Research
Journal of Marine and Engineering Technology (JOINMET) 2(2), 58-67, 2022
Recieved: 27-May-2022 Accepted: 02-Aug-2022

SAKARYA UNIVERSITY OF APPLIED SCIENCES

Numerical Simulations of the Mass-Spring-Damper System using High Resolution Schemes

Osman ÜNAL[1*], Nuri AKKAŞ[2]

[1] Maritime Vocational High School, Sakarya University of Applied Sciences, Sakarya 54050, Turkey, osmanunal@subu.edu.tr

[2] Maritime Vocational High School, Sakarya University of Applied Sciences, Sakarya 54050, Turkey, nuriakkas@subu.edu.tr

ABSTRACT

Minimizing time step intervals of numerical solutions for mass-spring-damper system is one technique to reduce numerical errors, but it considerably increases the total computation cost. Using high resolution schemes for first and second derivatives rather than the first order accurate techniques is another way to reduce numerical errors. The central difference approach is a second-order accurate scheme that can approximate first and second derivatives. Nevertheless, utilizing the central difference technique to approximate the first derivative does not contain current time step data. Approximation of the first derivative, on the other hand, should be based mostly on current time step data. When the current time step data are ignored to estimate the first derivative, considerable numerical inaccuracies arise, especially for some important points such as the usage of sharp external force. Approximation of the first derivative, the third order resolution scheme uses not only the next and previous time step values, but also the current and two prior time step data. To approximate the first and second derivatives, this paper presents combination of second order resolution scheme and third order accurate method. The proposed technique is more accurate than previous methods in terms of order of convergence.

Keywords: High resolution schemes, finite difference technique, mass-spring-damper system

Yüksek Çözünürlüklü Şemalar Kullanılarak Kütle-Yay-Damper Sisteminin Sayısal Simülasyonları

ÖZ

Kütle-yay-sönüm sistemi için sayısal çözümlerin zaman adımlarını küçültmek sayısal hataları azaltmak için bir tekniktir, ancak toplam hesaplama süresini önemli ölçüde artırır. Birinci dereceden doğruluktaki teknikler yerine birinci ve ikinci türevler için yüksek çözünürlüklü şemalar kullanmak, sayısal hataları azaltmanın başka bir yoludur. Merkezi fark yaklaşımı, birinci ve ikinci türevleri tahmin edebilen ikinci dereceden doğruluktaki bir şemadır. Bununla birlikte, birinci türevi tahmin etmek için merkezi fark tekniğini kullanmak, mevcut zaman adımı verilerini içermez. Birinci türevin yaklaşıklığı ise çoğunlukla mevcut zaman adımı verilerine dayanmaktadır. Birinci türevi tahmin etmek için mevcut zaman adımı verileri göz ardı edildiğinde, özellikle keskin dış kuvvet kullanımı gibi bazı önemli noktalarda önemli sayısal hatalar ortaya çıkmaktadır. Birinci türevin yaklaşıklığı,

* Corresponding Author's email: osmanunal@subu.edu.tr

üçüncü derece çözümleme şeması ile hesaplanırken, yalnızca sonraki ve önceki zaman adımı değerlerini değil, aynı zamanda mevcut ve iki önceki zaman adımı verileri de kullanılır. Birinci ve ikinci türevleri hesaplamak için, bu makale ikinci dereceden çözümleme şeması ve üçüncü dereceden doğruluktaki yöntemin bir kombinasyonunu sunmaktadır. Önerilen teknik, yakınsama derecesi açısından önceki yöntemlerden daha yüksek doğruluktadır.

Anahtar Kelimeler: Yüksek çözünürlüklü şemalar, sonlu farklar yöntemi, kütle-yay-damper system

