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Face-Counting Identities and Derivatives of Face Polynomials

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Abstract.
In this paper, we first introduce two incidence matrices related to the vertex-deck and the edge-deck of a given simplicial complex. Then, we obtain two face-counting identities based on these incidence matrices. Using these face-counting identities, we present combinatorial interpretations of the first and the second derivatives of face polynomials of simplicial complexes. We also propose several interesting open questions and conjectures. Finally, we conclude the paper with a discussion about some possible future research works.

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1. Introduction

One of the recent tools in the modern theory of data mining is the computational topology. The basic objects of study are discrete structures which are so-called *simplicial complexes* . A simplicial complex is a nonempty collection of sets which is closed under taking subset. It can be considered as a good discrete approximation of a surface (or in general a manifold) which arises in many areas of mathematical sciences and engineering disciplines. Considering the fact that the building blocks of simplicial complexes are *simplices* (or faces), problems related to counting faces have attracted the attention of many mathematicians and computer scientists [7].

In this paper, we take an algebraic approach to tackle these kind of problems through the lens of *face polynomials* or f-polynomials [1]. Indeed, we can associate with any given simplicial complex $\mathcal{K}$ a polynomial which can be considered as the generating function of the number of k-faces (faces with $k+1$ vertices) in $\mathcal{K}$. We will call this algebraic object the face polynomial of $\mathcal{K}$ and we will denote it by $f(\mathcal{K}, x)$. More precisely, if we denote the number of k-faces of $\mathcal{K}$ with $f_k(\mathcal{K})$, then we have

$$f(\mathcal{K}, x) = \sum_{k=0}^{\dim(\mathcal{K})+1} f_{k-1}(\mathcal{K}) x^k.$$

By convention, we will assume that $f_{-1}(\mathcal{K}) = 1$ for any arbitrary simplicial complex $\mathcal{K}$. Here, $\dim(\mathcal{K}) + 1$ denotes the number of vertices of the largest face of $\mathcal{K}$. In the literature, the vector

$$f(\mathcal{K}) = (f_{-1}(\mathcal{K}), f_0(\mathcal{K}), \ldots, f_{\dim(\mathcal{K})}(\mathcal{K})),$$

is called the *f-vector* [6] of the simplicial complex $\mathcal{K}$.
We recall that a non-empty *graph* G is a simplicial complex of dimension *one* and faces are called *cliques*. In this

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special case, the corresponding f-polynomial is called the *clique polynomial* of G and is denoted by $C(G, x)$. In [8], the author of this paper has shown graph-theoretical interpretations of the first and the second derivatives of the clique polynomial. To the best of our knowledge, the second derivative interpretation is new. Indeed, there are very few papers about the derivatives of graph polynomials [4, 5]. The main goal of this paper is to generalize the above results for the general class of simplicial complexes, based on the idea of face-counting identities. It is also important to note that, as far as we know, there is no *explicit* statement even about the first derivative of the face polynomial of a simplicial complex. Only in reference [3], the connection between the face polynomial of a given simplicial complex and links of its vertices is given in the *antiderivative* form (see Corollary 1 in [3]).
The paper will organize, as follows. In the next section, we first quickly review the basic theory of simplicial complexes. Next, in section three, we obtain two interesting face-counting identities using the idea of incidence matrices of k-faces versus the vertex-deck (or the edge-deck) of a given simplicial complex. Then, in section four, using these interesting identities, we obtain two combinatorial interpretations of the first and the second derivatives of the face polynomial of simplicial complexes. We finally conclude the paper with several interesting open questions and conjectures.

2. Simplicial Complexes

In this section, we briefly review some basic definitions and notations in the theory of *simplicial complexes*.
We first recall the well-known definition of the *affinely independent* set of points.

Definition 2.1. An n-family $(x_1, ..., x_n)$ of points from $\mathbb{R}^d$ is said to be *affinely independent* if a *linear combination* $\alpha_1 x_1 + \cdots + \alpha_n x_n$ with $\alpha_1 + \cdots + \alpha_n = 0$, can only have the value zero vector when $\alpha_1 = \cdots = \alpha_n = 0$.

