Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 1, 148–161.
https://doi.org/10.31197/atnaa.1122002
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

A new sequential proportional fractional derivative of hybrid differential equations with nonlocal hybrid condition

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Abstract

In this paper, we study the existence of solutions for a new problem of hybrid differential equations with nonlocal integro multi point boundary conditions by using the proportional fractional derivative. The presented results are obtained by using hybrid fixed point theorems for three Dhage operators. The application of theoretical conclusions is demonstrated through an example.

Keywords: Hybrid fixed point theorem Proportional fractional derivative Existence of solution.
2010 MSC: 34A38; 32A65; 26A33.

1. Introduction

In the last few decades, fractional differentiation and fractional integration have found many applications in various filelds of science and engineering. That is why this theory has gained widespread attention and significance; see, for example, the papers ([1]-[11]). Various approaches of fractional derivatives have been proposed and the most well-known types are Riemann-Liouville, Caputo, Hadamard, Caputo-Fabrizio, mean square fractional derivatives, and so on;(see [13]-[26]).

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Received May 26, 2022; Accepted: December 20, 20222; Online: January 4, 2023.

A new class of mathematical modelings based on hybrid fractional differential equations with hybrid or non hybrid boundary value conditions has piqued the interest of numerous academics; see [1, 6, 18, 19, 27]). Hybrid differential equations are significant because they incorporate a variety of dynamical systems as special instances. In addition, hybrid differential equations may be found in a wide range of applications in applied mathematics and physics; see [12, 14, 25].
M.I. Abbas and M.A.Ragusa [1] treated the following hybrid fractional differential equation problem:

$$\begin{cases} {}_a\mathfrak{D}^{\delta,\rho,\upsilon}\left(\frac{u(t)}{\Psi(t,u(t))}\right)=\Phi(t,u(t)),\ \text{and}\ t\in[a,b] \\ {}_a\mathfrak{J}^{\delta,\rho,\varphi}\left(\frac{u(t)}{\Psi(t,u(t))}\right)_{t=a}=\lambda\in\mathbb{R}, \end{cases}$$

where $0<\delta<1$, $\rho\in(0,1]$, ${}_a\mathfrak{D}^{\delta,\rho,\upsilon}$ is the proportional fractional derivative of order δ with respect to a certain continuously differentiable and increasing function υ with $\upsilon'(t)>0$ for all $t\in[a,b]$, ${}_a\mathfrak{J}^{1-\delta,\rho,\upsilon}$ is the left proportional fractional integral of order $(1-\delta)$ with respect to a continuously differentiable and increasing function υ; $\Psi: J\times\mathbb{R}\to\mathbb{R}^*$ and $\Phi: J\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

In the present research work, for $t\in[0,1]$, we study the following problem:

$$\begin{cases} {}_0\mathfrak{D}^{\alpha,\rho,\varphi}\left[{}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F})(t,u(t))}{\mathbb{G}(t,u(t))}\right)\right]=\mathbb{H}(t,u(t)),\ t\in J=[0,1], \\ {}_0\mathfrak{J}^{1-\alpha,\rho,\varphi}\left({}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F})(t,u(t))}{\mathbb{G}(t,u(t))}\right)\right)\Big|_{t=0^+}=\lambda, \\ {}_0\mathfrak{J}^{2-\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F})(t,u(t))}{\mathbb{G}(t,u(t))}\right)\Big|_{t=0^+}=0, \\ {}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F})(t,u(t))}{\mathbb{G}(t,u(t))}\right)\Big|_{t=0^+}=\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta),\ \zeta\in\,]0,1[\,. \end{cases} \tag{HP}$$

Here, we take $\rho\in(0,1]$, ${}_0\mathfrak{D}^{\alpha,\rho,\varphi}$, ${}_0\mathfrak{D}^{\beta,\rho,\varphi}$ as the proportional fractional derivatives of orders, $0<\alpha<1<\beta<2$, and ${}_0\mathfrak{J}^{1-\alpha,\rho,\varphi}$, ${}_0\mathfrak{J}^{1-\beta,\rho,\varphi}$, ${}_0\mathfrak{J}^{2-\beta,\rho,\varphi}$, ${}_0\mathfrak{J}^{\sigma,\rho,\varphi}$ are the left proportional fractional integrals of orders $(1-\alpha)$, $(1-\beta)$, $(2-\beta)$ and σ respectively, $\lambda\in\mathbb{R}$ and $\varphi: J\to\mathbb{R}$ is a function such that $\varphi'(t)>0$. For all $t\in[0,1]$, $\mathbb{F},\mathbb{H}: J\times\mathbb{R}\to\mathbb{R}$ and $\mathbb{G}: J\times\mathbb{R}\to\mathbb{R}^*$ are given functions satisfying some assumptions that will be specified later.

2. Preliminaries

Let $\mathcal{C}=C(J,\mathbb{R})$ be the Banach space of all continuous mappings from $[0,1]$ to $\mathbb{R}$ endowed with the norm $\|u\|_{\mathcal{C}}=\sup_{t\in[0,1]}u(t)$.
We introduce some notations and definitions of proportional fractional derivative, see [15, 20, 21]. Let $\varphi: J\to\mathbb{R}$ be an increasing function with $\varphi'(t)\neq 0$ for all $t\in J$ and for all $t,s\in J,(t>s)$, we pose

$$\varphi(t,s)=(\varphi(t)-\varphi(s)).$$

Definition 2.1. [20] Take $\rho\in(0,1],\alpha\in\mathbb{C}$, Re $Re(\alpha)>0$, $\varphi\in C^1[a,b]$, $\varphi'(t)>0$. The left and right fractional integrals of the function $x\in L^1[a,b]$ with respect to another function φ are defined by

$${}_a\mathfrak{J}^{\alpha,\rho,\varphi}x(t)=\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\alpha-1}\varphi'(s)x(s)ds, \tag{2.1}$$

$$\mathfrak{J}_b^{\alpha,\rho,\varphi}x(t)=\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_t^b e^{\frac{\rho-1}{\rho}\varphi(s,t)}\varphi(s,t^{\alpha-1}\varphi'(s)x(s)ds, \tag{2.2}$$

respectively.