1 Introduction

A variety of engineering field applications has made extensive use of damper mass spring systems (Aly and Salem, 2013; Qureshi, 2021; Bandivadekar and Jangid, 2012). They can be classified into active and passive mechanical equipment, with springs and dampers functioning as examples of the former while various types of actuators and sensors operate as examples of the latter (Agharkakli et. al., 2012; Kahveci and Kolmanovsky, 2010). The applications of damper mass spring systems have attracted the attention of many researchers, who have conducted numerous theoretical and experimental experiments. Some of these literatures concentrated on the investigation of the dynamic behavior of the mechanical components in the applied system, while others, which deal with the analysis of control system, relied on the relevant mathematical models to perform the numerical simulation of damper mass spring system (Badr et. al., 2020; Humaidi et. al., 2018; Abry et. al., 2013; Basri et. al., 2014; Rahmat et. al., 2011). In order to obtain well posed numerical solution of the motion equation of mass-spring-damper systems, first and second derivatives in this equation must be properly approximated. The poor approximation of first and second derivatives causes the numerical errors. A variety of studies have been conducted to approximate derivatives. The first derivative has been numerically calculated using first-order accurate scheme (Esfandiari, 2017; Ertekin et. al., 2001). It is based on the Taylor series' expansion. First-order upwinding method is one of the simplest ways to calculate the first derivative. However, it leads numerical dispersion significantly. The second-order approach (Schwendt and Pötz, 2020; Peaceman, 2000) is another method for predicting numerical derivatives. It can be used on first or second derivatives. The third-order scheme (Leonard, 1994; Wolcott, 1996; Peng et. al., 2013; Mazumder, 2015) is another approach for numerically approximating the first derivative that decreases numerical dispersion considerably. The main objective of this study is the application of third order accurate schemes for numerical solution of the motion equation of mass-spring-damper systems to minimize numerical errors compared to previously developed method.

2 Material and Method

Equation 1 indicates the equation of motion for mass-spring-damper systems:

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{1}$$

In equation 1, m represents mass, c represents damping coefficient, k represents stiffness coefficient, f(t) represents external load, $\ddot{x}$ represents acceleration, $\dot{x}$ represents velocity, and x represents displacement. This equation includes first and second derivatives. The numerical values of first and second derivatives must be calculated precisely to represent physical system of mass-spring-damper component correctly. The main objective of this study is the determination of these derivatives with minor numerical errors. There are there techniques to calculate first derivative numerically: first-order upstream method, second-order technique and third-order Leonard method. Equation2, Equation 3 and Equation 4 show first-order upstream method, second-order technique and third-order Leonard method respectively:

$$\dot{x}_t = \frac{x_t - x_{t-1}}{\Delta t} \tag{2}$$

$$\dot{x}_t = \frac{x_{t+1} - x_{t-1}}{2\Delta t} \tag{3}$$

$$\dot{x}_t = \frac{\left(\frac{x_{t+1} + x_t}{2} - \frac{x_{t+1} - 2x_t + x_{t-1}}{8}\right) - \left(\frac{x_t + x_{t-1}}{2} - \frac{x_t - 2x_{t-1} + x_{t-2}}{8}\right)}{\Delta t} \tag{4}$$

In Equation 2, Equation 3 and Equation 4 x_{t-2} represents two previous displacements, x_{t-1} represents a previous displacement, x_t represents current displacement and x_{t+1} represents a next displacement in time domain. Δt represents time step interval. The beauty of Equation 4 (third-order Leonard method) is that it depends two previous data, previous data, current data and next data. That is why it is more accurate compared to first and second order techniques. Therefore, third-order method decreases numerical errors significantly. Equation 5 indicates second derivative that is named as central difference method.

$$\ddot{x}_t = \frac{x_{t+1} - 2x_t + x_{t-1}}{\Delta t^2} \tag{5}$$

In this study, three different techniques for first derivative and central difference method for second derivative have been combined to compare each method.

Equation 6 represents numerical solution of the equation of motion for mass-spring-damper system using combination of first-order method for first derivative and central difference method for second derivative:

$$m\frac{x_{t+1} - 2x_t + x_{t-1}}{\Delta t^2} + c\frac{x_t - x_{t-1}}{\Delta t} + kx_t = f(t) \tag{6}$$

Equation 7 can be derived by rearranging equation 6 to leave the next time step value.

$$x_{t+1} = \left[f(t) - kx_t + c\frac{x_t - x_{t-1}}{\Delta t} + m\frac{2x_t - x_{t-1}}{\Delta t^2} \right] / \left(m\frac{1}{\Delta t^2} \right) \tag{7}$$

Similarly, Equation 8 represents numerical solution of the equation of motion for mass-spring-damper system using combination of second-order method for first derivative and central difference method for second derivative:

$$x_{t+1} = \left[f(t) - kx_t + c\frac{x_{t-1}}{2\Delta t} + m\frac{2x_t - x_{t-1}}{\Delta t^2} \right] / \left(m\frac{1}{\Delta t^2} + c\frac{1}{2\Delta t} \right) \tag{8}$$