Definition 2.2. For $n + 1$ affinely-independent set of points $S = \{v_0, \ldots, v_n\}$, the convex hull of S is defined as an n-simplex. The dimension of an n-simplex σ denoted by $\dim(\sigma)$ is defined to be n. A face τ of σ is a simplex that is a convex hull of a non-empty subset of S. A face with $k + 1$ vertices is called a k-face.

We note that a 0-face is called a *vertex* and a 1-face is also called an *edge*. We also denote the set of vertices and edges of a simplicial complex $\mathcal{K}$ with $V(\mathcal{K})$ and $E(\mathcal{K})$, respectively.

Definition 2.3. A simplicial complex is a set of simplices which is closed under the containment. In other words, a simplicial complex $\mathcal{K}$ is the set of simplices such that

$$\forall \sigma \in \mathcal{K}, \ \ \tau \subseteq \sigma \Rightarrow \tau \in \mathcal{K}.$$

The members of $\mathcal{K}$ are also called it's *faces*. The *dimension* of $\mathcal{K}$ is defined as the maximum of it's faces dimensions. The maximal faces (under inclusion) are called *facets*.

Definition 2.4. Any subset $\mathcal{K}' \subseteq \mathcal{K}$ which is itself a simplicial complex is called a *subcomplex* of $\mathcal{K}$.

Next, we introduce two important subcomplexes of a given simplicial complex.

Definition 2.5. Let $\mathcal{K}$ be a simplicial complex and σ be a face of $\mathcal{K}$. The *face-deleted* simplicial complex $\mathcal{K}\backslash\sigma$, is defined as

$$\mathcal{K}\backslash\sigma = \{\tau \in \mathcal{K} | \ \sigma \nsubseteq \tau\}.$$

Definition 2.6. Let $\mathcal{K}$ be a simplicial complex and σ be a face of $\mathcal{K}$. The *link* of σ, denoted by $lk_{\mathcal{K}}(\sigma)$, is defined as

$$lk_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K} | \ \sigma \cap \tau \neq \emptyset, \ \ \sigma \cup \tau \in \mathcal{K}\}.$$

Note that, the link of a face of a simplicial complex is itself a simplicial complex. It is also interesting to mention that the link of a vertex v in any graph G (as a simplicial complex of dimension at most one) is also called the (open) neighborhood of G and is denoted by $N_G(v)$. More precisely, we have

$$N_G(v) = \{u \in V(G) \mid uv \in E(G)\}.$$

We also note that, by a simple counting argument (addition rule), we can obtain the following key recurrence relations for the number of k-faces of any simplicial complex $\mathcal{K}$.

Lemma 2.7. *Let $\mathcal{K}$ be any simplicial complex with $v \in V(\mathcal{K})$ and $e \in E(\mathcal{K})$. Then, we have the following recurrence relations*

$$f_k(\mathcal{K}) = f_k(\mathcal{K}\backslash v) + f_{k-1}(lk_{\mathcal{K}}(v)) \qquad (k \geq 1),$$
$$f_k(\mathcal{K}) = f_k(\mathcal{K}\backslash e) + f_{k-2}(lk_{\mathcal{K}}(e)) \qquad (k \geq 2).$$

3. Incidence Matrices and Face-Counting Identities

In this section, we obtain two interesting face-counting identities related to the vertex-deck and the edge-deck of a given simplicial complex $\mathcal{K}$.

We recall that the *vertex-deck* of a simplicial complex $\mathcal{K}$ is defined as the collection $vdeck(\mathcal{K}) = \{\mathcal{K}\backslash v\}_{v\in V(\mathcal{K})}$ of all vertex-deleted subcomplexes of $\mathcal{K}$. Let $r = f_k(\mathcal{K})$ be the number of k-faces of $\mathcal{K}$ and $F_k(\mathcal{K}) = \{\tau_{k,1}, \ldots, \tau_{k,r}\}$ be the set of all those k-faces.