Definition 2.2. [20] Take $\rho \in (0,1], \alpha \in \mathbb{C}$, Re $Re(\alpha) > 0$, $\varphi \in C^1[a,b]$, $\varphi'(t) > 0$. The left fractional derivative of the function $x \in C^n[a,b]$ with respect to another function φ is defined by

$$\begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho,\varphi}x(t) &= \mathfrak{D}^{n,\rho,\varphi}\left({}_a\mathfrak{J}^{n-\alpha,\rho,\varphi}x\right)(t) \\ &= \frac{\mathfrak{D}^{n,\rho,\varphi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{n-\alpha-1}\varphi'(s)x(s)ds, \end{aligned} \tag{2.3}$$

and the right fractional derivative of x with respect to φ is defined by

$$\begin{aligned} \mathfrak{D}_b^{\alpha,\rho,\varphi}x(t) &= {}_*\mathfrak{D}^{n,\rho,\varphi}\mathfrak{J}_b^{n-\alpha,\rho,\varphi}x(t) \\ &= \frac{{}_*\mathfrak{D}^{n,\rho,\varphi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b e^{\frac{\rho-1}{\rho}\varphi(s,t)}\varphi(s,t)^{n-\alpha-1}\varphi'(s)x(s)ds, \end{aligned} \tag{2.4}$$

where $n = [\text{Re } Re(\alpha)] + 1, \mathfrak{D}^{n,\rho,\varphi} = \underbrace{\mathfrak{D}^{n,\rho,\varphi}\cdots\mathfrak{D}^{n,\rho,\varphi}}_{n \text{ times}}$ and

$$\begin{aligned} {}_*\mathfrak{D}^{\rho,\varphi}x(t) &= (1-\rho)x(t) - \rho\frac{x'(t)}{\varphi'(t)}, \\ {}_*\mathfrak{D}^{n,\rho,\varphi} &= \underbrace{{}_*\mathfrak{D}^{n,\rho,\varphi}\cdots{}_*\mathfrak{D}^{n,\rho,\varphi}}_{n \text{ times}}. \end{aligned}$$

Lemma 2.3. *[20] If $\rho \in (0,1], \alpha, \beta \in \mathbb{C}$, Re $Re(\alpha) > 0$, Re $Re(\beta) > 0$, then for $\varphi \in C^1[a,b]$, and $\varphi'(t) > 0$, we have*

$${}_a\mathfrak{J}^{\alpha,\rho,\varphi}\left({}_a\mathfrak{J}^{\beta,\rho,\varphi}x\right)(t) = {}_a\mathfrak{J}^{\beta,\rho,\varphi}\left({}_a\mathfrak{J}^{\alpha,\rho,\varphi}x\right)(t) = \left({}_a\mathfrak{J}^{\alpha+\beta,\rho,\varphi}x\right)(t), \tag{2.5}$$

$$\mathfrak{J}_b^{\alpha,\rho,\varphi}\left(\mathfrak{J}_b^{\beta,\rho,\varphi}x\right)(t) = \mathfrak{J}_b^{\beta,\rho,\varphi}\left(\mathfrak{J}_b^{\alpha,\rho,\varphi}x\right)(t) = \left(\mathfrak{J}_b^{\alpha+\beta,\rho,\varphi}x\right)(t), \tag{2.6}$$

Lemma 2.4. *[20] If $\rho \in (0,1], \alpha \in \mathbb{C}$, Re $Re(\alpha) > 0$, and $n = [\text{Re } Re(\alpha)] + 1$, then for $\varphi \in C^1[a,b]$, $\varphi'(t) > 0$, we have*

$${}_a\mathfrak{J}^{\alpha,\rho,\varphi}\left({}_a\mathfrak{D}^{\alpha,\rho,\varphi}x\right)(t) = x(t), \tag{2.7}$$

$$\mathfrak{J}_b^{\alpha,\rho,\varphi}\left(\mathfrak{D}_b^{\alpha,\rho,\varphi}x\right)(t) = x(t). \tag{2.8}$$

Lemma 2.5. *[21] Let $\alpha \in \mathbb{C}$, Re $Re(\alpha) > 0$,$\rho \in (0,1]$, $n = -[-\text{Re } Re(\alpha)]$, $x \in L^1[a,b]$ and $\mathfrak{J}_{a^+}^{\alpha,\rho,\varphi}x(t) \in AC^n[a,b]$. Then*

$${}_a\mathfrak{J}^{\alpha,\rho,\varphi}\left({}_a\mathfrak{D}^{\alpha,\rho,\varphi}x\right)(t) = x(t) - e^{\frac{\rho-1}{\rho}\varphi(t,a)}\sum_{i=1}^{n}\left({}_a\mathfrak{J}^{i-\alpha,\rho,\varphi}x\right)\left(a^+\right)\frac{\varphi(t,a)^{\alpha-i}}{\rho^{\alpha-i}\Gamma(\alpha+1-i)}. \tag{2.9}$$

As a particular case, for $0 < \alpha < 1$, we have

$${}_a\mathfrak{J}^{\alpha,\rho,\varphi}\left({}_a\mathfrak{D}^{\alpha,\rho,\varphi}x\right)(t) = x(t) - \frac{e^{\frac{\rho-1}{\rho}\varphi(t,a)}\varphi(t,a)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}\left({}_a\mathfrak{J}^{1-\alpha,\rho,\varphi}x\right)\left(a^+\right). \tag{2.10}$$

Lemma 2.6. *[21] Let $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha) > 0$,$Re(\beta) > 0 \geq 0$ and $n = [Re(\alpha)] + 1$, Then, for any $\rho > 0$, we have*

(1) $\left({}_a\mathfrak{J}^{\alpha,\rho,\varphi}e^{\frac{\rho-1}{\rho}\varphi(s)}\varphi(s,a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)e^{\frac{\rho-1}{\rho}\varphi(t)}}{\rho^{\alpha}\Gamma(\alpha+\beta)}\varphi(t,a)^{\alpha+\beta-1},\ Re(\beta) > 0,.$

(2) $\left(\mathfrak{J}_b^{\alpha,\rho,\varphi}e^{-\frac{\rho-1}{\rho}\varphi(s)}\varphi(b,s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)e^{-\frac{\rho-1}{\rho}\varphi(t)}}{\rho^{\alpha}\Gamma(\alpha+\beta)}\varphi(b,t)^{\alpha+\beta-1},\ Re(\beta) > 0,$

(3) $\left({}_a\mathfrak{D}^{\alpha,\rho,\varphi} e^{\frac{\rho-1}{\rho}\varphi(s)}\varphi(s,a)^{\beta-1}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)e^{\frac{\rho-1}{\rho}\varphi(t)}}{\Gamma(\beta-\alpha)}\varphi(t,a)^{\beta-\alpha-1}$,

(4) $\left(\mathfrak{D}_b^{\alpha,\rho,\varphi} e^{-\frac{\rho-1}{\rho}\varphi(s)}\varphi(b,s)^{\beta-1}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)e^{-\frac{\rho-1}{\rho}\varphi(t)}}{\Gamma(\beta-\alpha)}\varphi(b,t)^{\beta-\alpha-1}$.