Equation 9 represents numerical solution of the equation of motion for mass-spring-damper system using combination of third-order method for first derivative and central difference method for second derivative:

$$x_{t+1} = \left[f(t) - kx_t - c\frac{\frac{1}{2}x_t - x_{t-1} + \frac{1}{6}x_{t-2}}{\Delta t} + m\frac{2x_t - x_{t-1}}{\Delta t^2} \right] / \left(m\frac{1}{\Delta t^2} + c\frac{1}{3\Delta t} \right) \tag{9}$$

In this study, analytical solution must be used in order to validate all numerical solutions. For mass-spring-damper systems, an analytical solution is provided in literature if the external load is harmonic as in equation 10.

$$f(t) = \bar{f}\sin(w_e t) \tag{10}$$

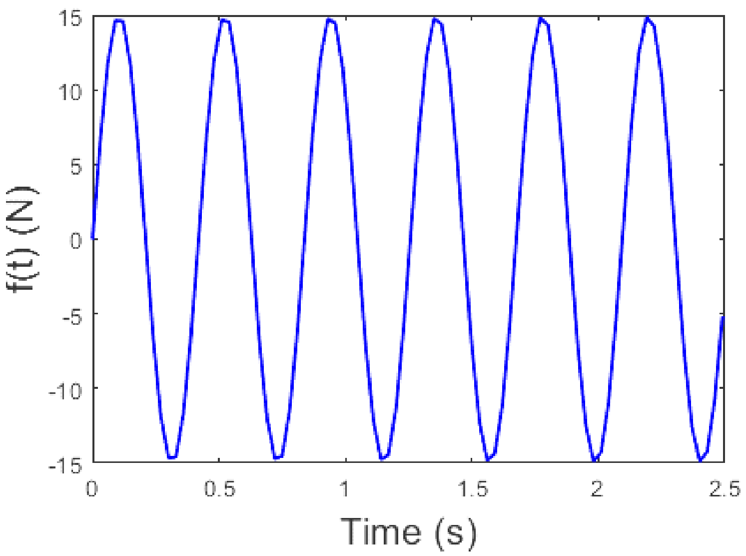
Equation 11 is utilized to obtain analytical solution for harmonic external load and free vibration response of mass-spring-damper systems (Wu, 2013).

$$x(t) = e^{-\xi\omega t}\left[\frac{\dot{x}(0) + \xi\omega x(0)}{\omega_d}\sin(\omega_d t) + x(0)\cos(\omega_d t)\right] + \frac{\bar{f}}{m\omega^2}\frac{1}{(1-\Omega^2)^2 + (2\xi\Omega)^2}$$
$$\times \left\{ \begin{array}{l} (1-\Omega^2)\left[-e^{-\xi\omega t}\left(\frac{\omega_e}{\omega_d}\right)\sin(\omega_d t) + \cos(\omega_e t)\right] \\ +2\xi\Omega\left[e^{-\xi\omega t}\left(\frac{\xi\omega}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right) - \cos(\omega_e t)\right] \end{array} \right\} \tag{11}$$

ω_d, ω, ξ, Ω, $\bar{f}$ and ω_e are represented by damped natural frequency, undamped natural frequency, damping ratio, undamped natural frequency over exciting frequency, amplitude of f(t) and exciting frequency in equation 11.

3 Results and Discussion

In this study, amplitude of sinusoidal force ($\bar{f}$) is selected as 15 Newton exciting frequency (w_e) is preferred as 15 rad/s. Figure 1 shows harmonic force using Equation 10.



<

Figure 1: Harmonic external force.

All simulations below assume an initial displacement of 0.15 m relative to the original position, an initial velocity of 1.2 m/s, a mass of 0.8 kg, a damping coefficient of 5 Ns/m and a stiffness coefficient of 20 N/m. Figure 2 exhibits analytical (exact) solution to validate numerical solutions of mass-spring-damper system with harmonic external force.

