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On Applications Shehu Variational Iteration Method to Time Fractional Initial Boundary Value Problems

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Abstract

The purpose of this study is to establish a semi analytical solution for time fractional linear or nonlinear mathematical problems by utilizing Shehu Variational Iteration Method (SVIM). SVIM is made up of two methods, called Shehu transform (ST) and variational iteration method (VIM). First of all, the time fractional differential equation is transformed into integer order differential equation by means of ST. Later, by taking VIM into account the solution of linear or nonlinear mathematical problem is acquired. The convergence analysis of the semi analytical solution is investigated and proves that SVIM is an accurate and effective method for fractional mathematical problems. The illustrated examples support analysis of this method.

Keywords: *Liouville-Caputo Derivative; Nonlinear Mathematical Problem; Shehu Transform; Variational Iteration Method*
2010 Mathematics Subject Classification: *26A33; 35A22*

1. Introduction

Recent years, modelling scientific processes such as reaction and diffusion processing, signal processing, electrical networks, etc. [16, 3] by fractional mathematical problems has been gained attention of researchers in various branches of science. Therefore, numerous numerical and analytical techniques such as reduced differential transform method (RDTM) [2], fractional difference method (FDM) [14], Adomian decomposition method have been developed to construct and investigate the solutions of fractional differential equations [9]. Main advantage of modelling by fractional differential equations is that the reflection of the processes is much more better compare to traditional differential equations. Moreover, existence, uniqueness and stability of the solution of fractional differential equations are analyzed by utilizing new developed methods.

ST, developed by Weidong Zhao and Shehu Maitama [11], is one of the most important linear integral transformations to play a significant role for constructing the solutions of differential equations. ST is regarded as modification of integral transformation Laplace. Laplace transformation is a special case of Shehu transformation. By means of ST, any kind of differential equation is transformed into an algebraic equation or simplistic differential equation. In addition to that the implementation of Shehu transformation for various ordinary and partial differential equations as well as fractional differential equations is easier compare to many other transformations. Furthermore, since integral transformations play a significant role in the solution of all kinds of differential equations, this makes Shehu transformation more valuable. New properties and an application of this transformation are presented in [5].

VIM and its modifications are very common to establish numerical solutions of initial value problems including differential equations. By utilizing initial condition the numerical solution is developed in VIM. As a result, it plays a significant role in construction of initial value problems [12, 6, 1, 8].

The purpose this research is to utilize a new method called SVIM to construct semi-analytical solutions of time fractional initial value problems. The algorithm of SVIM allows us to establish rapidly convergent numerical solutions to exact solutions of closed form. One of greatest advantages of SVIM, we do not deal with linearization or any restriction compare with other numerical techniques. The rest of paper is planned as follows: In section 2, fundamental notions and properties of the fractional calculus are presented. The implementation and converges analysis of SVIM are given in section 3. The examples of fractional linear diffusion problem and fractional nonlinear Fornberg-Whitham problem, playing significant roles in applied mathematics and physics, are presented to confirm the obtained results in section 4. Section 5 includes the conclusions on SVIM.

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2. Preliminary Results

The fundamental concepts and characteristic of the fractional calculus are presented in this section [14, 10].
The definition of Riemann-Liouville integral of a real valued function $u(x,t)$ with respect to time is introduced in the following form:

$$I_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x,s)ds, \tag{2.1}$$

where $\alpha > 0$ represents the order of the Riemann-Liouville integral. The definition of the Liouville-Caputo derivative of $u(x,t)$ is given as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = I_t^{m-\alpha}\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1} \frac{\partial^m u(x,y)}{\partial y^m} dy, \ m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, \ \alpha = m,. \end{cases} \tag{2.2}$$

where $\alpha > 0$ represents the order of the Liouville-Caputo fractional derivative.
The two parameters Mittag-Leffler function is introduced as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, Re(\alpha)>0,\ z,\beta \in \mathbb{C}, \tag{2.3}$$

where α and β are the parameters of the function. The domain of Shehu transformation is the following set:

$$\left\{ f(t) | \exists P, \tau_1,\ \tau_2 > 0, |f(t)| < Pe^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty) \right\}.$$