Remark 2.7. In view of Definition 2.2 and for $0<\beta<1$, it is noted that

$${}_0\mathfrak{D}_0^{\beta,\rho,\varphi}\left(e^{\frac{\rho-1}{\rho}\varphi(t)}\varphi(t,0)^{\beta-1}\right) = 0.$$

Lemma 2.8. *[15] Let $\mathcal{X}$ be a closed convex bounded nonempty subset of a Banach algebra $\mathcal{E}$; and let $\mathbb{T},\mathbb{N}:\mathcal{E}\to\mathcal{E}$ and $\mathbb{M}:\mathcal{X}\to\mathcal{E}$ be three operators such that:*
(a) $\mathbb{T}$ and $\mathbb{N}$ are Lipschitzian with constants τ and σ, respectively,
(b) $\mathbb{N}$ is compact and continuous,
(c) $u=\mathbb{T}u\mathbb{M}v+\mathbb{N}u \Longrightarrow u\in\mathcal{X}$ for all $v\in\mathcal{X}$,
(d) $\tau K+\sigma<1$, where $K=\mathbb{M}(\mathcal{X})$.
Then the operator equation $\mathbb{T}u\mathbb{M}u+\mathbb{N}u=u$ has a solution in $\mathcal{X}$.

3. Main Results

Definition 3.1. A function $u\in C(J,\mathbb{R})$ is said to be a mild solution of the hybrid fractional problem (HP) if the function

$$t\longmapsto \frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}$$

is continuous for each $u\in\mathbb{R}$ and u satisfies the fractional integral equation

$$\begin{aligned} u(t) &= g(t)\left\{\left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\right. \\ &\quad \left.+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)\right\}+\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t). \end{aligned} \tag{3.1}$$

Now, we consider the following linear issue of the hybrid fractional problem (HP):

$$\begin{cases} {}_0\mathfrak{D}^{\alpha,\rho,\varphi}\left[{}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\right]=h(t), \\ {}_0\mathfrak{J}^{1-\alpha,\rho,\varphi}\left({}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\right)\Big|_{t=0}=\lambda, \\ {}_0\mathfrak{J}^{2-\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\Big|_{t=0}=0, \\ {}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\Big|_{t=0}=\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta),\zeta\in\,]0,1[\,. \end{cases} \tag{HP}$$

where $f,h\in L^1(J,\mathbb{R})$ and $g\in L^1(J,\mathbb{R}^*)$.

Lemma 3.2. *Let $0<\alpha<1<\beta<2$; $f,h\in L^1(J,\mathbb{R})$ and $g\in L^1(J,\mathbb{R}^*)$. The linear hybrid fractional problem (HP) has a solution $u\in C(J,\mathbb{R})$ if and only if the fractional integral equation (3.1) is solvable and their solutions coincide.*

Proof. 1/ ($\Longrightarrow$) By (3.1) we get

$$\begin{aligned} &\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)} \\ &= \left\{\left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\right. \\ &\quad \left.+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)\right\}, \end{aligned}$$

and assume that u satisfies (HP). Then, $\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}$ is continuous and we get that ${}_0\mathfrak{D}^{\alpha,\rho,\varphi}\left[{}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\right]$ exists.
Applying the proportional fractional integral $\mathfrak{J}_{0^+}^{\alpha,\rho,\varphi}$ to both sides of (HP) and using Lemma 2.5 and Lemma 2.6, and by taking account the first condition on $t=0$, we obtain

$$\begin{aligned}
&{}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\\
=\;&\left({}_0\mathfrak{J}^{\alpha,\rho,\varphi}h\right)(t)\\
&+e^{\frac{\rho-1}{\rho}\varphi(t,0)}\left[{}_0\mathfrak{J}^{1-\alpha,\rho,\varphi}\left({}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\right)\right]_{t=0^=}\frac{(\varphi(t,0))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}\\
=\;&\left({}_0\mathfrak{J}^{\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)}(\varphi(t,0))^{\alpha-1}.
\end{aligned}$$

Applying now the proportional fractional integral $\mathfrak{J}_{0^+}^{\beta,\rho,\varphi}$ to both sides, we get

$$\begin{aligned}
&\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\\
=\;&{}_0\mathfrak{J}^{\beta,\rho,\varphi}\left(\left({}_0\mathfrak{J}^{\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)}(\varphi(t,0))^{\alpha-1}\right)\\
&+e^{\frac{\rho-1}{\rho}\varphi(t,0)}\left[{}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\right]_{t=0^+}\frac{(\varphi(t,0))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\\
&+e^{\frac{\rho-1}{\rho}\varphi(t,0)}\left[{}_0\mathfrak{J}^{2-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\right]_{t=0^+}\frac{(\varphi(t,0))^{\beta-2}}{\rho^{\beta-2}\Gamma(\beta-1)}.
\end{aligned}$$

By Lemma 2.6 and the second condition on $t=0$, we get

$$\begin{aligned}
&\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\\
=\;&\left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}h\right)(t)+\left({}_0\mathfrak{J}^{\beta,\rho,\varphi}\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)}(\varphi(t,0))^{\alpha-1}\right)\\
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)\\
=\;&\left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{1-\rho}{\rho}\varphi(0)}}{\rho^{\alpha-1}\Gamma(\alpha)}\left({}_0\mathfrak{J}^{\beta,\rho,\varphi}e^{\frac{\rho-1}{\rho}\varphi(t)}(\varphi(t,0))^{\alpha-1}\right)\\
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)\\
=\;&\left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}h\right)(t)+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\\
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta).
\end{aligned}$$

So, the fractional integral equation (3.1) is obtained.
2/ ($\Longleftarrow$) Conversely, assume that u satisfies (3.1). By definition, the function $t\longmapsto\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}$ is

continuous for each $u \in C(J, \mathbb{R})$. Then

$$\begin{aligned} & \frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \\ = & \left\{ \left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi} h\right)(t) + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \varphi(t,0)^{\alpha+\beta-1} \right. \\ & \left. + \frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)} \varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} u\right)(\zeta) \right\}. \end{aligned} \tag{3.2}$$

Operating the proportional fractional derivative ${}_0\mathfrak{D}^{\beta,\rho,\varphi}$ on both sides of (3.2), we get

$$\begin{aligned} & {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \right) \\ = & {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi} h\right)(t) + {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \varphi(t,0)^{\alpha+\beta-1} \right) \\ & + {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)} \varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} u\right)(\zeta) \right). \end{aligned} \tag{3.3}$$

