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A modified parallel monotone hybrid algorithm for a finite family of $\mathcal{G}$-nonexpansive mappings and application to a novel signal recovery

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Abstract

In this work, we aim to prove the convergence of the sequences generated by the shrinking projection method and the parallel monotone method to find a common fixed point of a finite family of $\mathcal{G}$-nonexpansive mappings endowed with graphs. We obtain strong convergence results under some mild conditions. We provide numerical examples and give applications to signal recovery. Moreover, numerical experiments of our algorithms which different blurred matrices on the algorithm to show the efficiency and the implementation for signal recovery.

Keywords: Shrinking projection method $G-$nonexpasive mapping Common fixed point Hilbert space, Signal recovery.
2010 MSC: 47H10, 65K05, 94A12.

1. Introduction

Many application in applied science such as the signal recovery, image restoration [18, 19, 20, 31, 32, 33, 34] can be explained by the linear equation system in one dimensional vector as follows:

$$v = \mathcal{F}u + \epsilon, \tag{1}$$

where $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is the blurred matrix, $u \in \mathbb{R}^N$ is an original signal with k nonzero components to be recovered, ϵ is additive noise and $v \in \mathbb{R}^M$ is the observed signal. It is known that solving the problem (1) can be seen as the well-known regularized least square problem which is called LASSO problem:

$$\min_{u\in\mathbb{R}^N} \left(\frac{1}{2}\|y - \mathcal{F}u\|_2^2 + \lambda\|u\|_1\right), \tag{2}$$

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where $\lambda > 0$. Moreover, the problem (2) can be generalized by convex optimization problems as the following form:

$$\text{minimize}(f(u) + g(u)), \tag{3}$$

where $u \in \mathcal{H}$, $f, g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ are two proper, lower-semicontinuous and convex functions, and $\mathcal{H}$ is a real Hilbert space. It is known that the problem (3) is equivalent to the fixed point problem as follows:

$$u = \text{prox}_{\alpha g}(u - \alpha \nabla f(u)), \tag{4}$$

where $\alpha > 0$, $\text{prox}_g = (I + \partial g)^{-1}$ is the proximal operator of g and ∂g is subdifferential of g and ∇f denotes the gradient of f. From this point of view, it is known that (4) can generate a classical forward-backward algorithm in the following manner:

$$u_{n+1} = \text{prox}_{\alpha_n g}(u_n - \alpha_n \nabla f(u_n)), \tag{5}$$

where α_n is a suitable stepsize. Many mathematicians have used this algorithm (5) to modify in many ways such as the proximal point algorithm [13, 21, 25, 28] and the gradient method [11, 30, 41, 42]. For its applications, there have been modifications of the algorithm (5) in many various areas of science and physic etc., (see [7, 8, 10, 16, 17, 23, 26, 44, 36]).

In signal processing, the sound may be disturbed by many noises. The goal in this paper is to remove noise without knowing the type of noise and different blurred matrixes. Here, we aim to focus on the following problem

$$\begin{aligned}
&\min_{u\in\mathbb{R}^N}\left(\frac{1}{2}\|\mathcal{F}_1 u - v_1\|_2^2 \quad + \quad \lambda_1\|u\|_1\right),\\
&\min_{u\in\mathbb{R}^N}\left(\frac{1}{2}\|\mathcal{F}_2 u - v_2\|_2^2 \quad + \quad \lambda_2\|u\|_1\right),\\
&\qquad\qquad\qquad\qquad\vdots\\
&\min_{u\in\mathbb{R}^N}\left(\frac{1}{2}\|\mathcal{F}_N u - v_N\|_2^2 \quad + \quad \lambda_N\|u\|_1\right).
\end{aligned} \tag{6}$$

where $\mathcal{F}_k$ is a bounded linear operator, u is original signal and v_k is observed signal with noisy for all $k = 1, 2, ..., N$.

Before we start solving the problem (6), we recall the concept of the fixed point problem of $\mathcal{G}$−nonexpansive mapping. Let $\mathcal{K}$ be a nonempty subset of a real Hilbert spaces $\mathcal{H}$. Let Δ be the diagonal of the cartesian product $\mathcal{K} \times \mathcal{K}$, i.e., $\Delta = \{(u, u) : u \in \mathcal{K}\}$ and $\mathcal{G}$ be a directed graph such that the set $V(\mathcal{G})$ is vertices coincides of graph $\mathcal{G}$ with $\mathcal{K}$ and the set $E(\mathcal{G})$ is edges of graph $\mathcal{G}$ with $\Delta \subseteq E(\mathcal{G})$. We assume $\mathcal{G}$ has on parallel edge, then the graph $\mathcal{G}$ is the pair $(V(\mathcal{G}), E(\mathcal{G}))$. A mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is said to be

1. *Contraction* if $\mathcal{S}$ satisfies the following way: there exists $\alpha \in (0, 1)$ such that
$$\|\mathcal{S}u - \mathcal{S}v\| \le \alpha\|u - v\|,$$
for all $u, v \in \mathcal{K}$.
2. *Nonexpansive* if $\mathcal{S}$ satisfies the conditions:
$$\|\mathcal{S}u - \mathcal{S}v\| \le \|u - v\|,$$
for all $u, v \in \mathcal{K}$.
3. $\mathcal{G}$ − *contraction* if $\mathcal{S}$ satisfies the conditions:
   (1) $\mathcal{S}$ preserves edges of $\mathcal{G}$, i.e.,
   $$(u, v) \in E(\mathcal{G}) \Rightarrow (Su, Sv) \in E(\mathcal{G}),$$
   for all $(u, v) \in E(\mathcal{G})$.
   (2) $\mathcal{S}$ decreases weights of edges of $\mathcal{G}$ in the following way: there exists $\alpha \in (0, 1)$ such that
   $$(u, v) \in E(\mathcal{G}) \Rightarrow \|\mathcal{S}u - \mathcal{S}v\| \le \alpha\|u - v\|,$$
   for all $(u, v) \in E(\mathcal{G})$.

4. $\mathcal{G}-$*nonexpansive* if $\mathcal{S}$ satisfies the conditions:
   (1) $\mathcal{S}$ preserves edges of $\mathcal{G}$, i.e.,

$$(u, v) \in E(\mathcal{G}) \Rightarrow (\mathcal{S}u, \mathcal{S}v) \in E(\mathcal{G}),$$

   for all $(u, v) \in E(\mathcal{G})$.
   (2) S non-indecreases weights of edges of $\mathcal{G}$ in the following way:

$$(u, v) \in E(\mathcal{G}) \Rightarrow \|\mathcal{S}u - \mathcal{S}v\| \leq \|u - v\|,$$

   for all $(u, v) \in E(\mathcal{G})$.