Moreover, the definition of Shehu transformation is presented in the following form:

$$\mathbb{S}[f(t)] = F(p,q) = \int_0^{\infty} e^{-\frac{p}{q}t} f(t)\,dt \tag{2.4}$$

which has the following property

$$\mathbb{S}[t^{\alpha}] = \int_0^{\infty} e^{-\frac{pt}{q}} t^{\alpha} dt = \Gamma(\alpha+1)\left(\frac{q}{p}\right)^{\alpha+1}, Re(\alpha)>0, \tag{2.5}$$

where the inverse Shehu transform of $\left(\frac{q}{p}\right)^{n\alpha+1}$ is computed as

$$\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{n\alpha+1}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, Re(\alpha)>0 \tag{2.6}$$

where $n > 0$ [11] [6]. The definition of the Shehu transformation of Liouville-Caputo time fractional derivative for the function $f(x,t)$ is computed as [4]:

$$\mathbb{S}\left[\frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}}\right] = \left(\frac{p}{q}\right)^{\alpha} \mathbb{S}[f(x,t)] - \sum_{k=0}^{n-1}\left[\left(\frac{p}{q}\right)^{\alpha-k-1} \frac{\partial^k f(x,0)}{\partial t^k}\right], n-1<\alpha\le n, n\in\mathbb{N} \tag{2.7}$$

where $\alpha > 0$ is the order of the Liouville-Caputo fractional derivative.

3. Main Results

3.1. SVIM for time fractional initial value problems

For the presentation of the implementation of SVIM, we consider the following fractional initial value problem:

$$\begin{aligned} &{}^{C}D_t^{\beta} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), m-1<\beta\le m, m=1,2,3,\ldots, \\ &\left[\frac{\partial^{m-1} u(x,t)}{\partial t^{m-1}}\right]_{t=0} = g_{m-1}(x), \end{aligned} \tag{3.1}$$

where we have Liouville-Caputo fractional derivative, $g(x,t)$, R and N represents the source function, linear and nonlinear parts of the differential equation, respectively.
Employing Shehu transformation to Eq. (3.1) leads to

$$\mathbb{S}[u(x,t)] = \sum_{k=0}^{m-1}\left[\left(\frac{q}{p}\right)^{k+1} \frac{\partial^k u(x,0)}{\partial t^k}\right] - \left(\frac{q}{p}\right)^{\beta} \mathbb{S}[Ru(x,t)+Nu(x,t)] + \left(\frac{q}{p}\right)^{\beta} \mathbb{S}[g(x,t)]. \tag{3.2}$$

Applying the inverse Shehu transformation to Eq. (3.2) results in

$$u(x,t) = k(x,t) - \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[Ru(x,t) + Nu(x,t)\right]\right]\right]$$

where $k(x,t) = \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[\sum_{k=0}^{m-1}\left[\left(\frac{p}{q}\right)^{k+1}\frac{\partial^k u(x,0)}{\partial t^k}\right]\right]\right] + \left(\frac{q}{p}\right)^{\beta}\mathbb{S}\left[g(x,t)\right]\right]$, and so

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[Ru(x,t) + Nu(x,t)\right]\right]\right] - \frac{\partial}{\partial t}k(x,t) = 0.$$

Utilizing VIM leads to the following recurrence relation:

$$u_{m+1}(x,t) = u_m(x,t) - \int_0^t\left[\frac{\partial u_m(x,\tau)}{\partial \tau} + \frac{\partial}{\partial \tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[Ru_m(x,\tau) + Nu_m(x,\tau)\right]\right]\right] - \frac{\partial}{\partial \tau}k(x,\tau)\right]d\tau.$$

Alternately

$$u_{m+1}(x,t) = k(x,t) - \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[Ru_m(x,t) + Nu_m(x,t)\right]\right]\right],$$

is called $(m+1)^{th}$ order of truncated solution.
If $u(x,t) = \lim_{m\to\infty} u_m(x,t)$ exists, the analytical solution $u(x,t)$ is obtained.

