq 0 \;\; or \;\; a_2 \neq 0. \tag{8}$$

By writing equation (8) into equation (7) and equating coefficients of equation (1) to zero, systems of equations may be found in the form

$$\begin{aligned}
&\left(\frac{1}{G'[\xi]}\right)^1: \; -c\lambda a_1 - \alpha\lambda^2 a_1 + c\beta\lambda^3 a_1 + \lambda a_0 a_1 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^2: \; -c\mu a_1 - 3\alpha\lambda\mu a_1 + 7c\beta\lambda^2\mu a_1 + \mu a_0 a_1 + \lambda a_1^2 - 2c\lambda a_2 - 4\alpha\lambda^2 a_2 + 8c\beta\lambda^3 a_2 + 2\lambda a_0 a_2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^3: \; -2\alpha\mu^2 a_1 + 12c\beta\lambda\mu^2 a_1 + \mu a_1^2 - 2c\mu a_2 - 10\alpha\lambda\mu a_2 + 38c\beta\lambda^2\mu a_2 + 2\mu a_0 a_2 + 3\lambda a_1 a_2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^4: \; 6c\beta\mu^3 a_1 - 6\alpha\mu^2 a_2 + 54c\beta\lambda\mu^2 a_2 + 3\mu a_1 a_2 + 2\lambda a_2^2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^5: \; 24c\beta\mu^3 a_2 + 2\mu a_2^2 = 0.
\end{aligned} \tag{9}$$

Using the system of equation (9) we can provide the following solutions with the help of computer technology.

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Case1. If

$$\beta = -\frac{6\alpha^2}{25c\left(c-a_0\right)}, \quad a_1 = 0, \quad \lambda = -\frac{5\left(c-a_0\right)}{6\alpha}, \quad \mu = -\frac{5\sqrt{ca_2 - a_0 a_2}}{6\sqrt{2}\alpha}, \tag{10}$$

replacing the values equation (10) into equation (8) and we found hyperbolic solution for equation (1):

$$u_1(x,t) = a_0 + \frac{a_2}{\left(A cosh[\theta] + A sinh[\theta] - \frac{\sqrt{ca_2 - a_0 a_2}}{\sqrt{2}\left(c-a_0\right)}\right)^2}, \quad \theta = \frac{5\left(-ct+x\right)\left(c-a_0\right)}{6\alpha}. \tag{11}$$

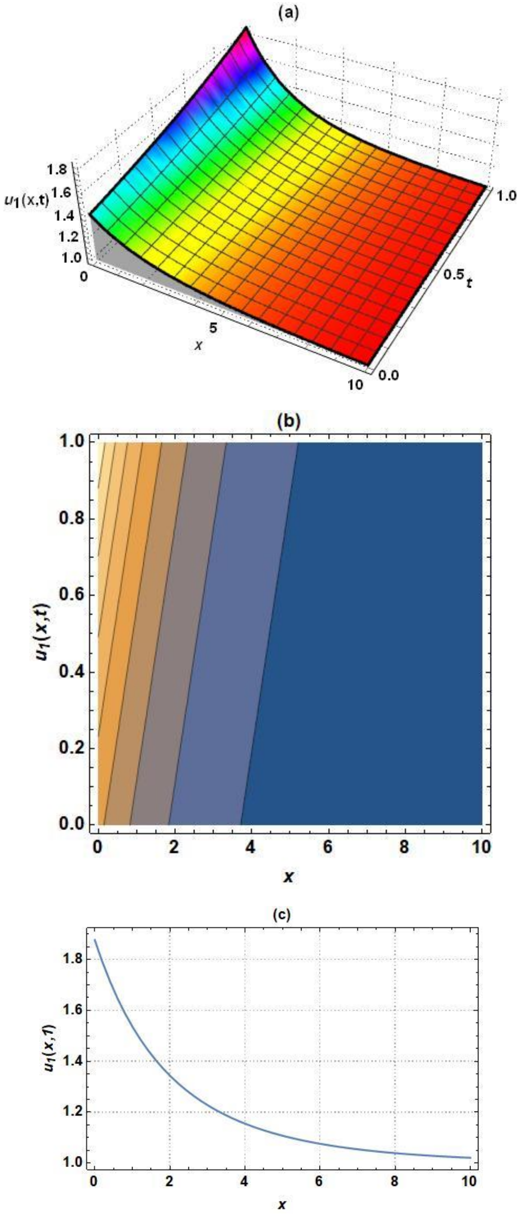

Figure 1. Graphs for $A = 4, a_0 = 1, a_2 = 2.5, c = 1.5, \alpha = 3.1$ values of equation (11).