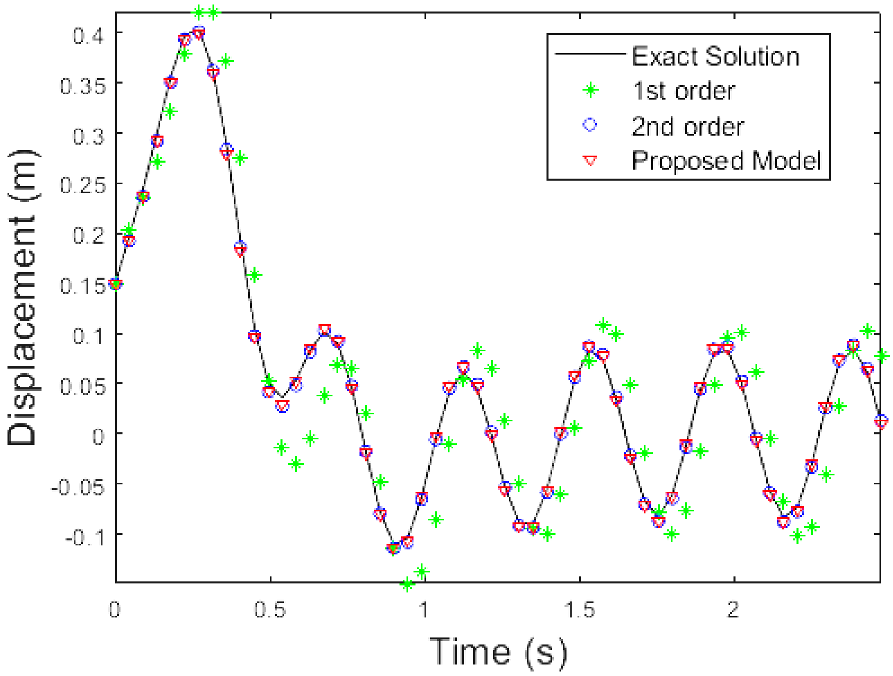


Figure 2: Simulation for harmonic external force.

Figure 3 indicates order of convergence for first-order upstream method, second-order method and proposed third-order method. According to Figure 3, while the inclination of proposed third-order method is 2.30, first and second methods have 1.17 and 1.98 gradient, respectively. For this reason proposed model is more accurate than other methods.

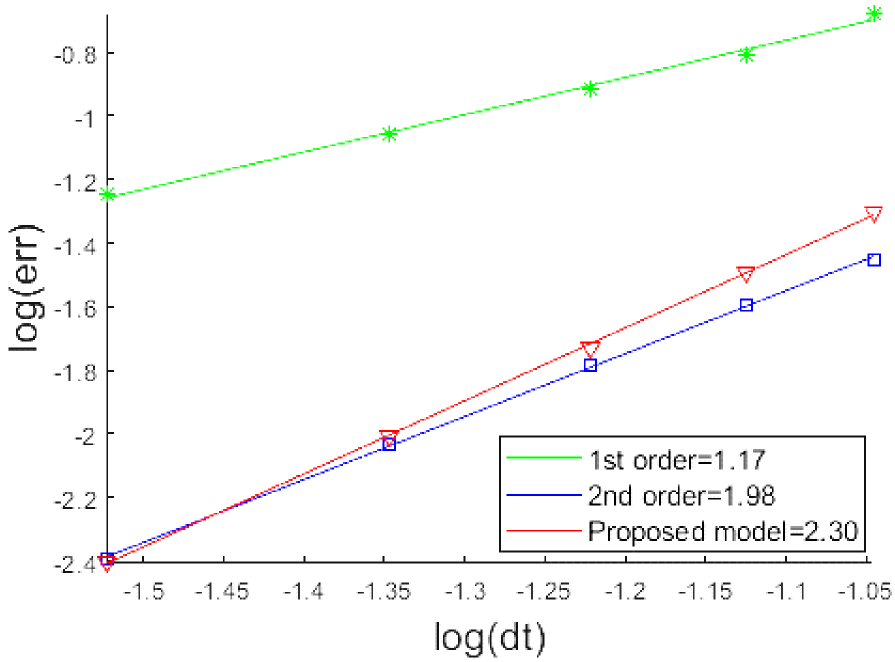


Figure 3: Order of convergence.