The set of all fixed points of $\mathcal{S}$ is denoted by $F(\mathcal{S})$, i.e., $F(\mathcal{S}) = \{z \in \mathcal{K} : \mathcal{S}z = z\}$. If we set $\mathcal{S}u = \text{prox}_{\alpha g}(u - \alpha \nabla f(u))$ where $\alpha \in (0, 2/L)$ and L is the Lipschitz constant of the gradient of functions f, then $\mathcal{S}$ is nonexpansive. It is known that if $\mathcal{S}$ is nonexpansive, then $\mathcal{S}$ is $\mathcal{G}-$nonexpansive. This is the reason that why we interested in studying $\mathcal{G}-$nonexpansive mapping.

In 2008, the Banach's contraction principle was studied and extended to complete metric spaces endowed with a graph by Jachymski [15]. In 2012, Aleomraninejed et al. [1] introduced some iterative scheme for $\mathcal{G}-$contraction with $\mathcal{G}-$nonexpansive mappings in Banach spaces endowed with a graph. Recently, Alfuradan [3] studied the existence of fixed point and proved the convergence result of monotone nonexpansive mapping on a Banach space endowed with a directed graph. Since 2012, the Browders convergence theorem for $\mathcal{G}-$nonexpansive mapping in a Hilbert space with a directed graph, weak and strong convergence of some iterations for $\mathcal{G}-$nonexpansive mappings were discussed by many authors (see for example [1, 2, 3, 39, 40]).

Finding a common solution to a problem system is very useful in real-world problems. Many authors [9, 12, 27, 38] have proposed many algorithms to solve it. One of that is a parallel monotone hybrid algorithm was proposed for solving common fixed point problems of a finite family of quasi ϕ-nonexpansive mappings $\{\mathcal{S}_i\}_{i=1}^N$ in Banach spaces by Anh and Hieu [4, 5]. It was modified by using parallel methods and the Shrinking projection method [24] . It can be seen in Hilbert spaces as follows:

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n^k = \alpha_n u_n + (1 - \alpha_n)\mathcal{S}_k u_n, k = 1, 2, ..., N, \\ k_n = \text{argmax}\{\|v_n^k - u_n\| : k = 1, 2, ..., N\}, \bar{v_n} := v_n^{k_n}, \\ C_{n+1} = \{t \in C_n : \|t - \bar{v_n}\| \leq \|t - u_n\|\}, \\ u_{n+1} = P_{C_{n+1}} u_0, n \geq 1, \end{cases} \tag{7}$$

where $0 < \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$. It can be seen that at the $n-$th iteration step of the algorithm (7), $\bar{v_n}$ is chosen from the parallel of v_n^k for $k = 1, 2, ..., N$. For the final step, the projection on to]the closed convex set C_{n+1} which can be more easily performed then existing algorithms specially when the number of variational inequalities N is large. The parallel algorithm can solve the problem which is divided into sub-problem and are executed in parallel to get individual outputs which are combined together to get the final desired out put, so it have be used to solve many problem (see [9, 14, 37]).

Motivated by the previous works, we proposed a new algorithm with the modified parallel monotone algorithm for finding a common fixed point. Using the Shrinking projection method, we obtain the strong convergence theorem under suitable conditions in Hilbert spaces endowed with a directed graph. Further, we give an example and numerical experiments for supporting our main results. Finally, we use our proposed algorithm for applying to solve the signal recovery problem (6).

2. Main results

In this section, we prove strong convergence theorems of the modified parallel hybrid algorithm for a finite family of $\mathcal{G}-$nonexpansive mappings in real Hilbert spaces. Throughout this paper, we denote $u_n \to u$ and $u_n \rightharpoonup u$ as strong and weak convergence of $\{u_n\}$ to u, respectively.

Algorithm 2.1. Let $\mathcal{K}$ be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$ and let $G = (V(\mathcal{G}), E(\mathcal{G}))$ be a directed graph such that $V(\mathcal{G}) = \mathcal{K}$ and $E(\mathcal{G})$ is convex. Let $\mathcal{S}_k : \mathcal{K} \to \mathcal{K}$ be a $\mathcal{G}-$nonexpansive mapping for all $k = 1, 2, ..., N$ such that $\Omega := \bigcap_{k=1}^{N} F(\mathcal{S}_k) \neq \emptyset$, Ω is closed and $F(\mathcal{S}_k) \times F(\mathcal{S}_k) \subseteq E(\mathcal{G})$ for all $k = 1, 2, ..., N$. Take $C_0 = C_1$ and $u_1 \in \mathcal{K}$ arbitrarily and
Step1: Compute

$$v_n = \alpha_n^0 u_n + \sum_{k=1}^{N} \alpha_n^k \mathcal{S}_k u_n.$$

Step2: Calculate

$$C_{n+1} = \{t \in C_n : \|t - v_n\| \leq \|t - u_n\|\}$$

and

$$u_{n+1} = P_{C_{n+1}} u_0, n \geq 1$$

where $\{\alpha_n^k\}$ is a sequence in $[0, 1]$ for all $k = 0, 1, ..., N$ and $\sum_{k=0}^{N} \alpha_n^k = 1$.

Theorem 2.2. *Let $\{u_n\}$ be generated by Algorithm 2.1. Assume that the following conditions hold:*

(1) $\{u_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup w \in \mathcal{K}$, then $(u_{n_i}, w) \in E(\mathcal{G})$;

(2) $\liminf_{n\to\infty} \alpha_n^0 \alpha_n^k > 0$ for all $k = 1, 2, ..., N$.

Then $\{u_n\}$ convergence strongly to $w = P_\Omega u_1$.