3.2. Analysis of Convergence

In this section, we investigate the convergence of VIM and its error estimate as well as the required conditions [15].
Let us define the following operator V as:

$$V = -\int_0^t\left[\frac{\partial u_m(x,\tau)}{\partial \tau} + \frac{\partial}{\partial \tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\left[\mathbb{S}\left[Ru_m(x,\tau) + Nu_m(x,\tau)\right]\right]\right] - \frac{\partial}{\partial \tau}k(x,\tau)\right]d\tau \tag{3.3}$$

where the components $v_k,\ k = 0,1,2,\ldots$ satisfy

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) = \sum_{k=0}^{\infty} v_k. \tag{3.4}$$

The following theorem is a result of Banach fixed point theorem.

Theorem 3.1. *[13] Let V, defined in (3.3), be an operator from a Banach space BS to BS. The series solution $u(x,t) = \lim_{m\to\infty} u_m(x,t) = \sum_{k=0}^{\infty} v_k$ as defined in (3.4), converges if $0 < p < 1$ exists such that $\|V[v_0 + v_1 + v_2 + \ldots + v_{k+1}]\| \leq p\|V[v_0 + v_1 + v_2 + \ldots + v_k]\|$, (i.e. $\|v_{k+1}\| \leq \|v_k\|$), $\forall k \in \mathbb{N}\cup\{0\}$.*

Utilization of Theorem 1 ensures the sufficiency of convergence for the series solution of VIM.

Theorem 3.2. *[13] The convergence of series solution $u(x,t) = \sum_{k=0}^{\infty} v_k$ in (3.4) ensure the existence of analytical solution for nonlinear problem (3.1).*

Theorem 3.3. *[13] Under the assumption that $\sum_{k=0}^{\infty} v_k$ in (3.4) converges to $u(x,t)$, the following inequality holds:*

$$E_j(x,t) \leq \frac{1}{1-p}p^{j+1}\|v_0\|).$$

where $E_j(x,t)$ denotes the maximum error of the truncated solution $\sum_{k=0}^{j} v_k$.
The sum $\sum_{k=0}^{\infty} v_k$ converges to an analytical solution $u(x,t)$, the parameters χ_i for $i \in \mathbb{N}\cup\{0\}$ are defined as

$$\chi_i = \begin{cases} \dfrac{\|v_{i+1}\|}{\|v_i\|}, \|v_i\| \neq 0, \\ 0, \|v_i\| = 0. \end{cases}$$

under the conditions $0 < \chi_i \leq 1, \forall i \in \mathbb{N}\cup\{0\}$.
Moreover, the maximum absolute truncation error fulfills the following inequality

$$\|u(x,t) - \sum_{k=0}^{\infty} v_k\| \leq \frac{1}{1-\chi}\chi^{j+1}\|v_0\|$$

where $\chi = max\{\chi_i,\ i = 0,1,2,\ldots,j\}$.

4. Illustrative Examples

In this section, illustrative examples are presented to show the implementation, accuracy and effectiveness of SVIM.

Example 4.1. *Consider the following fractional diffusion initial value problem:*

$$\begin{aligned} {}^{C}D_t^{\beta} u(x,t) &= \frac{x^2}{2} u_{xx}(x,t), 0<\beta\leq 1 \\ u(x,0) &= x^2. \end{aligned} \tag{4.1}$$

Step 1. *The implementation of Shehu transform for (4.1) leads to the following:*

$$\mathbb{S}[u(x,t)] = \left(\frac{q}{p}\right) x^2 + \left(\frac{q}{p}\right)^{\beta} \mathbb{S}\left[\frac{x^2}{2} u_{xx}(x,t)\right]. \tag{4.2}$$