In the first step, we introduce an incidence matrix related to the vertex-deck of a simplicial complex;

$$(I_{k,v}(\mathcal{K}))_{\tau_{k,i},\mathcal{K}\backslash v_j} = \begin{cases} 1 & \text{if } \tau_{k,i} \text{ is a face of } \mathcal{K}\backslash v_j \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus, by a simple *counting argument* we obtain the first *face-counting* identity.

Proposition 3.1. *Let $\mathcal{K}$ be a simplicial complex with the vertex set $V(\mathcal{K})$ and $|V(\mathcal{K})| = n$. Then, we have*

$$(n-k)f_{k-1}(\mathcal{K}) = \sum_{v\in V(\mathcal{K})} f_{k-1}(\mathcal{K}\backslash v), \quad (k \geq 1). \tag{3.1}$$

In the next step, we introduce another incidence matrix related to the edge-deck of a simplicial complex. We recall that, the edge-deck of a simplicial complex $\mathcal{K}$ is defined as the collection $edeck(\mathcal{K}) = \{\mathcal{K}\backslash e\}_{e\in E(\mathcal{K})}$ of all edge-deleted subcomplexes of $\mathcal{K}$;

$$(I_{k,e}(\mathcal{K}))_{\tau_{k,i},\mathcal{K}\backslash e_j} = \begin{cases} 1 & \text{if } \tau_{k,i} \text{ is a face of } \mathcal{K}\backslash e_j \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Once again, by a double-counting argument on the incidence matrix $I_{k,e}(\mathcal{K})$, we get another face-counting identity.

Proposition 3.2. *Let $\mathcal{K}$ be a simplicial complex with the edge set $E(\mathcal{K})$ and $|E(\mathcal{K})| = m$. Then, we have*

$$\left(m - \binom{k}{2}\right)f_{k-1}(\mathcal{K}) = \sum_{e\in E(\mathcal{K})} f_{k-1}(\mathcal{K}\backslash e), \quad (k \geq 2). \tag{3.2}$$

Remark 3.3. It is worth to note that Propostions 3.1 and 3.2 can be seen as the simplicial version of the special cases of the well-known *Kelly's lemmas* [2] related to the *reconstruction conjecture* in graph theory.

4. Derivatives of Face Polynomials

In this section, we are going to obtain two interesting combinatorial interpretations for the first and the second derivatives of face polynomials of simplicial complexes.

We first recall the definition of the face polynomial (or f-polynomial) of a simplicial complex

Definition 4.1. Let $\mathcal{K}$ be a simplicial complex of dimension d. The face polynomial of $\mathcal{K}$ denoted by $f(\mathcal{K}, x)$ is defined by

$$f(\mathcal{K}, x) = 1 + \sum_{k=1}^{d+1} f_{k-1}(\mathcal{K})x^k.$$

We also recall that based on Lemma 2.7, we can easily prove the following recurrence relations for face polynomials. The vertex-recurrence relation for face polynomials is the following;

$$f(\mathcal{K}, x) = f(\mathcal{K}\backslash v, x) + xf(lk_{\mathcal{K}}(v), x). \tag{4.1}$$

We have also the following edge-recurrence relation for face polynomials;

$$f(\mathcal{K}, x) = f(\mathcal{K}\backslash e, x) + x^2 f(lk_{\mathcal{K}}(e), x). \tag{4.2}$$

The next theorem shows the connection between the first derivative of the face polynomial of a simplicial complex and the links of its vertices.