Proof. We split the proof into five steps.
Step 1. Show that $P_{C_{n+1}}$ is well-defined for every $u_1 \in \mathcal{H}$. As shown in Theorem 3.2 of Tiammee et al. [2], $F(\mathcal{S}_k)$ is convex for all $k = 1, 2, ..., N$. It follows from our assumption that Ω is closed and convex. Hence, $P_\Omega u_1$ is well-defined. We see that $C_1 = \mathcal{K}$ is closed and convex. Assume that C_n is closed and convex. From the definition of C_{n+1} and Lemma 1.3 in [22], we get C_{n+1} is closed and convex. Let $p \in \Omega$. Since $\{u_n\}$ dominates p and $\mathcal{S}_k$ is edge-preserving for all $k = 1, 2, ..., N$, we have $(\mathcal{S}_k u_n, p) \in E(\mathcal{G})$ for all $k = 1, 2, ..., N$. This shows that $(v_n, p) = (\alpha_n^0 u_n + \sum_{k=1}^{N} \alpha_n^k \mathcal{S}_k u_n, p) \in E(\mathcal{G})$ by $E(\mathcal{G})$ is convex, we get

$$\begin{aligned} \|v_n - p\| &= \|\alpha_n^0 u_n + \sum_{k=1}^{N} \alpha_n^k \mathcal{S}_k u_n - p\| \\ &\leq \alpha_n^0 \|u_n - p\| + \sum_{k=1}^{N} \alpha_n^k \|\mathcal{S}_k u_n - p\| \\ &\leq \|u_n - p\|. \end{aligned}$$

Thus, we have $p \in C_{n+1}$. Therefore $\Omega \subset C_{n+1}$. This implies that $P_{C_{n+1}} u_1$ is well-defined.
Step 2. Show that $\lim_{n\to\infty} \|u_n - u_1\|$ exists. By the property of the metric projection P_Ω when Ω is a nonempty, closed and convex subset of $\mathcal{H}$, then there exists a unique $v \in \Omega$ such that $v = P_\Omega u_1$. From $u_{n+1} \in C_n$, for all $n \geq 1$, we get

$$\|P_{C_n} u_1 - u_1\| \leq \|u_{n+1} - u_1\|, \quad \forall n \geq 1. \tag{8}$$

On the other hand, as $\Omega \subset C_n$, we obtain

$$\|u_n - u_1\| \leq \|v - u_1\|, \quad \forall n \geq 1. \tag{9}$$

It follows from (8) and (9) that the sequence $\{u_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|u_n - u_1\|$ exists.

Step 3. Show that $u_n \to w \in \mathcal{K}$ as $n \to \infty$. For $j > n$, by the definition of C_n, since $u_j = P_{C_j} u_1 \in C_j \subset C_n$, it is from the metric projection that

$$\|u_j - u_n\|^2 \le \|u_j - u_1\|^2 - \|u_n - u_1\|^2.$$

Since $\lim_{n\to\infty} \|u_n - u_1\|$ exists, by Step 2, we have $u_j \to u_n$, as $n \to \infty$ i.e., $\{u_n\}$ is a Cauchy sequence. Hence, there exists $w \in \mathcal{K}$ such that $u_n \to w$ as $n \to \infty$. In particular, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{10}$$

Step 4. Show that $w \in \Omega$. Since $u_{n+1} \in C_{n+1} \subset C_n$, it follows from (10) that

$$\|v_n - u_n\| \le \|v_n - u_{n+1}\| + \|u_{n+1} - u_n\| \le 2\|u_{n+1} - u_n\| \to 0 \tag{11}$$

as $n \to \infty$. For $p \in \Omega$, it follows from Lemma 2.5 in [6] and $\{u_n\}$ dominates p that

$$\begin{aligned}
\|v_n - p\|^2 &= \|\alpha_n^0 u_n + \sum_{k=1}^{N} \alpha_n^k \mathcal{S}_k u_n - p\|^2 \\
&\le \alpha_n^0 \|u_n - p\|^2 + \sum_{k=1}^{N} \alpha_n^k \|\mathcal{S}_k u_n - p\|^2 - \sum_{k=1}^{N} \alpha_n^0 \alpha_n^k \|\mathcal{S}_k u_n - u_n\|^2 \\
&\le \|u_n - p\|^2 - \sum_{k=1}^{N} \alpha_n^0 \alpha_n^k \|\mathcal{S}_k u_n - u_n\|^2.
\end{aligned}$$

This implies that

$$\sum_{k=1}^{N} \alpha_n^0 \alpha_n^k \|\mathcal{S}_k u_n - u_n\|^2 \le \|u_n - p\|^2 - \|v_n - p\|^2.$$

It follows from the assumption (2) and (11), we obtain

$$\lim_{n\to\infty} \|\mathcal{S}_k u_n - u_n\| = 0$$

for all $k = 1, 2, ..., N$. From $u_n \to w$ as $n \to \infty$, the assumption (1) and Lemma 6 in [35], we have $w \in \Omega$.

Step 5. Show that $w = P_\Omega u_1$. By the property of the metric projection $P_{C_n} u_1$, we have

$$\langle u_1 - P_{C_n} u_1, P_{C_n} u_1 - p\rangle \ge 0, \quad \forall p \in C_n. \tag{12}$$

By taking the limit in (12), we obtain

$$\langle u_1 - w, w - p\rangle \ge 0, \quad \forall p \in C_n.$$

Since $\Omega \subset C_n$, so $w = P_\Omega u_1$. This completes the proof. □

We know that if $\mathcal{S}$ is nonexpansive, that $\mathcal{S}$ is $\mathcal{G}$−nonexpansive. Applying from Theorem 2.1, we obtain the following corollary.

Corollary 2.3. Let $\mathcal{K}$ be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{S}_k : \mathcal{K} \to \mathcal{K}$ be a nonexpansive mapping for all $k = 1, 2, ..., N$ such that $\Omega := \bigcap_{k=1}^{N} F(\mathcal{S}_k) \neq \emptyset$. Let $\{u_n\}$ be a sequence generated by

$$\begin{cases}
u_1 \in \mathcal{K}, \\
v_n = \alpha_n^0 u_n + \sum_{k=1}^{N} \alpha_n^k \mathcal{S}_k u_n, \\
C_{n+1} = \{t \in C_n : \|t - v_n\| \le \|t - u_n\|\}, \\
u_{n+1} = P_{C_{n+1}} u_0, n \ge 1,
\end{cases} \tag{13}$$

where $\{\alpha_n^k\}$ is a sequence in $[0,1]$ for all $k = 0,1,...,N$ and $\sum_{k=0}^{N} \alpha_n^k = 1$ with $\liminf_{n\to\infty} \alpha_n^0 \alpha_n^k > 0$ for all $k = 1,2,...,N$. Then $\{u_n\}$ convergence strongly to $w = P_\Omega u_1$.

3. Numerical Experiments

In this section, we provide the numerical experiments and apply the convex minimization problem 3 to signal restoration problem. All experiments and visualizations are performed on a computer (Intel(R) Core(TM) i7-2600 16 GB RAM/Windows 10/64-bit) with MATLAB 2022a.