Step 2. *Applying inverse Shehu transform to (4.2) leads to the following:*

$$u(x,t) = x^2 + \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta} \mathbb{S}\left[\frac{x^2}{2} u_{xx}(x,t)\right]\right]$$

and so

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial}{\partial t} x^2 + \frac{\partial}{\partial t} \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta} \mathbb{S}\left[\frac{x^2}{2} u_{xx}(x,t)\right]\right]. \\ \frac{\partial u(x,t)}{\partial t} &- \frac{\partial}{\partial t} \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta} \mathbb{S}\left[\frac{x^2}{2} u_{xx}(x,t)\right]\right] = 0. \end{aligned}$$

Step 3. *Employing variational iteration method, we*

$$u_{m+1}(x,t) = u_m(x,t) - \int_0^t \left[\frac{\partial u_m(x,\tau)}{\partial \tau} - \frac{\partial}{\partial \tau} \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta} \mathbb{S}\left[\frac{x^2}{2} u_{xx}(x,\tau)\right]\right]\right] d\tau \tag{4.3}$$

Based on the iteration formula (4.3), we have

$$\begin{aligned} u_0(x,t) &= x^2. \\ u_1(x,t) &= x^2 + x^2 \frac{t^{\beta}}{\Gamma(\beta+1)}. \\ u_2(x,t) &= x^2\left[1 + \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\right]. \\ u_3(x,t) &= x^2\left[1 + \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)}\right]. \end{aligned}$$

By means of the reiteration formula, the m^{th} approximate solution of (4.1) is computed in the following form:

$$u_m(x,t) = x^2 \sum_{k=0}^{m} \frac{t^{k\beta}}{\Gamma(k\beta+1)}, m = 0,1,2,... \tag{4.4}$$

Consequently, we reach the following result:

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) = x^2 E_{\beta,1}\left(t^{\beta}\right).$$

In Table 1, the values of truncated solutions, constructed by SVIM, for various β values are illustrated. From the analysis of them, we reach the conclusion that SVIM allows us to establish the truncated solution which tends to the exact solution of the fractional initial value problem. The solution for $\beta = 1$ and analytical solution are identical which indicates that SVIM works for time fractional initial value problems very well. Fig. 4.1 and Fig 4.2 also support the results that SVIM is one of the best method for time fractional initial value problems.

Example 4.2. *Let us consider fractional nonlinear Fornberg-Whitham equation*

$${}^{C}D_t^{\beta} u(x,t) = u_{xxt}(x,t) - u_x(x,t) + u(x,t) u_{xxx}(x,t) - u(x,t) u_x(x,t) + 3u_x(x,t) u_{xx}(x,t), \ 0<\beta\leq 1, \ t>0. \tag{4.5}$$

with the condition at $t = 0$

$$u(x,0) = e^{\frac{x}{2}}.$$

Table 1: The values of the exact and truncated solutions for various values β.

t	x		$\beta = 1$	$\beta = 3/4$	$\beta = 2/3$
		u_{exact}	u_{SVIM}	u_{SVIM}	u_{SVIM}
0.2	0.3	0,109926248234415	0,109926248234415	0,112601572771999	0,113362023042690
	0.6	0,439704992937661	0,439704992937661	0,450406291087996	0,453448092170761
	0.9	0,989336234109738	0,989336234109738	1,01341415494799	1,02025820738421
0.4	0.3	0,134264222787714	0,134264222787714	0,142709766064627	0,145544288235646
	0.6	0,537056891150857	0,537056891150857	0,570839064258509	0,582177152942582
	0.9	1,20837800508943	1,20837800508943	1,28438789458165	1,30989859412081
0.6	0.3	0,163990692035146	0,163990692035146	0,183216496981389	0,190632892411691
	0.6	0,655962768140583	0,655962768140583	0,732865987925555	0,762531569646766
	0.9	1,47591622831631	1,47591622831631	1,64894847283250	1,71569603170522
0.8	0.3	0,200298683564322	0,200298683564322	0,238246951321844	0,254896656569663
	0.6	0,801194734257288	0,801194734257288	0,952987805287378	1,01958662627865
	0.9	1,80268815207890	1,80268815207890	2,14422256189660	2,29406990912697
1	0.3	0,244645364561314	0,244645364561314	0,313727959804657	0,348088892677642
	0.6	0,978581458245256	0,978581458245256	1,25491183921863	1,39235557071057
	0.9	2,20180828105183	2,20180828105183	2,82355163824191	3,13280003409878

