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Novel Solutions of Perturbed Boussinesq Equation

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Abstract

In this article, we have worked on the perturbed Boussinesq equation. We have applied the generalized Kudryashov method (GKM) and sine-Gordon expansion method (SGEM) to the perturbed Boussinesq equation. So, we have obtained some new soliton solutions of the perturbed Boussinesq equation. Furthermore, we have drawn some 2D and 3D graphics of these results by using Wolfram Mathematica 12.

1. Introduction

Perturbed Boussinesq equation (BE) is a category of nonlinear evolution equations (NLEEs). NLEEs have very important applications in areas such as plasma physics, mathematical physics, optical fibers, mathematical chemistry, hydrodynamics, fluid dynamics, geochemistry, control theory, meteorology, optics, mechanics, chemical kinematics, biophysics, biogenetics, and so on. NLEE's important work is carried out by scientists in many disciplines, especially mathematics and physics [1]-[12].
Perturbed BE is given as:

$$u_{tt}-k^2u_{xx}+p(u^{2n})_{xx}+ru_{xxxx}=\beta u_{xx}+\rho u_{xxxx}, \tag{1.1}$$

where ρ is the higher-order stabilization term and β shows the coefficient of dissipation [13, 14]. The perturbed BE is defined for areas such as plasma waves, quantum mechanics, acoustic waves, nonlinear optics, the elasticity of longitudinal waves in bars. Recently perturbed BE has been studied by some researchers.
Ebadi et al. have worked exponential function method and G'/G method [13]. Akbar et al. have applied the modified auxiliary equation technique for the perturbed BE [14]. Daripa and Dash have used the Pseudospectral method [15]. Dash and Daripa have established weakly nonlocal solitary wave solutions of the regularized sixth-order BE [16]. Jiao have used approximate symmetry method for (2+1)-dimensional perturbed BE [17].
Our aim in this study is to detect soliton solutions of perturbed BE through GKM [18]-[21] and SGEM [22]-[25]. In part 2, GKM and SGEM's structures are given. In part 3, some soliton solutions of perturbed BE is obtained by applying GKM and SGEM.

2. Methods

2.1. Structure of GKM

We take notice of a general nonlinear partial differential equation (NLPDE) in the following form:

$$P(u,u_t,u_x,u_{xx},u_{xt},\ldots)=0. \tag{2.1}$$

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Step 1. Firstly, we regard the travelling wave transform like as in the below form;

$$u(x,t) = u(\xi), \xi = x - vt, \tag{2.2}$$

by inserting Eq. (2.2) into Eq. (2.1). We reduce the Eq. (2.1) to the ordinary differential equation form:

$$R(u, u^{'}, u^{''}, u^{'''}, \cdots) = 0. \tag{2.3}$$

Step 2. Solutions of the obtained ordinary differential equation are taken as follows;

$$u(\xi) = \frac{\sum_{i=0}^{T} p_i Z^i(\xi)}{\sum_{j=0}^{K} r_j Z^j(\xi)} = \frac{A[Z(\xi)]}{B[Z(\xi)]}, \tag{2.4}$$

where Z is $\frac{1}{1 \pm e^{\xi}}$. Z is a solution to the $Z_{\xi} = Z^2 - Z$ equation,
Step 3. We use the homogeneous balance principle to find the values of K and T in Eq. (2.4). For this purpose, we balance between the highest order derivative and highest order nonlinear term in Eq. (2.3).
Step 4. We put Eq. (2.4) into Eq. (2.3). So we get a polynomial $R(Z)$ of Z. By equating all coefficients of $R(Z)$ to zero, we get a system of algebraic equations. By solving obtained system, we find c and the variable coefficients of $p_0, p_1, p_2, \ldots, p_T, r_0, r_1, r_2, \ldots, r_K$. Finally we can get the solutions of Eq. (2.1).

2.2. Structure of SGEM

We will give the general basic of SGEM. For this, we first handle the sine-Gordon equation

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{2.5}$$

where m is a real constant and $u = u(x,t)$ is a function.
Performing wave transformation $u(x,t) = u(\xi), \xi = \mu(x - kt)$ to Eq. (2.5),

$$u'' = \frac{m^2}{\mu^2 \left(1 - k^2\right)} \sin(u) \tag{2.6}$$

is obtained. Integrating Eq. (2.6) and setting the integration constant to zero, we have,

$$\left[\left(\frac{u}{2}\right)'\right]^2 = \frac{m^2}{\mu^2 \left(1 - k^2\right)} \sin^2\left(\frac{u}{2}\right). \tag{2.7}$$