In this experiments, firstly, we give an example in Euclidian space $\mathbb{R}^3$ which shows numerical experiment for supporting our main theorem.

Example 3.1. Let $\mathcal{H} = \mathbb{R}^3$ and $\mathcal{K} = [0,\infty) \times (-\infty, 5] \times [-5,5]$. Assume that $(u,v) \in E(\mathcal{G})$ if and only if $1 \le u_1, v_1$, $u_2, v_2 \le 0$, $-1.5 \le u_3, v_3 \le -0.5$ for all $u = (u_1,u_2,u_3)$, $v = (v_1,v_2,v_3) \in \mathcal{K}$. Define mappings $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ by

$$\begin{aligned}
\mathcal{S}_1 u &= (\log \frac{u_1}{3} + 3, -1, -1);\\
\mathcal{S}_2 u &= (3, -1, \frac{\tan(u_3+1)}{3} - 1);\\
\mathcal{S}_3 u &= (3, \frac{e^{u_2+1}}{2} - \frac{3}{2}, -1)
\end{aligned}$$

for all $u = (u_1,u_2,u_3) \in \mathcal{K}$. It is easy to check that $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are $\mathcal{G}$-nonexpansive such that $F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) = \{(3,-1,-1)\}$. On the other hand, $\mathcal{S}_1$ is not nonexpansive since for $u = (0.23,-5,5)$ and $v = (0.15,-5,5)$. This implies that $\|\mathcal{S}_1 u - \mathcal{S}_1 v\| > 0.1 > \|u - v\|$. $\mathcal{S}_2$ is not nonexpansive since for $u = (10,-7,0.1)$ and $v = (10,-7,0.29)$. We have $\|\mathcal{S}_2 u - \mathcal{S}_2 v\| > 0.3 > \|u - v\|$. Moreover, $\mathcal{S}_3$ is not nonexpansive since for $u = (0.5,0.48,1)$ and $v = (0.5,0.49,1)$. We have $\|\mathcal{S}_3 u - \mathcal{S}_3 v\| > 0.5 > \|u - v\|$. We use the mean squared error (MSE) to measure quantitatively, which is defined by

$$\text{MSE} = \frac{1}{N}\|u^k - u_*\|^2 < \varepsilon,$$

where u^k is an estimated point of u_*. In our experiment, we give cases as follows in Table 1. The numerical results are reported by Table 2.

Table 1: Choose parameters α for S_i, $i = 1, 2, 3$ and stop condition(Cauchy error) $< 10^{-9}$.

Initail point	Cases		Inputting						
			S_1	S_2	S_3	S_1S_2	S_1S_3	S_2S_3	$S_1S_2S_3$
$(7.61, -1.29, -7.98)$	C1	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.5	0.5	-	0.2
		α_2	-	0.9	-	0.4	-	0.5	0.3
		α_3	-	-	0.9	-	0.4	0.4	0.4
	C2	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.6	0.6	-	0.2
		α_2	-	0.9	-	0.3	-	0.6	0.4
		α_3	-	-	0.9	-	0.3	0.3	0.3
	C3	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.5	0.5	-	0.3
		α_2	-	0.9	-	0.4	-	0.5	0.2
		α_3	-	-	0.9	-	0.4	0.4	0.4
	C4	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.7	0.7	-	0.3
		α_2	-	0.9	-	0.2	-	0.7	0.4
		α_3	-	-	0.9	-	0.2	0.2	0.2
	C5	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.7	0.7	-	0.4
		α_2	-	0.9	-	0.2	-	0.7	0.3
		α_3	-	-	0.9	-	0.2	0.2	0.2
	C6	α_0	0.1	0.1	0.1	0.1	0.1	0.1	0.1
		α_1	0.9	-	-	0.6	0.6	-	0.4
		α_2	-	0.9	-	0.3	-	0.6	0.2
		α_3	-	-	0.9	-	0.3	0.3	0.3
	C7	α_0	0.25	0.25	0.25	0.25	0.25	0.25	0.25
		α_1	0.75	-	-	0.5	0.5	-	0.25
		α_2	-	0.75	-	0.25	-	0.5	0.25
		α_3	-	-	0.75	-	0.25	0.25	0.25
$(11, -0.9, -0.1)$	C8	α_0	0.25	0.25	0.25	0.25	0.25	0.25	0.25
		α_1	0.75	-	-	0.5	0.5	-	0.25
		α_2	-	0.75	-	0.25	-	0.5	0.25
		α_3	-	-	0.75	-	0.25	0.25	0.25
	C9	α_0	0.4	0.4	0.4	0.4	0.4	0.4	0.4
		α_1	0.6	-	-	0.3	0.3	-	0.1
		α_2	-	0.6	-	0.3	-	0.3	0.2
		α_3	-	-	0.6	-	0.3	0.3	0.3
	C10	α_0	0.4	0.4	0.4	0.4	0.4	0.4	0.4
		α_1	0.6	-	-	0.5	0.5	-	0.2
		α_2	-	0.6	-	0.1	-	0.5	0.3
		α_3	-	-	0.6	-	0.1	0.1	0.1
	C11	α_0	0.4	0.4	0.4	0.4	0.4	0.4	0.4
		α_1	0.6	-	-	0.5	0.5	-	0.2
		α_2	-	0.6	-	0.1	-	0.5	0.3
		α_3	-	-	0.6	-	0.1	0.1	0.1
	C12	α_0	0.4	0.4	0.4	0.4	0.4	0.4	0.4
		α_1	0.6	-	-	0.5	0.5	-	0.3
		α_2	-	0.6	-	0.1	-	0.5	0.2
		α_3	-	-	0.6	-	0.1	0.1	0.1
	C13	α_0	0.7	0.7	0.7	0.7	0.7	0.7	0.7
		α_1	0.3	-	-	0.2	0.2	-	0.1
		α_2	-	0.3	-	0.1	-	0.2	0.1
		α_3	-	-	0.3	-	0.1	0.1	0.1

Table 2: The convergence behavior of Table 1-**??**.