4 Conclusions

In numerical simulations of the mass-spring-damper system, there are two typical techniques for decreasing numerical errors. Using small time increments is the first technique. Very long total simulation duration is encountered by using small time step intervals. Calculating first and second derivatives using higher order finite difference approximation rather than first order accurate techniques is the second method for reducing numerical errors. The central difference method uses to approximate the first and second derivatives with second-order precision. However, the central difference approximation of the first derivative does not employ the current time step data. Discarding those results in significant numerical errors since the first derivative's estimate largely depends on the current time step values. On the other hand, the Leonard approach's numerical estimate of the first derivative is third order accurate and depends not only on the data for the current time step and the two time steps before it, but also on the data for the next and previous time steps. This research presents a unique model that combines the third-order approach with central difference techniques to estimate the first and second derivatives for numerical solution of mass-spring-damper system. The proposed model provides the best degree of convergence compared to previous methods.

5 Declerations

5.1 Competing Interests

There is no conflict of interest in this study.

5.2 Authors’ Contributions

Osman ÜNAL: To create ideas or hypotheses for research and/or article, to plan materials and methods to reach conclusions, to take responsibility for the logical explanation and presentation of findings, to take responsibility for literature review during research, to take responsibility for the creation of the entire article, to rework not only in terms of spelling and grammar, but also in terms of intellectual content before submitting the article.

Nuri AKKAŞ: To create ideas or hypotheses for research and/or article, to plan materials and methods to reach conclusions, to take responsibility for the logical explanation and presentation of findings, to take responsibility for literature review during research, to take responsibility for the creation of the entire article, to rework not only in terms of spelling and grammar, but also in terms of intellectual content before submitting the article.

5.3 Funding source

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors

References

Abry, F., Brun, X., Sesmat, S., & Bideaux, E. (2013, July). Non-linear position control of a pneumatic actuator with closed-loop stiffness and damping tuning. In 2013 European control conference (ECC) (pp. 1089-1094). IEEE.

Agharkakli, A., Sabet, G. S., & Barouz, A. (2012). Simulation and analysis of passive and active suspension system using quarter car model for different road profile. International Journal of Engineering Trends and Technology, 3(5), 636-644.

Aly, A. A., & Salem, F. A. (2013). Vehicle suspension systems control: a review. International journal of control, automation and systems, 2(2), 46-54.

Badr, M. F., Karam, E. H., & Mjeed, N. M. (2020). Control design of damper mass spring system based on backstepping controller scheme. International Review of Applied Sciences and Engineering.

Bandivadekar, T. P., & Jangid, R. S. (2012). Mass distribution of multiple tuned mass dampers for vibration control of structures. International Journal of Civil and Structural Engineering, 3(1), 70.

Basri, M. A. M., Husain, A. R., & Danapalasingam, K. A. (2014). Robust chattering free backstepping sliding mode control strategy for autonomous quadrotor helicopter. International Journal of Mechanical & Mechatronics Engineering, 14(3), 36-44.

Ertekin, T., Abou-Kassem, J. H., & King, G. R. (2001). Basic applied reservoir simulation (Vol. 7). Richardson, TX: Society of Petroleum Engineers.

Esfandiari, R. S. (2017). Numerical methods for engineers and scientists using MATLAB®. Crc Press.

Humaidi, A. J., Hameed, M. R., & Hameed, A. H. (2018). Design of block-backstepping controller to ball and arc system based on zero dynamic theory. Journal of Engineering Science and Technology, 13(7), 2084-2105.

Kahveci, N. E., & Kolmanovsky, I. V. (2010). Control design for electromagnetic actuators based on backstepping and landing reference governor. IFAC Proceedings Volumes, 43(18), 393-398.

Leonard, B. P. (1994). Comparison of truncation error of finite-difference and finite-volume formulations of convection terms. Applied mathematical modelling, 18(1), 46-50.

Mazumder, S. (2015). Numerical methods for partial differential equations: finite difference and finite volume methods. Academic Press.

Peaceman, D. W. (2000). Fundamentals of numerical reservoir simulation. Elsevier.

Peng, Y., Liu, C., & Shi, L. (2013, August). Soution of Convection-Diffusion Equations. In International Conference on Information Computing and Applications (pp. 546-555). Springer, Berlin, Heidelberg.

Qureshi, S. (2021). Fox H-functions as exact solutions for Caputo type mass spring damper system under Sumudu transform. Journal of Applied Mathematics and Computational Mechanics, 20(1), 83-89.

Rahmat, M. F., Sunar, N. H., Salim, S. N. S., Abidin, M. S. Z., Fauzi, A. M., & Ismail, Z. H. (2011). Review on modeling and controller design in pneumatic actuator control system. International journal on smart sensing and intelligent systems, 4(4).