By using Lemma 2.3, Lemma 2.4, Lemma 2.6 and Remark 2.7, we obtain

$$\begin{aligned} & {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \right) \\ = & {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left({}_0\mathfrak{J}_0^{\beta,\rho,\varphi} \mathfrak{J}^{\alpha,\rho,\varphi} h\right)(t) + \frac{\lambda e^{\frac{1-\rho}{\rho}\varphi(0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \; {}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(e^{\frac{\rho-1}{\rho}\varphi(t)} \varphi(t,0)^{\alpha+\beta-1} \right) \\ & + \frac{e^{\frac{1-\rho}{\rho}\varphi(0)} \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} u\right)(\zeta)}{\rho^{\beta-1}\Gamma(\beta)} \; {}_0\mathfrak{D}_0^{\beta,\rho,\varphi} \left(e^{\frac{\rho-1}{\rho}\varphi(t)} \varphi(t,0)^{\beta-1} \right) \\ = & \left({}_0\mathfrak{J}^{\beta,\rho,\varphi} h\right)(t) + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)} \varphi(t,0)^{\alpha-1}. \end{aligned} \tag{3.4}$$

Operating the proportional fractional derivative ${}_0\mathfrak{D}^{\alpha,\rho,\varphi}$ on both sides of (3.4), we get

$$\begin{aligned} & {}_0\mathfrak{D}^{\alpha,\rho,\varphi} \left[{}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \right) \right] \\ = & {}_0\mathfrak{D}^{\alpha,\rho,\varphi} \left({}_0\mathfrak{J}^{\alpha,\rho,\varphi} h\right)(t) + {}_0\mathfrak{D}^{\alpha,\rho,\varphi} \left(\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)} \varphi(t,0)^{\alpha-1} \right). \end{aligned} \tag{3.5}$$

By Lemma 2.4 and Remark 2.7, we obtain

$$ {}_0\mathfrak{D}^{\alpha,\rho,\varphi} \left[{}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \right) \right] = h(t). $$

By (3.5) and Lemma 2.6, we have

$$\begin{aligned} & {}_0\mathfrak{J}^{1-\alpha,\rho,\varphi} \left[{}_0\mathfrak{D}^{\beta,\rho,\varphi} \left(\frac{u(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi} f\right)(t)}{g(t)} \right) \right] \\ = & {}_0\mathfrak{J}^{1,\rho,\varphi} h(t) + \lambda \, {}_0\mathfrak{J}^{1-\alpha,\rho,\varphi} \left(\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha-1}\Gamma(\alpha)} \varphi(t,0)^{\alpha-1} \right) \\ = & {}_0\mathfrak{J}^{1,\rho,\varphi} h(t) + \lambda. \end{aligned}$$

Substitution $t \to 0$ leads to ${}_0\mathfrak{J}^{1-\alpha,\rho,\varphi}\left({}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{u(t)-({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f)(t)}{g(t)}\right)\right)\Big|_{t=0^+} = \lambda.$
By (3.3), Lemma 2.6 and Remark 2.7, we have

$$\begin{aligned}
&{}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\\
=\ & \left({}_0\mathfrak{J}^{\alpha,\rho,\varphi}h\right)(t) + {}_0\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\right)\\
& + \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)_0\,\mathfrak{D}^{\beta,\rho,\varphi}\left(\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\right)\\
=\ & \left({}_0\mathfrak{J}^{\alpha,\rho,\varphi}h\right)(t) +_0 \mathfrak{J}^{\sigma,\rho,\varphi}u(\zeta)\\
& + \frac{\lambda}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\ {}_0\mathfrak{J}^{\alpha,\rho,\varphi}\left({}_0\mathfrak{D}^{\alpha+\beta,\rho,\varphi}\left(e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\alpha+\beta-1}\right)\right),
\end{aligned}$$

and

$${}_0\mathfrak{J}^{2-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right) =_0 \mathfrak{J}^{1,\rho,\varphi}\left({}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\right).$$

Substitution $t \to 0$ leads to

$${}_0\mathfrak{J}^{1-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\Bigg|_{t=0} =_0 \mathfrak{J}^{\sigma,\rho,\varphi}u(\zeta)$$

and

$${}_0\mathfrak{J}^{2-\beta,\rho,\varphi}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}f\right)(t)}{g(t)}\right)\Bigg|_{t=0} = 0.$$

This finishes the proof. □

3.1. Hypotheses

$(\mathbb{HP}_0)$ The functions $\mathbb{F}, \mathbb{G}$ and $\mathbb{H}$ are continuous functions.
$(\mathbb{HP}_1)$ There exist three positive functions $\mathcal{K}_{\mathcal{F}}, \mathcal{K}_{\mathcal{G}}$, and $\mathcal{K}_{\mathcal{H}}$ such that:

$$|\mathbb{F}(t,u)-\mathbb{F}(t,\widetilde{u})| \le \mathcal{K}_{\mathcal{F}}|u-\widetilde{u}|,$$

$$|\mathbb{G}(t,u)-\mathbb{G}(t,\widetilde{u})| \le \mathcal{K}_{\mathcal{G}}|u-\widetilde{u}|,$$

for all $t \in [0,1]$ and $u, \widetilde{u} \in \mathbb{R}$.
$(\mathbb{HP}_2)$ There exist two positive functions $\mathcal{H}_1, \mathcal{H}_2$, such that:

$$|\mathbb{H}(t,u)| \le \mathcal{H}_1(t) + \mathcal{H}_2(t)|u|,$$

for all $t \in [0,1]$ and $u \in \mathbb{R}$.
$(\mathbb{HP}_3)$ There exist four positive $\mathcal{K}^*_{\mathcal{F}}, \mathcal{K}^*_{\mathcal{G}}, \mathcal{H}^*_1$ and $\mathcal{H}^*_2$ such that:

$$\begin{aligned}
\mathcal{K}^*_{\mathcal{F}} &= \sup_{t\in[0,1]}|\mathbb{F}(t,0)|,\ \mathcal{K}^*_{\mathcal{G}} = \sup_{t\in[0,1]}|\mathbb{G}(t,0)|, \mathcal{H}^*_1 = \sup_{t\in[0,1]}|\mathcal{H}_1(t)|\\
\text{and }\ \mathcal{H}^*_2 &= \sup_{t\in[0,1]}|\mathcal{H}_2(t)| < 1.
\end{aligned}$$

$(\mathbb{HP}_4)$ The following inequality holds:

$$\mathcal{B}_3 = \Upsilon\mathcal{K}_{\mathcal{G}} + \frac{\mathcal{K}^{\sigma}_{\varphi}e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\sigma}\Gamma(\sigma+1)}\mathcal{K}_{\mathcal{F}} < 1, \tag{3.6}$$

where

$$\begin{aligned}
\mathcal{K}_{\varphi} &= \varphi(1,0),\\
\mathcal{A} &= \left[\left|\frac{\rho}{\rho-1}\frac{\mathcal{K}_{\varphi}^{\sigma-1}}{\rho^{\sigma}\Gamma(\sigma)}\right| e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}\right]\mathcal{K}_{\mathcal{F}},\\
\mathcal{B}_1 &= \frac{\mathcal{K}_{\varphi}^{\beta+\alpha}e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}\mathcal{H}_1^*}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)} + \frac{\lambda e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\mathcal{K}_{\varphi}^{\alpha+\beta-1},\\
\mathcal{B}_2 &= \frac{\mathcal{K}_{\varphi}^{\beta+\alpha}e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}\mathcal{H}_2^*}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)} + \frac{e^{2\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}\mathcal{K}_{\varphi}^{\sigma+\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma+1)},\\
\Upsilon &= \mathcal{B}_1 + \mathcal{B}_2 r,\\
\mathcal{B}_4 &= \frac{\mathcal{K}_{\varphi}^{\sigma}e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\sigma}\Gamma(\sigma+1)}\mathcal{K}_{\mathcal{F}}^* + \Upsilon\mathcal{K}_{\mathcal{G}}^*.
\end{aligned}$$

Theorem 3.3. *Assume ($\mathbb{HP}_0$)–($\mathbb{HP}_4$) hold. Then, the problem (HP) has a solution defined on* $[0,1]$.