Subsituting $w(\xi) = \frac{u}{2}$ and $b^2 = \frac{m^2}{\mu^2(1-k^2)}$ in Eq. (2.7), we get,

$$w' = b \sin(w). \tag{2.8}$$

If we receive $b = 1$ in Eq. (2.8), we have,

$$w' = \sin(w). \tag{2.9}$$

From the Eq. (2.9), we get,

$$\sin(w) = \sin(w(\xi)) = \left.\frac{2de^{\xi}}{d^2 e^{2\xi} + 1}\right|_{d=1} = \operatorname{sech}(\xi), \tag{2.10}$$

$$\cos(w) = \cos(w(\xi)) = \left.\frac{d^2 e^{2\xi} - 1}{d^2 e^{2\xi} + 1}\right|_{d=1} = \tanh(\xi), \tag{2.11}$$

To find the solution of the following nonlinear partial differential equation;

$$F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0, \tag{2.12}$$

we handle the equation given below,

$$u(\xi) = \sum_{i=1}^{n} \tanh^{i-1}(\xi) \left[B_i \operatorname{sech}(\xi) + A_i \tanh(\xi)\right] + A_0. \tag{2.13}$$

Considering the Eqs. (2.10) and (2.11), we can write the Eq. (2.13) as follows:

$$u(w) = \sum_{i=1}^{n} \cos^{i-1}(w) \left[B_i \sin(w) + A_i \cos(w)\right] + A_0. \tag{2.14}$$

Here we specify the value of n in Eq. (2.14) by means of balance principle, replace Eq. (2.14) into Eq. (2.12), and comparison the terms, we get a system of equations. By solving obtained system of equations, we acquire travelling wave solutions of the Eq. (2.12).

3. Application of Methods

3.1. GKM

To get the exact solutions of Eq. (1.1) we take account of the following transformation:

$$u(x,t) = u(\xi), \xi = x - vt. \tag{3.1}$$

Replacing Eq. (3.1) into Eq. (1.1) and integrating by taking the integration constant as zero, we get the following equation,

$$(v^2 - k^2 - \beta)u + p(u^{2n}) + (r - \rho)u^{''} = 0. \tag{3.2}$$

In Eq. (3.2), $u = q^{\frac{2}{2n-1}}$ transformation is applied. Thus, Eq. (3.2) is converted into the following form.

$$(v^2 - k^2 - \beta)q^2 + pq^4 + (r - \rho)\frac{2(3-2n)}{(2n-1)^2}\left(q^{'}\right)^2 + (r - \rho)\frac{2}{(2n-1)}qq^{''} = 0. \tag{3.3}$$

By using balance principle in Eq. (3.3), we get $T = K + 1$. Takes the value $T = 2$ for $K = 1$, so we get

$$u(\xi) = \frac{a_0 + a_1 Z + a_2 Z^2}{b_0 + b_1 Z}, \tag{3.4}$$

$$u^{'}(\xi) = \left(Z^2 - Z\right)\left[\frac{(a_1 + 2a_2 Z)(b_0 + b_1 Z) - b_1\left(a_0 + a_1 Z + a_2 Z^2\right)}{(b_0 + b_1 Z)^2}\right], \tag{3.5}$$

$$\begin{aligned} u^{''}(\xi) = {} & \frac{(Z^2 - Z)(2Z - 1)}{(b_0 + b_1 Z)}\left[(a_1 + 2a_2 Z)(b_0 + b_1 Z) - b_1\left(a_0 + a_1 Z + a_2 Z^2\right)\right] \\ & + \frac{(Z^2 - Z)^2}{(b_0 + b_1 Z)^3}\left[2a_2(b_0 + b_1 Z)^2 - 2b_1(a_1 + 2a_2 Z)(b_0 + b_1 Z) + 2b_1^2\left(a_0 + a_1 Z + a_2 Z^2\right)\right]. \end{aligned} \tag{3.6}$$

We find the solution cases as follows;
Case 1:

$$\begin{aligned} & a_0 = 0, \quad a_2 = -a_1, \quad b_0 = \frac{i(-1+2n)\sqrt{p}a_1}{2\sqrt{2}\sqrt{(1+2n)(r-\rho)}}, \quad b_1 = -\frac{i(-1+2n)\sqrt{p}a_1}{\sqrt{2r + 4nr - 2\rho - 4n\rho}}, \\ & v = -\frac{\sqrt{k^2(1-2n)^2 - 4r + \beta + 4(-1+n)n\beta + 4\rho}}{(1-2n)}. \end{aligned} \tag{3.7}$$

Soliton solutions of Eq. (1.1) are found by writing values in (3.7) into Eq. (3.4).

$$u_1(x,t) = \left(\frac{\sqrt{2}\sqrt{(1+2n)(r-\rho)}csc[ivt - ix]}{(1-2n)\sqrt{p}}\right)^{\frac{2}{2n-1}}. \tag{3.8}$$