Schwendt, M., & Pötz, W. (2020). Transparent boundary conditions for higher-order finite-difference schemes of the Schrödinger equation in (1+ 1) D. Computer Physics Communications, 250, 107048.

Wolcott, D. S., Kazemi, H., & Dean, R. H. (1996, October). A practical method for minimizing the grid orientation effect in reservoir simulation. In SPE annual technical conference and exhibition. OnePetro.

Wu, J. S. (2013). Analytical and numerical methods for vibration analyses. John Wiley & Sons.

Appendices

```
tic
clc
clearvars
close all
% Mass-Spring-Damper System
% SDOF
%% Input Data
p=5;                %Points for order of convergence
err=zeros(p,4);     %Error matrix
dt_i=0.03;          %Initial time step
dt_f=0.09;          %Final time step
i=1;                %First time step loop

%Time step loop (second)
for dt=linspace(dt_i,dt_f,p)
m=1;                %kg
k=25;               %N/m
c=5;                %N s/m (xi=0.05)
w=sqrt(k/m);        %Undamped natural frequency
xi=c/(2*m*w);       %Damping ratio
wd=w*sqrt(1-xi^2);  %Damped natural frequency
x0=0.1;             %Initial condition (displacement)
dx0=1;              %Initial condition (velocity)
tf=2.5;             %Final time (second)
t=0:dt:tf;          %Time (second)
%External Force
F_ex='Harmonic';    %Harmonic or Arbitrary
f=0;                %N (Amplitude)
if isequal(F_ex,'Harmonic')
we=0;               %rad/s
omega=we/w;         %Frequency ratio
ft=@(tn) f*sin(we*tn);
elseif isequal(F_ex,'Arbitrary')
prd=10000000;
pw=1;
pd=1;
st=0.9;
ft=@(tn) ( (tn>pd*st)-(tn>(pd+pw)*st) )*f;
end
%% Output Data
%% Exact solution
if isequal(F_ex,'Harmonic')
x_exact=zeros(1,length(t));
for n=1:length(t)
x_exact(n)=exp(-xi*w*t(n))*((dx0+xi*w*x0)/wd*sin(wd*t(n))...
    +x0*cos(wd*t(n)))+f/(m*w^2*((1-omega^2)^2+(2*xi*omega)^2))...
    *((1-omega^2)*(-exp(-xi*w*t(n))*we/wd*sin(wd*t(n))...
    +sin(we*t(n)))+2*xi*omega*(exp(-xi*w*t(n))*(xi*w/wd...
    *sin(wd*t(n))+cos(wd*t(n)))-cos(we*t(n))));
end
end
%% Explicit Euler Method
A=[m 0;0 1];
B=[c k;-1 0];
Y0=[dx0;x0];
Y=zeros(2,length(t));
Y(:,1)=Y0;
for n=2:length(t)
F=[ft(t(n-1));0];
Y(:,n)=Y(:,n-1)+dt*A^-1*(F-B*Y(:,n-1));
end
x_eem=Y(2,:);
%% Central Difference Method
x_dt=x0-dt*dx0+1/2*dt^2*(ft(t(1))-c*dx0-k*x0)/m;
x_cdm=zeros(1,length(t));
```

```
x_cdm(1)=x0;
x_cdm(2)=(ft(t(1))-(k-2/dt^2*m)*x_cdm(1)...
    -(1/dt^2*m-1/(2*dt)*c)*x_dt)/(1/dt^2*m+1/(2*dt)*c);
for n=3:length(t)
x_cdm(n)=(ft(t(n-1))-(k-2/dt^2*m)*x_cdm(n-1)...
    -(1/dt^2*m-1/(2*dt)*c)*x_cdm(n-2))/(1/dt^2*m+1/(2*dt)*c);
end
%% Quick Difference Method
x_dt=x0-dt*dx0+1/2*dt^2*(ft(t(1))-c*dx0-k*x0)/m;
x_qdm=zeros(1,length(t));
x_qdm(1)=x0;
x_qdm(2)=(ft(t(1))-(k-2/dt^2*m)*x_qdm(1)...
    -(1/dt^2*m-1/(2*dt)*c)*x_dt)/(1/dt^2*m+1/(2*dt)*c);