Cases		Inputting operators						
		S_1	S_2	S_3	S_1S_2	S_1S_3	S_2S_3	$S_1S_2S_3$
C1	Time	0.0585	0.0069	0.0074	0.0073	0.0069	0.0076	0.0074
	*Iter	55	53	34	46	41	58	48
C2	Time	0.0491	0.0088	0.0073	0.0068	0.0073	0.0070	0.0074
	*Iter	53	53	39	50	43	46	55
C3	Time	0.0571	0.0067	0.0067	0.0072	0.0082	0.0073	0.0082
	*Iter	53	56	34	48	47	53	39
C4	Time	0.0454	0.0068	0.0073	0.0068	0.0069	0.0086	0.0070
	*Iter	59	53	39	46	45	47	40
C5	Time	0.0499	0.0071	0.0072	0.0071	0.0074	0.0080	0.0070
	*Iter	53	54	34	42	54	63	45
C6	Time	0.0499	0.0074	0.0072	0.0069	0.0073	0.0074	0.0067
	*Iter	52	55	34	41	48	44	42
C7	Time	0.0513	0.0068	0.0103	0.0106	0.0084	0.0080	0.0073
	*Iter	71	63	50	54	55	59	47
C8	Time	0.0503	0.0071	0.0071	0.0068	0.0074	0.0081	0.0070
	*Iter	66	61	47	50	54	50	46
C9	Time	0.0545	0.0079	0.0086	0.0073	0.0076	0.0078	0.0086
	*Iter	81	73	54	66	70	65	64
C10	Time	0.0549	0.0077	0.0073	0.0074	0.0073	0.0073	0.0075
	*Iter	83	75	57	66	65	60	58
C11	Time	0.0548	0.0069	0.0071	0.0075	0.0073	0.0077	0.0069
	*Iter	84	74	54	66	75	77	66
C12	Time	0.0497	0.0069	0.0100	0.0079	0.0079	0.0075	0.0073
	*Iter	84	75	55	69	95	67	67
C13	Time	0.0489	0.0071	0.0078	0.0077	0.0077	0.0073	0.0077
	*Iter	161	146	114	128	148	126	129

The results are presented in Table **??**, where the CPU time and number of iterations for all cases under the three operators S_1, S_2 and S_3 by using the main algorithm. It is shown that in the CPU time and number of iterations of proposed algorithm decrease when α set by case C7 and C8, it has an effect on the number of iterations for input many mapping S_i.

Next,we apply the Algorithm 2.1 to solve the LASSO problem in signal recovery (6) by setting

$$S_i u = \text{prox}_{\lambda_i g_i}(u_n - \lambda_i \nabla f_i(u_n)).$$

In our experiment, the sparse vector $u \in \mathbb{R}^N$ is generated from uniform distribution in the interval [-2,2] with n nonzero elements. The matrix $A_i \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and invariance one. The observation v_i is generated by with Gaussian noise white signal-to-noise ratio (SNR).

In what follows, let the initial point is picked randomly. Let the step size $\alpha_n^k = 1/4$ in Algorithm 2.1. The numerical results are shown in Table 3.

Table 3: The computational result for solving the LASSO problem (6).

Cases	Size	Inputting	$m = 10$		$m = 20$		$m = 40$	
			Time	Iter	Time	Iter	Time	Iter
1-A1	$M = 512$	A_1	15.7565	7573	14.5070	7406	29.4638	10643
1-A2	$N = 256$	A_2	13.4425	7141	15.1371	7594	29.0689	10459
1-A3		A_3	16.0106	7798	12.7354	6917	27.2869	10206
1-A12		A_1A_2	0.6685	1521	0.6461	1489	0.6962	1627
1-A13		A_1A_3	0.8720	1672	0.6313	1411	0.9151	1731
1-A23		A_2A_3	0.8203	1594	0.6989	1440	1.0820	1843
1-A123		$A_1A_2A_3$	0.1480	481	0.1172	441	0.1437	546
2-A1	$M = 1024$	A_1	87.6031	12791	85.5404	12693	139.0744	16075
2-A2	$N = 512$	A_2	85.2670	13402	102.9699	13979	139.0744	16075
2-A3		A_3	99.7328	13691	109.2934	14155	149.6947	16789
2-A12		A_1A_2	6.0965	2634	6.8648	2819	6.9998	2859
2-A13		A_1A_3	6.1098	2587	6.7377	2654	7.0165	2837
2-A23		A_2A_3	5.8777	2383	8.3323	2923	8.736	3004
2-A123		$A_1A_2A_3$	1.644	617	1.2883	698	1.2076	635
3-A1	$M = 512$	A_1	13.4005	7094	16.1148	7781	32.4653	11103
3-A2	$N = 256$	A_2	13.8487	7258	16.6190	7811	34.5990	11543
3-A3		A_3	11.6239	6647	15.8769	7696	40.2163	12264
3-A12		A_1A_2	0.5897	1390	0.6758	1489	0.7678	1671
3-A13		A_1A_3	0.7108	1569	0.7120	1460	0.5935	1385
3-A23		A_2A_3	0.8484	1600	1.2158	1816	0.9180	1428
3-A123		$A_1A_2A_3$	0.1400	461	0.1888	539	0.1531	474

The Table 3, we set Case1-2 by input A_i, $i = 1, 2, 3$, SNR=40 and in Case 3 input A_1, SNR=40, A_2, SNR=60 and A_3, SNR=70. It is shown that the recovered signal by inputting $A_i, i = 1, 2, 3$ has less number of iterations and CPU time than inputting $A_i, i = 1, 2$ and $A_i, i = 1$ for all cases. Next, we give some numerical experiments for two cases in Table 3 to illustrate the convergence behavior of cases in comparison. We plot the number of iterations versus MSE< 10^{-5} are shown in Figure 11 and the original signal, observation data and recovered signal are shown in Figure 1, Figure 2, Figure 3-5, respectively.

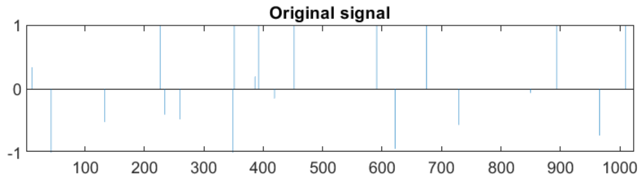

Figure 1: The original signal size $N = 512$, $M = 256$ and 20 spikes.

The different types of three blurred matrices are shown in Figure 2.