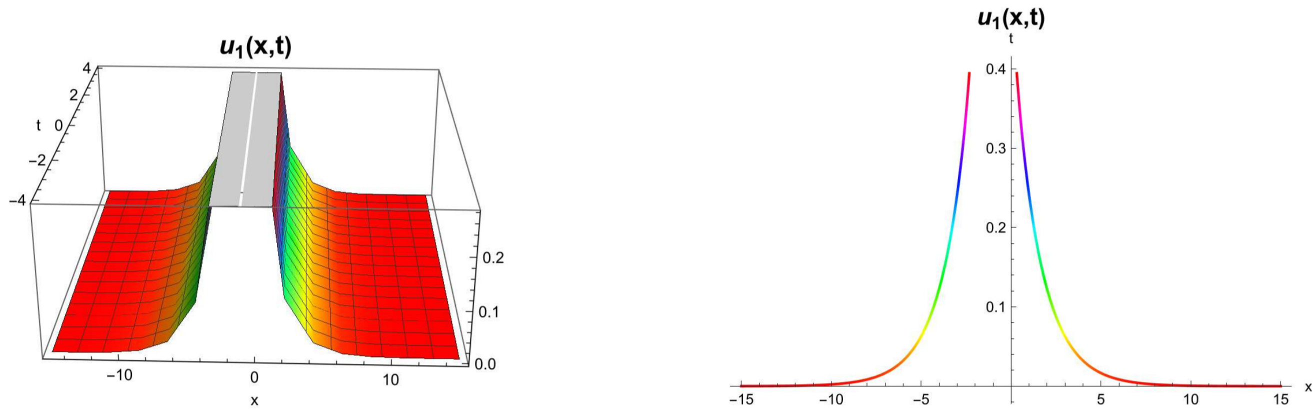

Figure 3.1: The 3D graph of the solution (3.8) for $n = 2,\ r = 2,\ \rho = 1.5,\ v = -1,\ p = 3, -15 \le x \le 15,\ -4 \le t \le 4$ and 2D graph for this values and $t = 1$.

Case 2:

$$a_0=0,\quad a_1=-a_2,\quad b_0=-\frac{b_1}{2},\quad \beta=\frac{-k^2(1-2n)^2+4r+(1-2n)^2v^2-4\rho}{(1-2n)^2},\quad p=\frac{2(1+2n)(-r+\rho)b_1^2}{(1-2n)^2a_2^2}. \tag{3.9}$$

Soliton solutions of Eq. (1.1) are found by writing values in (3.9) into Eq. (3.4).

$$u_2(x,t)=\left(\frac{csch[x-vt]a_2}{b_1}\right)^{\frac{2}{2n-1}}. \tag{3.10}$$

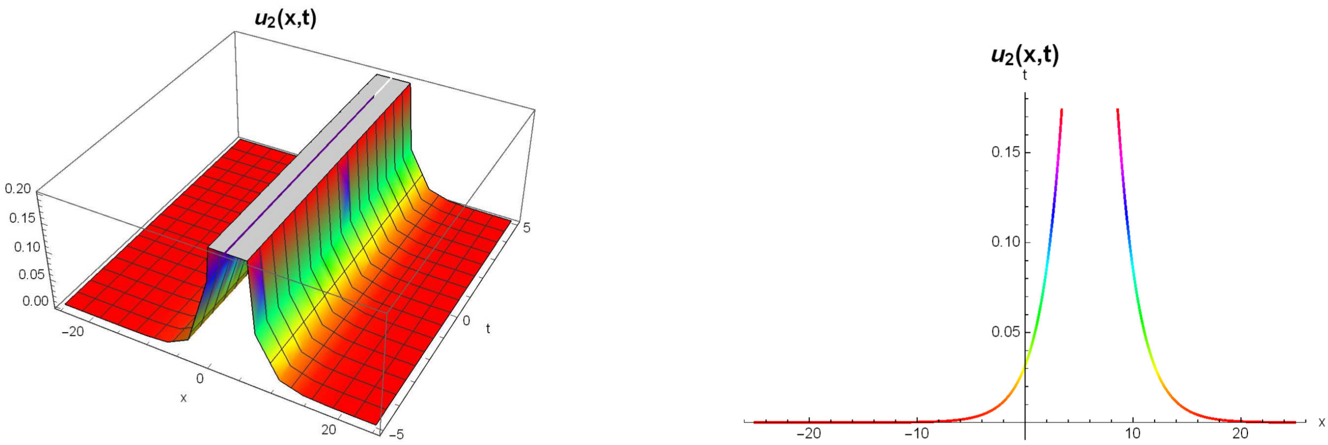

Figure 3.2: The 3D graph of the solution (3.10) for $n=2.5,\ v=1, a_2=1,\ b_1=5,\ -25\leq x\leq 25,\ -5\leq t\leq 5$ and 2D graph for this values and $t=3$.