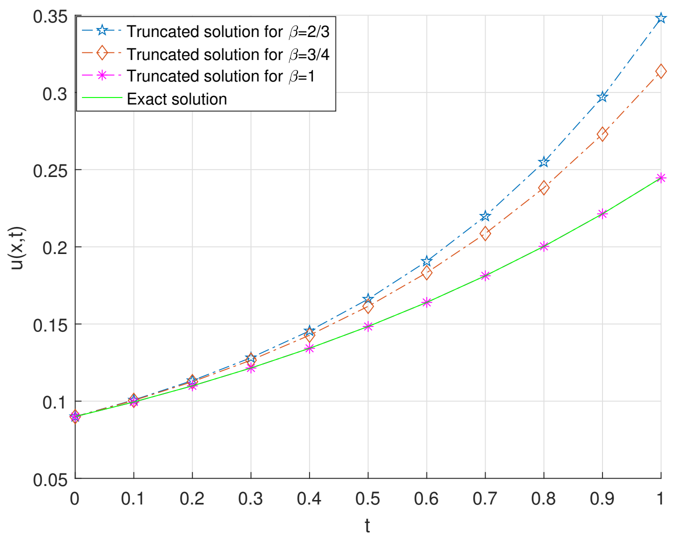

Figure 4.1: The graphs of analytical solution and truncated solutions of order 10 for various β values at $x = 0.3$ for example 4.1.

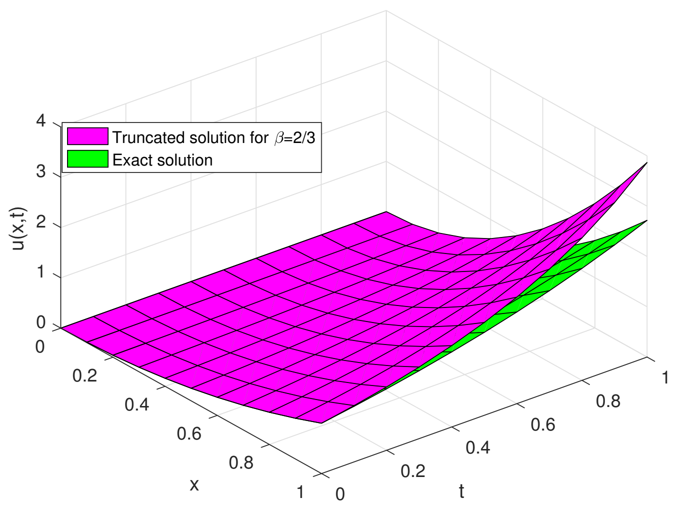

Figure 4.2: The graphs of analytical solution and truncated solutions of order 10 at $\beta = 2/3$ for example 4.1.

Step 1. *Utilizing Shehu transform for (4.5) leads to*

$$\mathbb{S}[u(x,t)]-\left(\frac{q}{p}\right)e^{\frac{x}{2}}=\left(\frac{q}{p}\right)^{\beta}\mathbb{S}\Big[u_{xxt}(x,t)-u_{x}(x,t)+u(x,t)u_{xxx}(x,t)-u(x,t)u_{x}(x,t)+3u_{x}(x,t)u_{xx}(x,t)\Big] \tag{4.6}$$

Step 2. *Carrying out inverse Shehu transform for (4.6), we have*

$$u(x,t)=e^{\frac{x}{2}}+\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\mathbb{S}\Big[u_{xxt}(x,t)-u_{x}(x,t)+u(x,t)u_{xxx}(x,t)-u(x,t)u_{x}(x,t)+3u_{x}(x,t)u_{xx}(x,t)\Big]\right]$$

and so

$$0=\frac{\partial u(x,t)}{\partial t}-\frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\beta}\mathbb{S}\Big[u_{xxt}(x,t)-u_{x}(x,t)+u(x,t)u_{xxx}(x,t)-u(x,t)u_{x}(x,t)+3u_{x}(x,t)u_{xx}(x,t)\Big]\right]$$