Proof. By Lemma 3.2, the solution of the problem (HP) is given by:

$$\begin{aligned}
u(t) &= \mathbb{G}(t,u(t))\left\{{}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}\mathbb{H}(t,u(t)) + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\right.\\
&\quad \left. + \frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta)\right\} + \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F}\right)(t,u(t)).
\end{aligned} \tag{3.7}$$

Choose $r>0$, so that:

$$r = \mathcal{B}_4\left(1-\mathcal{B}_3\right)^{-1}. \tag{3.8}$$

Define the set $\mathcal{X} = \{u \in \mathcal{C}, \|u\|_{\mathcal{C}} \le r\}$. Clearly, $\mathcal{X}$ is a closed convex bounded subset of the Banach space $\mathcal{C}$. Taking into account Lemma 2.8, we define the operators $\mathbb{T},\mathbb{N} : \mathcal{C} \to \mathcal{C}$ and $\mathbb{M} : \mathcal{X} \to \mathcal{C}$ by

$$\begin{aligned}
\mathbb{T}(u)(t) &= \mathbb{G}(t,u(t)), \qquad (3.9)\\
\mathbb{M}(u)(t) &= \left({}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}\mathbb{H}\right)(t,u(t)) + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1}\\
&\quad + \frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}u\right)(\zeta),\\
\mathbb{N}(u)(t) &= \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F}\right)(t,u(t)),
\end{aligned}$$

where

$$\begin{aligned}
{}_0\mathfrak{J}^{\beta+\alpha,\rho,\varphi}\mathbb{H}(t,u(t)) &= \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\beta+\alpha-1}\varphi'(s)\mathbb{H}(s,u(s))ds,\\
{}_0\mathfrak{J}^{\sigma,\rho,\varphi}u(\zeta) &= \frac{1}{\rho^{\sigma}\Gamma(\sigma)}\int_0^{\zeta} e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)}\varphi(\zeta,s)^{\sigma-1}\varphi'(s)u(s)ds,\\
&\text{and}\\
{}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F}(t,u(t)) &= \frac{1}{\rho^{\sigma}\Gamma(\sigma)}\int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\sigma-1}\varphi'(s)\mathbb{F}(s,u(s))ds.
\end{aligned}$$

Then the integral equation (3.7) can be written in the operator form as

$$u(t) = \mathbb{T}(u)(t)\mathbb{M}(u)(t) + \mathbb{N}(u)(t), t \in [0,1]. \tag{3.10}$$

We will show that the operators $\mathbb{T},\mathbb{M}$, and $\mathbb{N}$ satisfy all the conditions of Lemma 2.8. This will be achieved in the following series of claims.

Step1. We show that $\mathbb{T}$ and $\mathbb{N}$ are Lipschitzian on $\mathcal{C}$. Let $u, \widetilde{u} \in \mathcal{C}$. Then by $(\mathbb{HP}_1)$, for $t \in [0,1]$, we have

$$|\mathbb{T}(u)(t) - \mathbb{T}(\widetilde{u})(t)| = |\mathbb{G}(u)(t) - \mathbb{G}(\widetilde{u})(t)| \leq \mathcal{K}_{\mathcal{G}} \left| u - \widetilde{u} \right|.$$

So

$$\left\| \mathbb{T}(u) - \mathbb{T}(\widetilde{u}) \right\|_{\mathcal{C}} \leq \mathcal{K}_{\mathcal{G}} \left\| u - \widetilde{u} \right\|_{\mathcal{C}}.$$

Therefore $\mathbb{T}$ is Lipschitzian on $\mathcal{C}$ with Lipschitz constant $\mathcal{K}_{\mathcal{G}}$, for all $u, \widetilde{u} \in \mathcal{C}$.
Also, by $(\mathbb{HP}_1)$, fort $t \in [0,1]$, and $u, \widetilde{u} \in \mathcal{C}$ we have

$$\begin{aligned}
& |\mathbb{N}(u)(t) - \mathbb{N}(\widetilde{u})(t)| \\
= & \left| \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F} \right)(u)(t) - \left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F} \right)(\widetilde{u})(t) \right| \\
\leq & \frac{1}{\rho^{\sigma}\Gamma(\sigma)} \left| \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\sigma-1} \varphi'(s) \mathbb{F}(s,u(s)) ds \right. \\
& \left. - \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\sigma-1} \varphi'(s) \mathbb{F}(s,\widetilde{u}(s)) ds \right| \\
\leq & \frac{1}{\rho^{\sigma}\Gamma(\sigma)} \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\sigma-1} \varphi'(s) \left| \mathbb{F}(s,u(s)) - \mathbb{F}(s,\widetilde{u}(s)) \right| ds \\
\leq & \left[\left| \frac{\rho}{\rho-1} \frac{\mathcal{K}_{\varphi}^{\sigma-1}}{\rho^{\sigma}\Gamma(\sigma)} \right| e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}} \right] \left| \mathbb{F}(s,u(s)) - \mathbb{F}(s,\widetilde{u}(s)) \right| \\
\leq & \left[\left| \frac{\rho}{\rho-1} \frac{\mathcal{K}_{\varphi}^{\sigma-1}}{\rho^{\sigma}\Gamma(\sigma)} \right| e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}} \right] \mathcal{K}_{\mathcal{F}} \left| u - \widetilde{u} \right| \\
\leq & \mathcal{A} \left| u - \widetilde{u} \right|,
\end{aligned}$$

so

$$\left\| \mathbb{N}(u) - \mathbb{N}(\widetilde{u}) \right\|_{\mathcal{C}} \leq \mathcal{A} \left\| u - \widetilde{u} \right\|_{\mathcal{C}}.$$

Hence $\mathbb{N}$ is Lipschitzian on $\mathcal{C}$ with Lipschitz constant $\mathcal{A}$, for all $u, \widetilde{u} \in \mathcal{C}$.