3.2. SGEM

By using balance principle in Eq. (3.3), we find $N=1$. Using the value of $N=1$ in Eq. (2.14), we get:

$$u(w)=B_1\sin(w)+A_1\cos(w)+A_0, \tag{3.11}$$

$$u'(w)=B_1\cos(w)\sin(w)-A_1\sin^2(w), \tag{3.12}$$

$$u''(w)=B_1\cos^2(w)\sin(w)-B_1\sin^3(w)-2A_1\sin^2(w)\cos(w). \tag{3.13}$$

Placing Eq. (3.11), (3.12) and (3.13) into Eq. (3.3), we are generating trigonometric equations. We obtain an equation system by performing some mathematical operations in these trigonometric equations. Solving the obtained system of equations, we can result:

$$A_0=0,\quad A_1=0,\quad B_1=-\frac{\sqrt{2(1+2n)(r-\rho)}}{\sqrt{(1-2n)^2p}},\quad k=-\frac{\sqrt{4r+(1-2n)^2(v^2-\beta)-4\rho}}{\sqrt{(1-2n)^2}}. \tag{3.14}$$

For values (3.14) we get the following result:

$$u_3(x,t)=\left(-\frac{\sqrt{2(1+2n)(r-\rho)}\,\mathrm{sech}[x-vt]}{\sqrt{(1-2n)^2p}}\right)^{\frac{2}{2n-1}}. \tag{3.15}$$

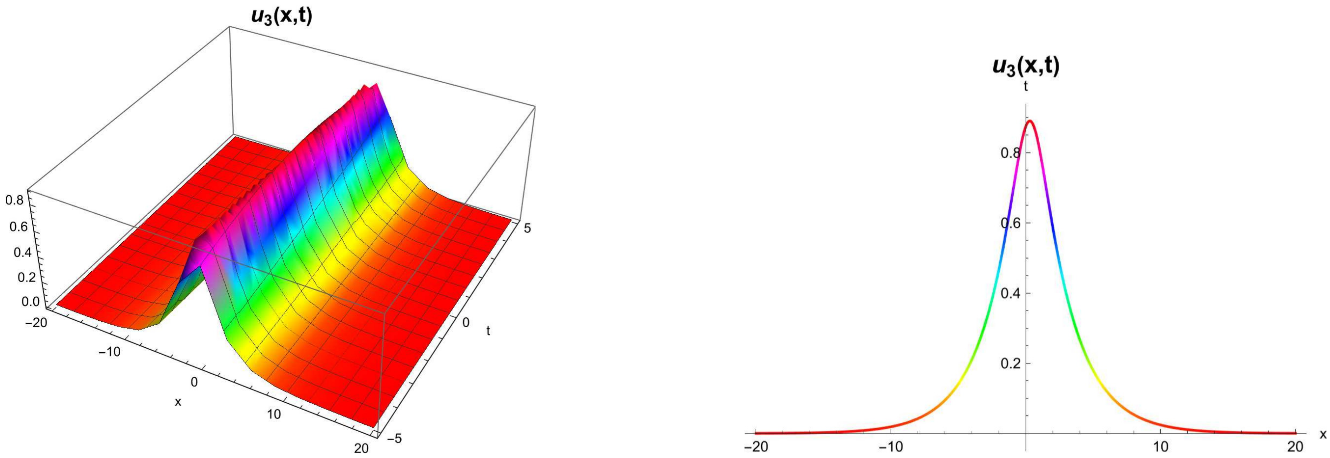

Figure 3.3: The 3D graph of the solution (3.15) for $n = 3, r = 2, \rho = 3, v = 0.2, p = -1, -20 \leq x \leq 20, -5 \leq t \leq 5$ and 2D graph for this values and $t = 1.5$.

4. Results and Discussion

In this study, the perturbed BE is discussed. GKM and SGEM have been applied to this equation and thus the solutions of the equation have been sought. As a result, bright soliton solutions of the equation have been acquired. As far as we researched, these obtained bright soliton solutions are new and have not been demonstrated before compared to previous studies. Both 2D and 3D graphical representations have been made for the physical representation of these obtained solutions.

5. Conclusion

In this study, the perturbed BE was studied. First, it is reduced to an ordinary differential equation by applying the traveling wave transform to the equation. Afterward, some n-dimensional soliton solutions of the equation were found by applying GKM and SGEM to this ordinary differential equation. 2D and 3D graphics were drawn thanks to Wolfram Mathematica 12 by giving certain values to the acquired solutions. According to our study, GKM and SGEM appear to be effective and reliable methods for finding NLEEs solutions. Thus, it is seen that GKM and SGEM are methods that facilitate the solution of NLEEs emerging in mathematical physics, applied mathematics and engineering.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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