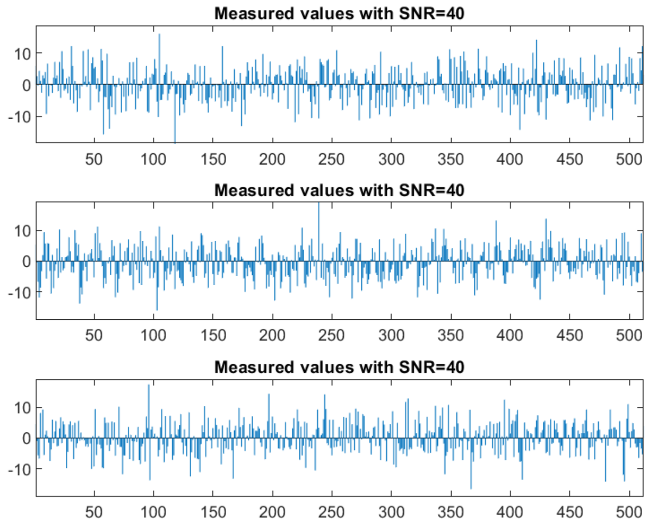

Figure 2: The measurred values with $A_i, i = 1, 2, 3$, SNR=40.

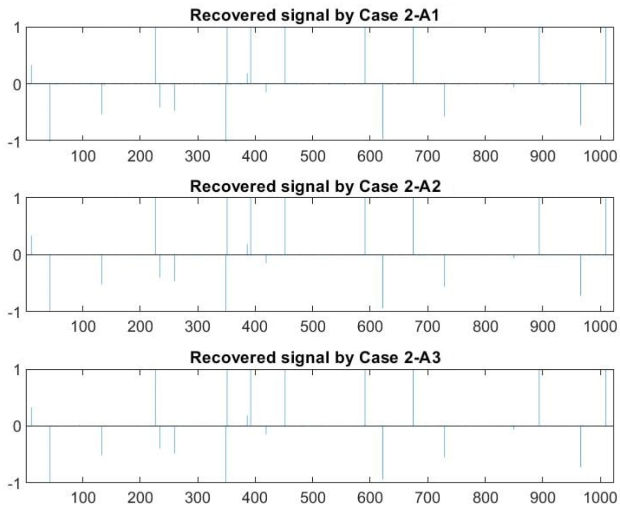

Figure 3: The recovered signal by Table 3 with $m = 20$ in case 2-A1 (12693 Iter, CPU=85.5404), case 2-A2 (13979 Iter, CPU=102.9699), case 2-A3 (14155 Iter, CPU=109.2934), respectively.

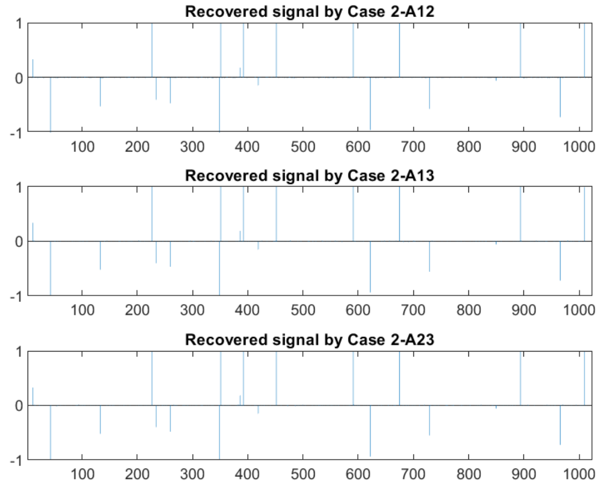

Figure 4: The recovered signal by Table 3 with $m = 20$ in case 2-A12 (2819 Iter, CPU=6.8648), case 2-A13 (2654 Iter, CPU=6.7377), case 2-A23 (2923 Iter, CPU=8.3323), respectively.

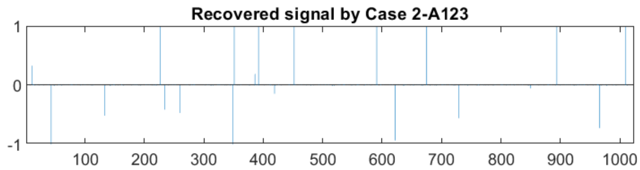

Figure 5: The recovered signal by Table 3 with $m = 20$ in case 2-A123 (698 Iter, CPU=1.2883).

Next, we provide the signal recovery by different type of inputting SNR. The original signal, observation data and recovered signal are shown in Figure 6, Figure 7, Figure 8-10, respectively.

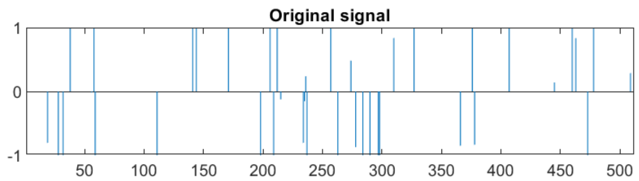

Figure 6: The original signal size $N = 1024$, $M = 512$ and 40 spikes.

The different types of three blurred matrices are shown in Figure 7.

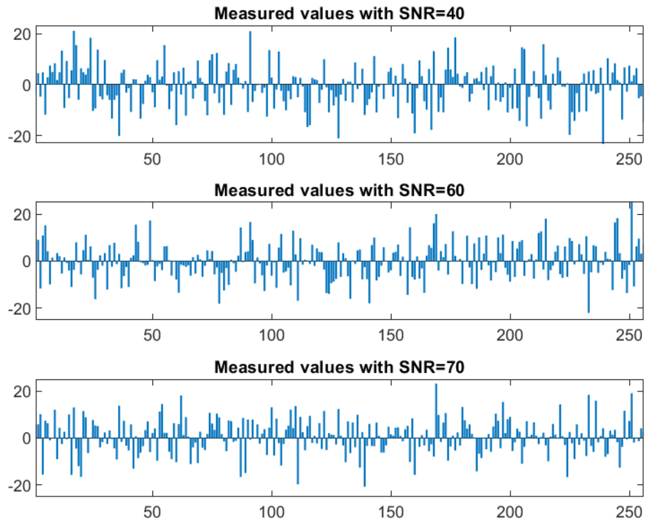

Figure 7: The measurred values with input A_1, SNR=40 A_2, SNR=60, A_3, SNR=70, respacetively.

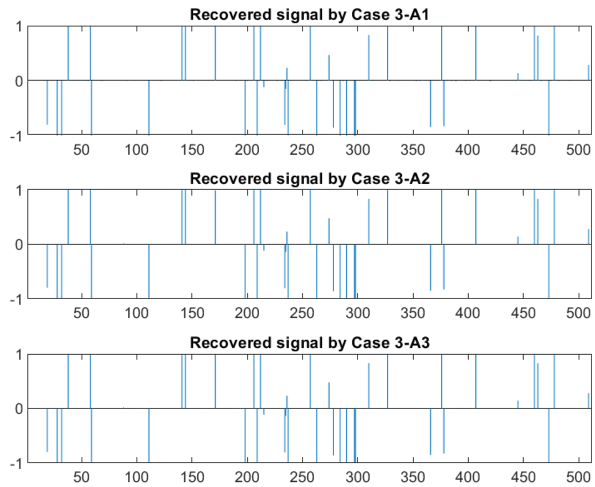

Figure 8: The recovered signal by Table 3 with $m = 40$ in case 3-A1 (11103 Iter, CPU=32.4653), case 3-A2 (11543 Iter, CPU=34.5990), case 3-A3 (12264 Iter, CPU=40.2163), respectively.