x_qdm(3)=(ft(t(2))-(k-2/dt^2*m)*x_qdm(2)...
    -(1/dt^2*m-1/(2*dt)*c)*x_qdm(1))/(1/dt^2*m+1/(2*dt)*c);
for n=4:length(t)
x_qdm(n)=(ft(t(n-1))-m/dt^2*(-2*x_qdm(n-1)+x_qdm(n-2))...
    -c/dt*(1/2*x_qdm(n-1)-x_qdm(n-2)+1/6*x_qdm(n-3))...
    -k*x_qdm(n-1))/(m/dt^2+c/(3*dt));
end
%% Simulink Heun Method
if isequal(F_ex,'Harmonic')
sim('simH')
elseif isequal(F_ex,'Arbitrary')
sim('simA')
end
%% Plot Solution
figure(i*2-1)
set(gcf, 'Units', 'Normalized', 'OuterPosition', [0.2 0.2 0.35 0.55]);
plot(t,ft(t),'b-','lineWidth',1.5)
xlabel('Time (s)','fontsize',16)
ylabel('f(t) (N)','fontsize',16)
figure(i*2)
set(gcf, 'Units', 'Normalized', 'OuterPosition', [0.2 0.2 0.35 0.55]);
if isequal(F_ex,'Harmonic')
plot(t,x_exact,'k-','lineWidth',0.8)
end
hold on
plot(t,x_sim,'bo','MarkerSize',5)
plot(t,x_qdm,'rv','MarkerSize',4)
xlabel('Time (s)','fontsize',16)
ylabel('Displacement (m)','fontsize',16)
axis tight
if isequal(F_ex,'Harmonic')
legend({'Exact Solution','Simulink','Hybrid Method'},...
    'Location','Best','fontsize',11)
else
legend({'Simulink','Hybrid Method'},'Location','Best','fontsize',11)
end
%% Error
if isequal(F_ex,'Harmonic')
err(i,1)=norm(x_exact-x_eem,inf);
err(i,2)=norm(x_exact-x_cdm,inf);
err(i,3)=norm(x_exact-x_qdm,inf);
err(i,4)=norm(x_exact-x_sim',inf);
end
i=i+1;
end
%% Order of Convergence
if isequal(F_ex,'Harmonic')
o_eem=polyfit(log10(linspace(dt_i,dt_f,p)'),log10(err(:,1)),1);
o_cdm=polyfit(log10(linspace(dt_i,dt_f,p)'),log10(err(:,2)),1);
o_qdm=polyfit(log10(linspace(dt_i,dt_f,p)'),log10(err(:,3)),1);
o_sim=polyfit(log10(linspace(dt_i,dt_f,p)'),log10(err(:,4)),1);
figure(i*2-1)
set(gcf, 'Units', 'Normalized', 'OuterPosition', [0.2 0.2 0.35 0.55]);
hold on
plot(log10(linspace(dt_i,dt_f,p*20)),...
```

```
    polyval(o_eem,log10(linspace(dt_i,dt_f,p*20))),'g-')
plot(log10(linspace(dt_i,dt_f,p*20)),...
    polyval(o_cdm,log10(linspace(dt_i,dt_f,p*20))),'b-')
plot(log10(linspace(dt_i,dt_f,p*20)),...
    polyval(o_qdm,log10(linspace(dt_i,dt_f,p*20))),'r-')
plot(log10(linspace(dt_i,dt_f,p*20)),...
    polyval(o_sim,log10(linspace(dt_i,dt_f,p*20))),'b--')
plot(log10(linspace(dt_i,dt_f,p)),log10(err(:,1)),'g*')
plot(log10(linspace(dt_i,dt_f,p)),log10(err(:,2)),'bs')
plot(log10(linspace(dt_i,dt_f,p)),log10(err(:,3)),'rv')
plot(log10(linspace(dt_i,dt_f,p)),log10(err(:,4)),'bo')
xlabel('log(dt)','fontsize',16)
ylabel('log(err)','fontsize',16)
axis tight
legend({sprintf('Euler=%.2f',o_eem(1)),...
    sprintf('Central=%.2f',o_cdm(1)),sprintf('Hybrid=%.2f',o_qdm(1)),...
    sprintf('Simulink=%.2f',o_sim(1))},'Location','Best','fontsize',11)
end
```

toc

In order to reach Matlab files, please use following Google Drive link:

https://drive.google.com/drive/folders/1RQn9rgrtEjv1IG9URDReIGww6OjrImYV?usp=sharing