Step 3. *Enforcing the variational iteration method leads to the following:*

$$\begin{aligned}u_{m+1}(x,t)=u_{m}(x,t)-\int_{0}^{t}\Bigg[\frac{\partial u_{m}(x,\tau)}{\partial\tau}-\frac{\partial}{\partial\tau}\mathbb{S}^{-1}\Bigg[\left(\frac{q}{p}\right)^{\beta}\mathbb{S}\Big[(u_{m}(x,\tau))_{xxt}-(u_{m}(x,\tau))_{x}\\+u_{m}(x,\tau)(u_{m}(x,\tau))_{xxx}-u_{m}(x,\tau)(u_{m}(x,\tau))_{x}+3(u_{m}(x,\tau))_{x}(u_{m}(x,\tau))_{xx}\Big]\Bigg]\Bigg]d\tau\end{aligned} \tag{4.7}$$

Based on the iteration formula (4.7), we have

$$\begin{aligned}u_{0}(x,t)&=e^{\frac{x}{2}}.\\u_{1}(x,t)&=e^{\frac{x}{2}}\left[1-\frac{t^{\beta}}{2\Gamma(\beta+1)}\right].\\u_{2}(x,t)&=e^{\frac{x}{2}}\left[1-\frac{1}{2}\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{1}{4}\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{1}{8}\frac{t^{2\beta-1}}{\Gamma(2\beta)}\right].\\u_{3}(x,t)&=e^{\frac{x}{2}}\left[1-\frac{1}{32}\frac{t^{3\beta-2}}{\Gamma(3\beta-1)}-\frac{1}{8}\frac{t^{2\beta-1}}{\Gamma(2\beta)}+\frac{2}{16}\frac{t^{3\beta-1}}{\Gamma(3\beta)}-\frac{1}{2}\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{1}{4}\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{1}{8}\frac{t^{3\beta}}{\Gamma(3\beta+1)}\right].\end{aligned}$$

The recurrence relation allows us to construct the m^{th} order of numerical solution of (4.5) in following form:

$$u_{m}(x,t)=e^{\frac{x}{2}}\left[1+\sum_{k=1}^{m}\sum_{l=0}^{k-1}\binom{k-1}{l}\left(-\frac{1}{2}\right)^{k+l}\frac{1}{\Gamma(k\beta+1-l)}t^{k\beta-l}\right]. \tag{4.8}$$

Hence, as m tends to infinity, (4.8) leads to the exact solution of time fractional initial value problem (4.5):

$$u(x,t)=\lim_{m\to\infty}u_{m}(x,t).$$

Note that as the fractional order β tends to 1 the exact solution becomes:

$$u(x,t)=e^{\frac{x}{2}-\frac{2t}{3}}.$$

The analysis of the Table 2 allow us to reach the conclusion that utilizing the method SVIM for the establishment of truncated solutions for nonlinear time fractional initial value problems produce better results compare to results obtained by VIM and HPM in [7]. Fig 4.3 and Fig 4.4 also confirm the results, we have from the analysis of Table 2, that, SVIM enables us to acquire better truncated solutions.

5. Conclusions

In this study, SVIM, made up of ST and VIM, is presented for constructing semi-analytical solutions of linear or non-linear time fractional initial value problems. Implementation, accuracy and effectiveness of SVIM are the greatest advantages of it. The converges analysis of the semi-analytical solutions is also presented. Furthermore, the presented examples illustrates that the obtained results are correct.
Note that the implementation of Shehu transformation for various ordinary and partial differential equations as well as fractional differential equations is easier compare to many other transformations. Moreover, since integral transformations play a significant role in the solution of all kinds of differential equations, this makes Shehu transformation more valuable.
In the future works, the combination of other integral transformation with various numerical method are developed to establish numerical and exact solutions of fractional mathematical problems.