Step2a. We show that $\mathbb{M}$ is completely continuous operator from $\mathcal{X}$ into $\mathcal{C}$.

Let a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $\{u_n\}_{n\in\mathbb{N}} \to u \in \mathcal{X}$. Then, Lebesgue's dominated convergence result yields:

$$\begin{aligned}
& \lim_{n\to+\infty} \mathbb{M}(u_n)(t) \\
= & \lim_{n\to+\infty} \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\beta+\alpha-1} \varphi'(s) \mathbb{H}(s,u_n(s)) ds \\
& + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \varphi(t,0)^{\alpha+\beta-1} \\
& + \lim_{n\to+\infty} \frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)} \varphi(t,0)^{\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma)} \int_0^{\zeta} e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)} \varphi(\zeta,s)^{\sigma-1} \varphi'(s) u_n(s) ds \\
= & \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\beta+\alpha-1} \varphi'(s) \lim_{n\to+\infty} \mathbb{H}(s,u_n(s)) ds \\
& + \frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \varphi(t,0)^{\alpha+\beta-1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma)}\int_0^{\zeta} e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)}\varphi(\zeta,s)^{\sigma-1}\varphi'(s)\lim_{n\to+\infty}u_n(s)ds \\
=\;& \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\beta+\alpha-1}\varphi'(s)\mathbb{H}(s,u(s))ds \\
&+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1} \\
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma)}\int_0^{\zeta} e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)}\varphi(\zeta,s)^{\sigma-1}\varphi'(s)u(s)ds \\
=\;& \mathbb{M}(u)(t).
\end{aligned}
$$

Hence, $\mathbb{M}(u_n)$ converges to $\mathbb{M}(u)$ pointwise on $[0,1]$.

Step2b. Next, we will show that the operator $\mathbb{M}$ is compact on $\mathcal{X}$. Firstly, to ensure the uniform boundedness, let $u\in\mathcal{X}$ and by applying $(\mathbb{HP}_3)$, we get:

$$
\begin{aligned}
|\mathbb{M}(u)(t)| \leq\;& \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\beta+\alpha-1}\varphi'(s)\left|\mathbb{H}(s,u(s))\right|ds \\
&+\frac{\lambda e^{\frac{\rho-1}{\rho}\varphi(t,0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\varphi(t,0)^{\alpha+\beta-1} \\
&+\frac{e^{\frac{\rho-1}{\rho}\varphi(t,0)}\varphi(t,0)^{\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma)}\int_a^{\zeta} e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)}\varphi(\zeta,s)^{\sigma-1}\varphi'(s)\left|u(s)\right|ds \\
\leq\;& \frac{\mathcal{K}_\varphi^{\beta+\alpha}e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\left(\mathcal{H}_1^*+\mathcal{H}_2^*r\right)}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)}+\frac{\lambda e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\mathcal{K}_\varphi^{\alpha+\beta-1} \\
&+\frac{re^{2\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\mathcal{K}_\varphi^{\sigma+\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma+1)} \\
\leq\;& \frac{\mathcal{K}_\varphi^{\beta+\alpha}e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\mathcal{H}_1^*}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)}+\frac{\lambda e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\mathcal{K}_\varphi^{\alpha+\beta-1} \\
&+\left(\frac{\mathcal{K}_\varphi^{\beta+\alpha}e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\mathcal{H}_2^*}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)}+\frac{re^{2\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\mathcal{K}_\varphi^{\sigma+\beta-1}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma+1)}\right)r.
\end{aligned}
$$

Taking the supremum in terms of u in above, we get:

$$
\|\mathbb{M}(u)(t)\| \leq \mathcal{B}_1+\mathcal{B}_2 r=\Upsilon<\infty, \quad \text{for all } u\in\mathcal{X}. \tag{3.11}
$$

Thus $\mathbb{M}$ is a uniformly bounded operator on $\mathcal{X}$.

Step2c. Now, we will show that $\mathbb{M}(\mathcal{X})$ is an equicontinuous set in $\mathcal{C}$.

Let $t, \widetilde{t} \in [0,1]$ such that $t < \widetilde{t}$. Then for any $u \in \mathcal{X}$, and by $\mathbb{HP}_2, \mathbb{HP}_3$ we get

$$\begin{aligned}
&\left|\mathbb{M}(u)(\widetilde{t}) - \mathbb{M}(u)(t)\right| \\
\leq\ & \frac{1}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha)} \left| \int_0^{\widetilde{t}} e^{\frac{\rho-1}{\rho}\varphi(\widetilde{t},s)} \varphi(\widetilde{t},s)^{\beta+\alpha-1}\varphi'(s)\mathbb{H}(s,u(s))ds \right. \\
& \left. - \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)} \varphi(t,s)^{\beta+\alpha-1}\varphi'(s)\mathbb{H}(s,u(s))ds \right| \\
& + \frac{\lambda e^{\frac{1-\rho}{\rho}\varphi(0)}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \left(e^{\frac{\rho-1}{\rho}\varphi(\widetilde{t})}\varphi(\widetilde{t},0)^{\alpha+\beta-1} - e^{\frac{\rho-1}{\rho}\varphi(t)}\varphi(t,0)^{\alpha+\beta-1} \right) \\
& + \frac{e^{\frac{1-\rho}{\rho}\varphi(0)}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma)} \left(e^{\frac{\rho-1}{\rho}\varphi(\widetilde{t})}\varphi(\widetilde{t},0)^{\beta-1} - e^{\frac{\rho-1}{\rho}\varphi(t)}\varphi(t,0)^{\beta-1} \right) \\
& \int_0^\zeta e^{\frac{\rho-1}{\rho}\varphi(\zeta,s)}\varphi(\zeta,s)^{\sigma-1}\varphi'(s)u(s)ds \\
\leq\ & \frac{e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}\left(\mathcal{H}_1^* + \mathcal{H}_2^* r\right)}{\rho^{\beta+\alpha}\Gamma(\beta+\alpha+1)} \varphi(\widetilde{t},t)^{\beta+\alpha} \\
& + \frac{\lambda e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \left(\varphi(\widetilde{t},0)^{\alpha+\beta-1} - \varphi(t,0)^{\alpha+\beta-1} \right) \\
& + \frac{\mathcal{K}_\varphi^\sigma r e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma+1)} \left(\varphi(\widetilde{t},0)^{\beta-1} - \varphi(t,0)^{\beta-1} \right) \\
& + \left[\frac{\mathcal{K}_\varphi^{\sigma+\beta-1} e^{\frac{1-\rho}{\rho}\varphi(1)} r}{\rho^{\sigma+\beta-1}\Gamma(\beta)\Gamma(\sigma+1)} + \frac{\lambda e^{\frac{1-\rho}{\rho}\varphi(0)}\mathcal{K}_\varphi^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \right] \left(e^{\frac{\rho-1}{\rho}\varphi(\widetilde{t})} - e^{\frac{\rho-1}{\rho}\varphi(t)} \right),
\end{aligned}$$

then

$$\left|\mathbb{M}(u)(\widetilde{t}) - \mathbb{M}(u)(t)\right| \to 0, \text{ as } \widetilde{t} \to t,$$

which implies that

$$\left\|\mathbb{M}(u)(\widetilde{t}) - \mathbb{M}(u)(t)\right\|_{\mathcal{C}} \to 0$$

uniformly for all $u \in \mathcal{X}$. This means that $\mathbb{M}$ is equicontinuous on $\mathcal{C}$. In consequence, $\mathbb{M}$ is relatively compact. As a result of the Arzelà–Ascoli theorem, we deduce that $\mathbb{M}$ is completely continuous. Thus, $\mathbb{M}$ is compact on $\mathcal{X}$.