Theorem 4.2. *Let $\mathcal{K}$ be a simplicial complex with the vertex set $V(\mathcal{K})$. Then, we have*

$$\frac{d}{dx} f(\mathcal{K}, x) = \sum_{v\in V(\mathcal{K})} f(lk_{\mathcal{K}}(v), x).$$

Proof. The proof is proceed, as follows. We first multiply both sides of identity (3.1) by x^k and then we sum over all non-negative integers k. Hence, we get

$$\sum_{k\geq 0}(n-k)f_{k-1}(\mathcal{K})x^k = \sum_{k\geq 0}\sum_{v\in V(\mathcal{K})} f_{k-1}(\mathcal{K}\backslash v)x^k,$$

which is equivalent to (by interchanging summation) the following equality:

$$n\sum_{k\geq 0} f_{k-1}(\mathcal{K})x^k - x\sum_{k\geq 0} kf_{k-1}(\mathcal{K})x^{k-1} = \sum_{v\in V(\mathcal{K})}\left(\sum_{k\geq 0} f_{k-1}(\mathcal{K}\backslash v)x^k\right). \tag{4.3}$$

From the definition of the face polynomial, we conclude that

$$\frac{d}{dx}f(\mathcal{K},x) = \sum_{k\geq 1} kf_{k-1}(\mathcal{K})x^{k-1}. \tag{4.4}$$

Therefore, considering the equation (4.4), we can rewrite the equality (4.3) as follows:

$$n.f(\mathcal{K},x) - x\frac{d}{dx}f(\mathcal{K},x) = \sum_{v\in V(K)} f(\mathcal{K}\backslash v, x),$$

or equivalently,

$$\frac{d}{dx}f(\mathcal{K},x) = \sum_{v\in V(K)} \frac{f(\mathcal{K},x) - f(\mathcal{K}\backslash v,x)}{x}. \tag{4.5}$$

Thus, we finish the proof considering relations (4.5) and (4.1). □

The next theorem states the connection between the second derivative of a given simplicial complex and the links of its edges. To the best of our knowledge this is a new formula.

Theorem 4.3. *For a given simplicial complex $\mathcal{K}$ with the edges set $E(\mathcal{K})$, we have*

$$\frac{d^2}{dx^2}f(\mathcal{K},x) = 2!\sum_{e\in E(\mathcal{K})} f(lk_{\mathcal{K}}(e),x).$$

Proof. By multiplying the both sides of relation (3.2) by x^k and summing over all non-negative integers k, we get

$$\sum_{k\geq 0}\left(m-\binom{k}{2}\right)f_{k-1}(\mathcal{K})x^k = \sum_{k\geq 0}\sum_{e\in E(\mathcal{K})} f_{k-1}(\mathcal{K}\backslash e)x^k,$$

or equivalently (by interchanging the summation), we obtain

$$m\sum_{k\geq 0} f_{k-1}(\mathcal{K})x^k - x^2\sum_{k\geq 0}\binom{k}{2}f_{k-1}(\mathcal{K})x^{k-2} = \sum_{k\geq 0}\sum_{e\in E(\mathcal{K})} f_{k-1}(\mathcal{K}\backslash e)x^k. \tag{4.6}$$

Next, by the definition of the second derivative of the face polynomial, we have

$$\frac{d^2}{dx^2}f(\mathcal{K},x) = 2!\sum_{k\geq 2}\binom{k}{2}f_{k-1}(\mathcal{K})x^{k-2}. \tag{4.7}$$

Hence, considering relation (4.7) and the definition of a face polynomial, we can rewrite equation (4.6) as follows

$$m.f(\mathcal{K},x) - x^2\left(\frac{1}{2!}\frac{d^2}{dx^2}f(\mathcal{K},x)\right) = \sum_{e\in E} f(\mathcal{K}\backslash e, x)$$

or equivalently,

$$\frac{d^2}{dx^2}f(\mathcal{K},x) = 2!\sum_{e\in E(K)} \frac{f(\mathcal{K},x) - f(\mathcal{K}\backslash e,x)}{x^2}. \tag{4.8}$$

Finally, relations (4.8) and (4.2) yield the desired result. □

5. Open Questions and Conjectures

In this section, we propose several interesting open questions and conjectures.
It seems that one possible way to find a combinatorial interpretation of the third derivative of face polynomials is to find the analogues of the two face-counting identities (3.1) and (3.2) for the so-called *triangle-deck* of simplicial complexes. Therefore, the following open question naturally arises.