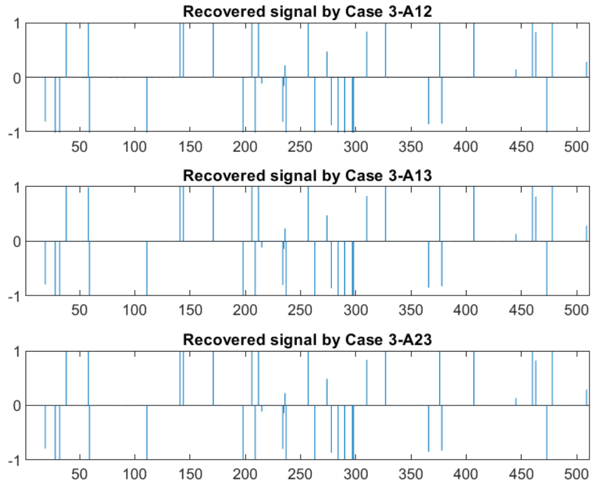

Figure 9: The recovered signal by Table 3 with $m = 40$ in case 3-A12 (1671 Iter, CPU=0.7678), case 3-A13 (1385 Iter, CPU=0.5935), case 3-A23 (1428 Iter, CPU=0.9180), respectively.

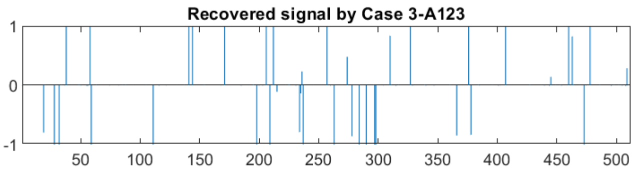

Figure 10: The recovered signal by Table 3 with $m = 40$ in case 3-A123 (474 Iter, CPU=0.5131).

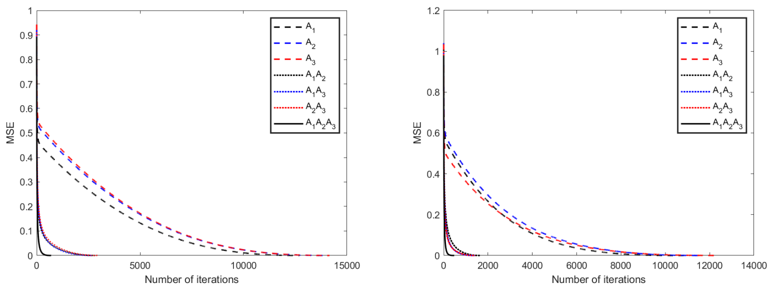

Figure 11: The MSE of Algorithm 2.1 with Table 3 with case 2, $m = 20$ and case 3, $m = 40$, respectively.

From Table 3 and Figure 11, we see that the CPU time and the numbers of iterations of $A_i, i = 1, 2, 3$ of Algorithm 2.1 are better than inputting $A_i, i = 1, 2$ and $A_i, i = 1$ of all cases for solving the LASSO problem in signal recovery.

Next, we provide a comparison among Algorithms 2.1, MSP algorithm [29] and NMTS algorithm [43]. Let the stepsize $\alpha_i = 1/4$ in Algorithms 2.1 and let $\alpha_n = \beta_n = \gamma_n = 0.1$ in SP algorithm [29] and NMTS algorithm [43]. The numerical results are shown in Table 4.

Table 4: The computational result for solving the LASSO problem (6).

Cases	Size	Inputting	$m = 10$		$m = 20$		$m = 40$	
			Time	Iter	Time	Iter	Time	Iter
1	$M = 512$	Algorithm 2.1	0.1480	481	0.1172	441	0.1437	546
	$N = 256$	MSP	0.2044	669	0.1738	614	0.2451	756
		NMTS	0.1979	663	0.1598	630	0.2537	746
2	$M = 1024$	Algorithm 2.1	1.644	617	1.2883	698	1.2076	635
	$N = 512$	MSP	1.5547	857	1.9789	959	1.6240	879
		NMTS	2.2776	1096	2.431	1136	2.3807	1180
3	$M = 512$	Algorithm 2.1	13.4005	7094	16.1148	7781	0.1531	474
	$N = 256$	MSP	0.2111	628	0.2262	731	0.2580	677
		NMTS	0.1759	604	0.1802	628	0.2466	738
4	$M = 1024$	Algorithm 2.1	1.640	622	1.2736	664	1.1695	624
	$N = 512$	MSP	1.5744	858	1.6933	911	1.5577	860
		NMTS	1.8619	982	2.5307	1215	2.5862	1233

Next, we give two cases in Table 3 to show the efficiency of Algorithm 2.1 in case A123 with tho other algorithms by MSE versus the number of iterations.

The original signal are shown in Figure 12-15, the observation signal are shown in Figure 13-16 and the recovered signal are shown in Figure 14-17, respectively.

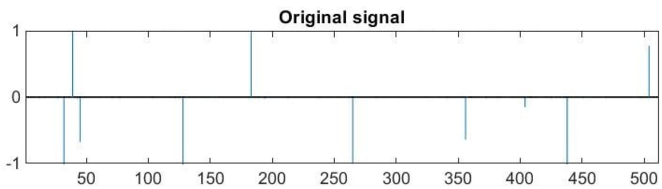

Figure 12: The original signal size $N = 512, M = 256$ and 10 spikes.