Case2.

$$\beta = \frac{6\alpha^2}{25c\left(c-a_0\right)}, \quad a_1 = -2\sqrt{2}\sqrt{-\left(c-a_0\right)a_2}, \quad \lambda = -\frac{5\left(c-a_0\right)}{6\alpha}, \quad \mu = -\frac{5\sqrt{-ca_2+a_0a_2}}{6\sqrt{2}\alpha}, \tag{12}$$

by writing the values in equation (12) in equation (8), and found hyperbolic solution for equation (1)

$$u_2\left(x,t\right) = a_0 + \frac{a_2}{\left(A\cosh\left[\theta\right] + A\sinh\left[\theta\right] - \frac{\sqrt{-ca_2+a_0a_2}}{\sqrt{2}\left(c-a_0\right)}\right)^2} - \frac{2\sqrt{2}\sqrt{\left(-c+a_0\right)a_2}}{A\cosh\left[\theta\right] + A\sinh\left[\theta\right] - \frac{\sqrt{-ca_2+a_0a_2}}{\sqrt{2}\left(c-a_0\right)}}, \tag{13}$$

where $\theta = \dfrac{5\left(-ct+x\right)\left(c-a_0\right)}{6\alpha}$.

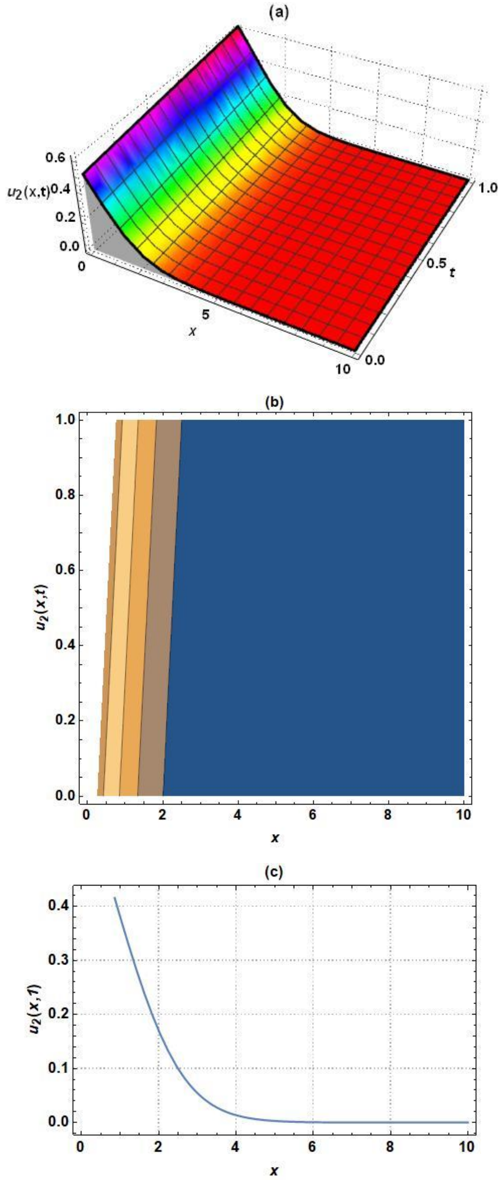

Figure 2. Graphs for $a_0 = 1, a_2 = 1.5, A = 3, c = 0.5, \alpha = 0.5$ values of equation (13).

Case3.

$$a_0 = \frac{72\alpha^2\mu^2 + 25ca_2}{25a_2}, \quad \beta = -\frac{a_2}{12c\mu^2}, \quad a_1 = \frac{24\alpha\mu}{5}, \quad \lambda = \frac{12\alpha\mu^2}{5a_2}, \tag{14}$$

modifying the values in (14) in (8), and we get hyperbolic type solution of equation (1)

$$u_3(x,t) = \frac{72\alpha^2\mu^2 + 25ca_2}{25a_2} + \frac{a_2}{\left(A\cosh[\psi] - A\sinh[\psi] - \frac{5a_2}{12\alpha\mu}\right)^2} + \frac{24\alpha\mu}{5\left(A\cosh[\psi] - A\sinh[\psi] - \frac{5a_2}{12\alpha\mu}\right)}, \tag{15}$$

where $\psi = \dfrac{12(-ct + x)\alpha\mu^2}{5a_2}$.