Step3. We show that the hypothesis (c) of Lemma 2.8 holds. For any $u \in \mathcal{X}$, $t \in [0,1]$, by $\mathbb{HP}_2$ and $\mathbb{HP}_3$, we have

$$\begin{aligned}
|\mathbb{T}(t,u(t))| &= |\mathbb{G}(t,u)| \leq |\mathbb{G}(t,u) - \mathbb{G}(t,0)| + |\mathbb{G}(t,0)| \\
&\leq \mathcal{K}_{\mathcal{G}}|u(t)| + \mathcal{K}_{\mathcal{G}}^*,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
|\mathbb{N}(u)(t)| &= \left|\left({}_0\mathfrak{J}^{\sigma,\rho,\varphi}\mathbb{F}\right)(u)(t)\right| \\
&\leq \frac{1}{\rho^\sigma\Gamma(\sigma)} \left| \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\sigma-1}\varphi'(s)\mathbb{F}(s,u(s))ds \right| \\
&\leq \frac{1}{\rho^\sigma\Gamma(\sigma)} \int_0^t e^{\frac{\rho-1}{\rho}\varphi(t,s)}\varphi(t,s)^{\sigma-1}\varphi'(s)\left(\mathcal{K}_{\mathcal{F}}|u| + \mathcal{K}_{\mathcal{F}}^*\right)ds \\
&\leq \frac{\mathcal{K}_\varphi^\sigma e^{\frac{\rho-1}{\rho}\mathcal{K}_\varphi}}{\rho^\sigma\Gamma(\sigma+1)}\left(\mathcal{K}_{\mathcal{F}}|u(t)| + \mathcal{K}_{\mathcal{F}}^*\right).
\end{aligned} \tag{3.13}$$

Let $u \in \mathcal{C}$ and $v \in \mathcal{X}$ be arbitrary elements such that

$$u = \mathbb{T}u\mathbb{M}v + \mathbb{N}u.$$

Then, by (3.11) (3.12) and (3.13), we have

$$|u(t)| \leq |\mathbb{T}u(t)|\,|\mathbb{M}v(t)| + |\mathbb{N}u(t)|\,;$$

so

$$\begin{aligned} \|u\|_{\mathcal{C}} &\leq \Upsilon\left(\mathcal{K}_{\mathcal{G}}\|u\|_{\mathcal{C}} + \mathcal{K}_{\mathcal{G}}^{*}\right) + \frac{\mathcal{K}_{\varphi}^{\sigma} e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\sigma}\Gamma(\sigma+1)}\left(\mathcal{K}_{\mathcal{F}}\|u\|_{\mathcal{C}} + \mathcal{K}_{\mathcal{F}}^{*}\right) \\ &\leq \mathcal{B}_3\|u\|_{\mathcal{C}} + \mathcal{B}_4. \end{aligned}$$

Hence, by (3.6), we get

$$\|u\|_{\mathcal{C}} \leq \frac{\mathcal{B}_4}{1-\mathcal{B}_3} \leq r.$$

Step4. Finally, by

$$\mathcal{B}_3 = \Upsilon\mathcal{K}_{\mathcal{G}} + \frac{\mathcal{K}_{\varphi}^{\sigma} e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\sigma}\Gamma(\sigma+1)}\mathcal{K}_{\mathcal{F}} < 1,$$

we see that $\tau K + \sigma < 1$ holds, where $\tau = \mathcal{K}_{\mathcal{G}}$ and $\sigma = \frac{\mathcal{K}_{\varphi}^{\sigma} e^{\frac{\rho-1}{\rho}\mathcal{K}_{\varphi}}}{\rho^{\sigma}\Gamma(\sigma+1)}\mathcal{K}_{\mathcal{F}}$.
Thus, all the conditions of Lemma 2.8 are satisfied. Hence the operator equation $u = \mathbb{T}u\mathbb{M}u + \mathbb{N}u$ has a solution in $\mathcal{X}$. As a result, the problem (HP) has a solution on $[0,1]$. □

3.2. Example

Consider the nonlinear equations

$$\begin{cases} {}_0\mathfrak{D}^{\frac{6}{8},\frac{1}{2},t^2}\left[{}_0\mathfrak{D}^{\frac{3}{2},\frac{1}{2},t^2}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}\mathbb{F}\right)(t,u(t))}{\mathbb{G}(t,u(t))}\right)\right] = \mathbb{H}(t,u(t)), t \in J = [0,1], \\ {}_0\mathfrak{J}^{\frac{1}{4},\frac{1}{2},t^2}\left({}_0\mathfrak{D}^{\frac{3}{2},\frac{1}{2},t^2}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}\mathbb{F}\right)(t,u(t))}{\mathbb{G}(t,u(t))}\right)\right)\Bigg|_{t=0^+} = \frac{1}{9}, \\ {}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}\mathbb{F}\right)(t,u(t))}{\mathbb{G}(t,u(t))}\right)\Bigg|_{t=0^+} = 0, \\ {}_0\mathfrak{J}^{\frac{-1}{2},\frac{1}{2},t^2}\left(\frac{u(t)-\left({}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}\mathbb{F}\right)(t,u(t))}{\mathbb{G}(t,u(t))}\right)\Bigg|_{t=0^+} = \left({}_0\mathfrak{J}^{\frac{1}{2},\frac{1}{2},t^2}u\right)(\zeta), \zeta \in]0,1[; \end{cases} \tag{3.14}$$

and

$$\varphi(t) = t^2, \lambda = \frac{1}{9}, \alpha = \frac{6}{8}, \beta = \frac{3}{2}, \rho = \sigma = \frac{1}{2},$$

$$\begin{aligned} \mathbb{H}(t,u(t)) &= \frac{1}{9+t^2} + \frac{t^2}{99}|u(t)|, \\ \mathbb{G}(t,u(t)) &= \frac{1}{8+e^t} + \frac{e^{-3t}}{1+9e^t}\frac{|u(t)|}{1+u^2(t)}, \\ \mathbb{F}(t,u(t)) &= \frac{1}{9+t^2} + \frac{t^2}{9(1+t^2)}\frac{|u(t)|}{1+u^2(t)}. \end{aligned}$$