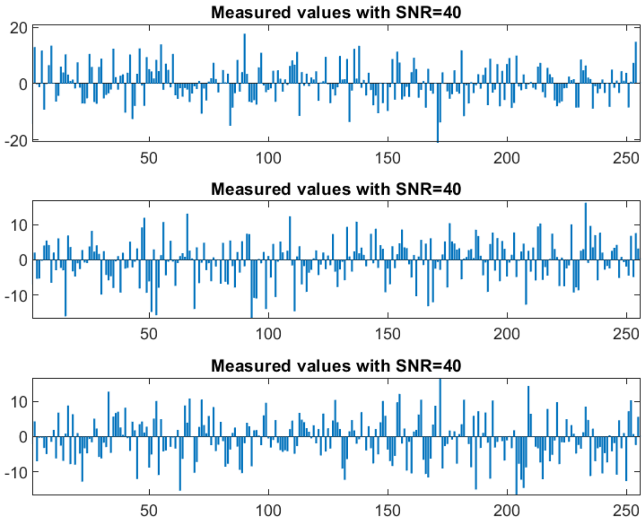

Figure 13: The measurred values with $A_i, i = 1, 2, 3$, SNR=40.

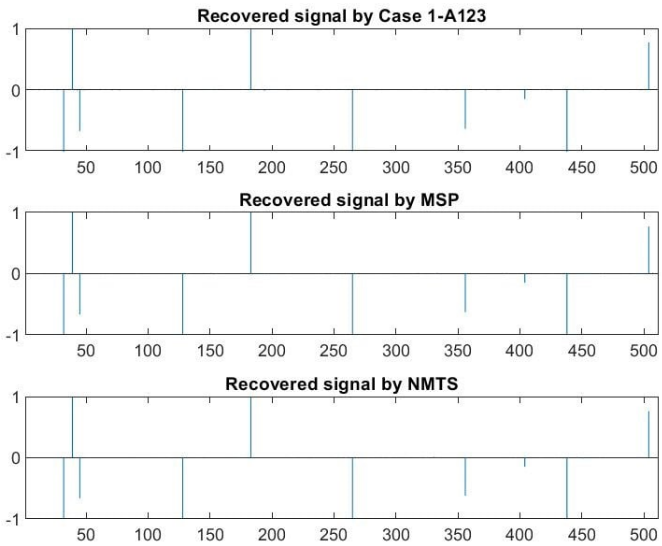

Figure 14: The recovered signal by Table 4 case 1 with $m = 10$ in Algorithm 2.1 (481 Iter, CPU=0.1480), MSP algorithm (669 Iter, CPU=0.2044), NMTS algorithm (1233 Iter, CPU=0.1979), respectively.

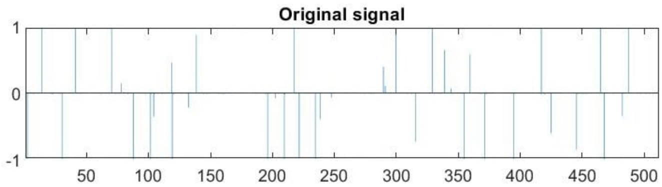

Figure 15: The original signal size $N = 1024$, $M = 512$ and 40 spikes.

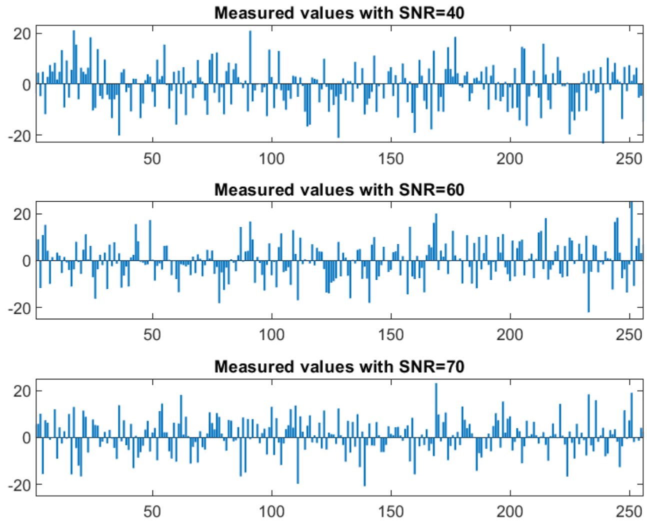

Figure 16: The measurred values with input A_1, SNR=40 A_2, SNR=60, A_3, SNR=70, respacetively.

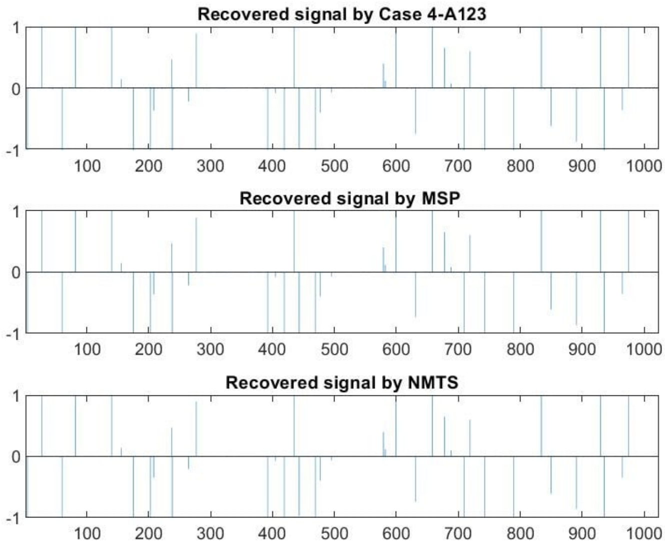

Figure 17: The recovered signal by Table 4 case 4 with $m = 40$ in Algorithm 2.1 (624 Iter, CPU=1.1695), MSP algorithm (860 Iter, CPU=1.5577), NMTS algorithm (1233 Iter, CPU=2.5862), respectively.

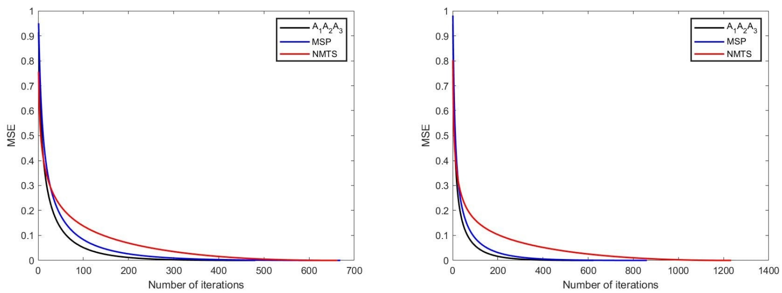

Figure 18: The MSE of Algorithm 2.1 by Table 4 with case 1, $m = 10$ and case 4, $m = 40$, respectively.

From Table 4 and Figure 18, we see that the CPU time and the numbers of iterations of Algorithm 2.1 are better than those of MSP algorithm and NMTS algorithm.

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