Open Question 5.1. *For which classes of simplicial complexes the following identity is true?*

$$\left(t-\binom{k}{3}\right)f_{k-1}(\mathcal{K})=\sum_{\delta\in T(\mathcal{K})}f_{k-1}(\mathcal{K}\backslash\delta),\qquad (k\geq 3).\tag{5.1}$$

Here, t is the number of triangles (2-simplces) of $\mathcal{K}$ and $T(\mathcal{K})$ denotes the set of all triangles of $\mathcal{K}$.

We first note that, the identity (5.1) is not true in general. Here is a counter-example. Consider the simplicial complex $\mathcal{K}_1$, shown in figure Fig1.

$\mathcal{K}_1$:

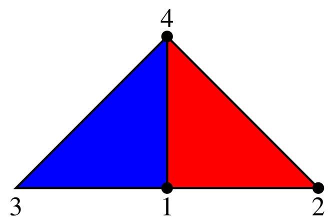

Fig1. The simplicial complex $\mathcal{K}_1$ which contradicts the identity (5.1).

Then, for $k = 3$, the left-hand side is equal to $(2-1)2 = 2$ but the right-hand side equals to $0 + 0 = 0$.
For any simplicial complex $\mathcal{K}$, we can associate a discrete object which we call it the *triangle simplicial complex* of $\mathcal{K}$, as follows.

Definition 5.1. Let $\mathcal{K}$ be a simplicial complex and $T(\mathcal{K})$ be the set of its 2-faces. Then, we define the *triangle simplicial complex* of $\mathcal{K}$ denoted by $Tri(\mathcal{K})$ as a simplicial complex with the vertex set $T(\mathcal{K})$ and $\tau = \{i_1, \ldots, i_k\}$ is a face of $Tri(\mathcal{K})$ whenever triangles i_l $(l = 1, \ldots, k)$ of $\mathcal{K}$ share an edge.

For example, the triangle simplicial complex of $\mathcal{K}_1$ in figure Fig1 is an edge (or a 1-simplex).
Based on the above definition, we made the following conjecture. We recall that, an *empty* simplicial complex is the one with non-empty vertex set V and no k-faces with $k > 0$.

Conjecture 5.1. *If the triangle simplicial complex of a given simplicial complex $\mathcal{K}$ is an empty simplicial complex, then the identity (5.1) is true for $\mathcal{K}$.*

In the same direction, we also come up with the following question.

Open Question 5.2. *For which classes of simplicial complexes of the following triangle-recurrence relation is true?*

$$f(\mathcal{K},x)=f(\mathcal{K}\backslash\delta,x)+x^3f(lk_{\mathcal{K}}(\delta),x).\tag{5.2}$$

We note that, this identity is also not valid in general. One can easily check that it is not true for a *tetrahedron* (or a 3-simplex) and any facet (triangle) of it.
Next, we note that despite the fact that identities (5.1) and (5.2) are not true in general, but we still believe that the following combinatorial interpretation of the third derivative of face polynomials is true for any simplicial complex.

Conjecture 5.2. *For a given simplicial complex $\mathcal{K}$ with the set of triangles $T(\mathcal{K})$, we have*

$$\frac{d^3}{dx^3}f(\mathcal{K},x)=3!\sum_{\delta\in T(\mathcal{K})}f(lk_{\mathcal{K}}(\delta),x).$$

6. Concluding Remarks and Future Works

In this paper, we obtained two combinatorial interpretations of the first and second derivatives of a face polynomial of a simplicial complex based on two corresponding face-counting identities.
One possible line of research for future is to get combinatorial interpretations of higher-order derivatives of f-polynomials. Another possible idea is to define the following new version of the face polynomial of a simplicial complex $\mathcal{K}$ (see [3]):

$$f_{\mathcal{K}}(x, y) = \sum_{i,j} f_{ij}(\mathcal{K}) x^i y^j,$$

where $f_{ij}(\mathcal{K})$ denotes the number of intersecting i and j-faces in $\mathcal{K}$. Then, our goal will be to find some possible combinatorial interpretations of *partial derivatives* of the above multivariate face polynomial of $\mathcal{K}$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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