Then, we have

$$\mathcal{K}_{\varphi} = 1, \mathcal{K}_{\mathcal{F}} = \frac{1}{9}, \mathcal{K}_{\mathcal{G}} = \frac{1}{10},$$

$$\mathcal{H}_1 = \frac{1}{9\left(1+t^2\right)}, \mathcal{H}_2(t) = \frac{t^2}{99},$$

$$\mathcal{K}_{\mathcal{F}}^* = \mathcal{K}_{\mathcal{G}}^* = \mathcal{H}_1^* = \frac{1}{9} \text{ and } \mathcal{H}_2^* = \frac{1}{99} < 1.$$

Hence, the hypotheses $(\mathbb{HP}_0)$-$(\mathbb{HP}_4)$ are satisfied. In fact, we have

$$\begin{aligned}
\mathcal{A} &= \frac{\sqrt{2}e^{-1}}{9\Gamma\left(\frac{1}{2}\right)} = \frac{1}{9e}\sqrt{\frac{2}{\pi}} \simeq 0,032614, \\
\mathcal{B}_1 &= \frac{\frac{e^{-1}}{9}2^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + \frac{\frac{e^{-1}}{9}2^{\frac{5}{4}}}{\Gamma\left(\frac{9}{4}\right)} \simeq 0,84853, \\
\mathcal{B}_2 &= \frac{2^{\frac{9}{4}}\frac{e^{-1}}{99}}{\Gamma\left(\frac{13}{4}\right)} + \frac{e^{-2}2^{\frac{5}{4}}}{\Gamma^2\left(\frac{3}{2}\right)} \simeq 0,41677, \\
\Upsilon &= \mathcal{B}_1 + \mathcal{B}_2 r \simeq 0,84853 + 0,41677r, \\
\mathcal{B}_3 &\simeq 0.150083 + 0,041677r, \\
\mathcal{B}_4 &= \frac{\frac{2e^{-1}}{9}\sqrt{2}}{\sqrt{\pi}} + \frac{\Upsilon}{9} \simeq 0,15951 + 0,04631r.
\end{aligned}$$

By (3.8), $r \simeq 0.2005$; and then $\mathcal{B}_3 < 1$. Accordingly, all the conditions of Theorem 3.3 are fulfilled. Then, the hybrid fractional problem (3.14) has at least one solution on $[0, 1]$.

References

[1] M.I. Abbas, M.A. Ragusa, On the Hybrid fractional fifferential equations with fractional proportional derivatives of a function with respect to a certain function. Symmetry 2021, 13, 264. https://doi.org/10.3390/sym13020264

[2] M.I. Abbas, Existence results and the Ulam stability for fractional differential equations with hybrid proportional-Caputo derivatives. J. Nonlinear Funct. Anal. 2020, 2020, 1-14.

[3] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, Implicit differential and integral equations: existence and stability, Walter de Gruyter, London, 2018.

[4] S. Abbas, M. Benchohra and G. M. N'Guérékata, Advanced fractional differential and integral equations, Nova Science Publishers, New York, 2014.

[5] S. Abbas, M. Benchohra and G. M. N'Guérékata, Topics in fractional differential equations, Springer-Verlag, New York, 2012.

[6] R. S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solutions of fractional differential equations via geraghty type hybrid contractions, Appl. Comput. Math., V.20, N.2, 2021,313-333.

[7] H. Afshari and E. Karapinar, A solution of the fractional differential equations in the setting of b-metric space. Carpathian Math. Publ. 13 (2021), 764-774. https://doi.org/10.15330/cmp.13.3.764-774.

[8] H. Afshari, E. Karapınar, A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces, Adv. Difference Equ. 2020, 616. https://doi.org/10.1186/s13662-020-03076-z.

[9] G.A. Anastassiou, Generalized fractional calculus: New advancements and applications, Springer International Publishing, Switzerland, 2021.

[10] H. Beddani and Z. Dahmani, Solvability for nonlinear differential problem of Langevin type via ϕ-Caputo approch, Eur. J. Math. Appl. (2021) 1:11, DOI: 10.28919/ejma.2021.1.11

[11] H. Beddani and M. Beddani, Solvability for a differential systems via ϕ-Caputo approach. J. Sci. Arts. 56(3)2021

[12] A. Bharucha-Reid, Random integral equations, Academic Press, New York, 1972.

[13] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 2015, 1, 73-85.

[14] B.C. Dhage, Fixed point theorems in ordered Banach algebras and applications. Panamer. Math. J. 1999, 9, 93–102.

[15] B.C. Dhage, A fixed point theorem in Banach algebras with applications to functional integral equations. Kyungpook Math. J. 44, 145-155 (2004).

[16] B.C. Dhage, On a fixed point theorem in banach algebras with applications, Applied Mathematics Letters, 18, (2005) p: 273-280.

[17] B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, differential eEquations and applications, Vol 2, Number 4 (2010), p: 465-486.
[18] S. Etemad, S. Rezapour, M. E. Samei, On fractional hybrid and non-hybrid multi-term integrodifferential inclusions with three-point integral hybrid boundary conditions, Adv. Differ. Equ., 2020 (2020), 161. doi: 10.1186/s13662-020-02627-8.
[19] S. Ferraoun, and Z. Dahmani, Existence and stability of solutions of a class of hybrid fractional differential equations involving RL-operator. J. Interdisciplinary Math., vol 23 no 4(2020), 885-903. https://doi.org/10.1080/09720502.2020.1727617
[20] F. Jarad, M.A. Alqudah, T. Abdeljawad, On more general forms of proportional fractional operators. Open Math. 2020, 18, 167–176.
[21] F. Jarad, T. Abdeljawad, S. Rashid, Z. Hammouch, More properties of the proportional fractional integrals and derivatives of a function with respect to another function. Adv. Differ. Equ. 2020, 2020, 303
[22] A. Keten, M. Yavuz, D. Baleanu, Nonlocal Cauchy problem via a fractional operator involving power kernel in banach spaces. Fractal Fract. 2019, 3, 27.
[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, vol. 204. Elsevier Science, Amsterdam, 2006
[24] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.
[25] S.G. Samko, A. A. Kilbas and O. I. Mariche, Fractional integrals and derivatives, translated from the 1987 Russian original. Yverdon: Gordon and Breach, (1993).
[26] M. Yavuz, European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels. Numer. Methods Partial. Differ. Equ. 2021, 37.
[27] Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid di erential equations, Comput. Math. Appl., 62 (2011), 1312–1324. doi: 10.1016/j.camwa.2011.03.041.