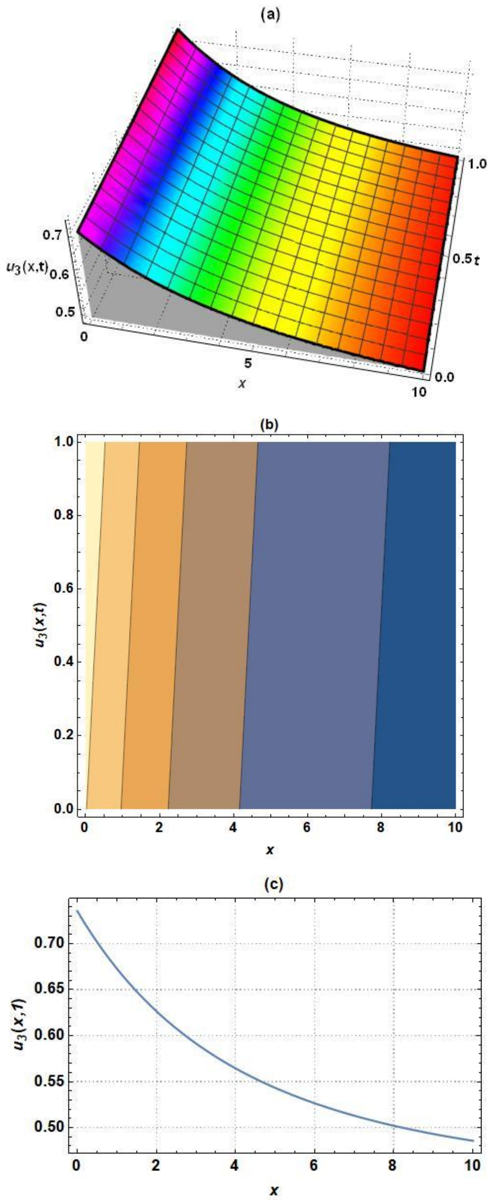

Figure 3. Graphs for $a_2 = 2, A = 2.5, c = 0.5, \alpha = 1, \mu = 0.2$ values of equation (15).

IV. RESULTS AND DISCUSSION

There are different methods to reach the exact solution of NPEs. The (1/G')-expansion technique is one of these techniques. In this manuscript, we attained the traveling wave solutions of the Oskolkov equation by using this expansion method. The solutions attained in this work are hyperbolic type wave solutions. The graphs shown represent the standing wave. While these graphics are attained, special values are given to the constants. The advantage of this method is that an easier algebraic equation system is attained compared to other methods. The only disadvantage of the method is that it produces a uniform solution function. In the work, it has been shown that expansion method is easier than other analytical methods in terms of process complexity. Thus, this method is easy and an effective method to find the solution. This expansion method can be easily used to NPEs.

Each method generates a unique set of solutions because of how it is structured, as is well known. This approach results in solutions for hyperbolic type propagating waves. This approach is intended to present many forms of solutions to the literature and has not been used for this equation. This approach has the benefit of being highly reliable and simple to use. The drawback is that it only generates one kind of solution. These solutions do, however, play an important role in the examination of the shock wave structure. Additionally, asymptotic behavior reviewers find the single point feature appealing.

V.CONCLUSIONS

Traveling wave solutions of Oskolkov equation have been presented using (1/G')-expansion method in this article. This method produces hyperbolic type oscillating wave solution.

Different types of solutions of the Oskolkov equation have been obtained using different methods in the literature. For example; Alam et al. have been presented periodic respiratory waves, kink wave, cusp wave and periodic wave solutions in their studies [1]. Roshid and Bashar have been presented kinky periodic wave and breather wave solutions using simple equation method [2]. Ghanbari has been obtained the exponential and hyperbolic type solutions of the Oskolkov equation [22].

In this study, solutions of Oskolkov equation in hyperbolic form are produced. This expansion method generates solutions in Eq. (6) format. In the solutions obtained, the 3-D, contour and 2-D graphs are presented by giving special values to the parameters. In addition, the advantage and disadvantage of applied method is given in the section 4.

The Mathematica ready package program was used to present these solutions. The obtained results demonstrate the algorithm's reliability and its applicability to nonlinear domains.

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