Table 2: Comparison of SVIM with the methods VIM and HPM for various β.

height$t$	x		$\beta=1$	$\beta=1$	$\beta=1$	$\beta=3/4$	$\beta=3/4$	$\beta=3/4$	$\beta=2/3$	$\beta=2/3$	$\beta=2/3$
		u_{exact}	u_{SVIM}	u_{HPM}	u_{VIM}	u_{SVIM}	u_{HPM}	u_{VIM}	u_{SVIM}	u_{HPM}	u_{VIM}
0.2	-5	0.07184	0.07184	0.07185	0.07184	0.06430	0.06413	0.06373	0.06184	0.06143	0.06018
	0	0.87517	0.87517	0.87534	0.87521	0.78337	0.78121	0.77639	0.75338	0.74837	0.73313
	5	10.66179	10.66179	10.66309	10.6620	9.54345	9.51690	9.45800	9.17802	9.11620	8.93100
0.4	-5	0.06287	0.06287	0.06287	0.06287	0.05682	0.05666	0.05540	0.05552	0.05541	0.05225
	0	0.76593	0.76593	0.76587	0.76589	0.69227	0.69024	0.67485	0.67634	0.67504	0.63658
	5	9.33092	9.33092	9.32980	9.33010	8.43357	8.40960	8.22100	8.23954	8.22310	7.75480
0.6	-5	0.05502	0.05502	0.05499	0.05502	0.05138	0.05133	0.05032	0.05101	0.05113	0.04873
	0	0.67032	0.67032	0.66996	0.67024	0.62590	0.62536	0.61306	0.62139	0.62288	0.59359
	5	8.16617	8.16617	8.16150	8.16490	7.62503	7.61780	7.46830	7.57008	7.58760	7.23110
0.8	-5	0.04815	0.04815	0.04811	0.04813	0.04707	0.04715	0.04686	0.04747	0.04774	0.04705
	0	0.58665	0.58665	0.58615	0.58656	0.57340	0.57440	0.57090	0.57827	0.58165	0.57324
	5	7.14681	7.14681	7.14050	7.14550	6.98541	6.99690	6.95470	7.04473	7.08580	6.98320
1	-5	0.04214	0.04214	0.04211	0.04214	0.04351	0.04370	0.04421	0.04455	0.04488	0.04607
	0	0.51342	0.51342	0.51302	0.51341	0.53009	0.53236	0.53861	0.54276	0.54674	0.56129
	5	6.25470	6.25470	6.24960	6.25440	6.45783	6.48650	6.56130	6.61223	6.66030	6.83760

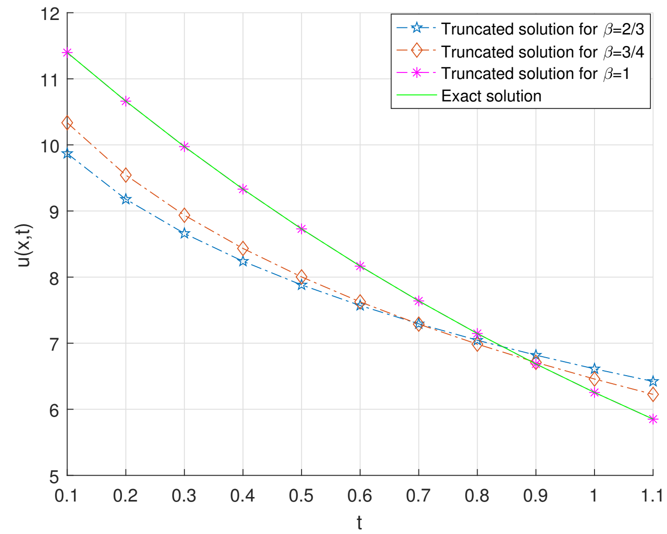

Figure 4.3: The graphs of analytical solution and truncated solutions of order 10 for various β values at $x = 0.3$ for example 4.2.

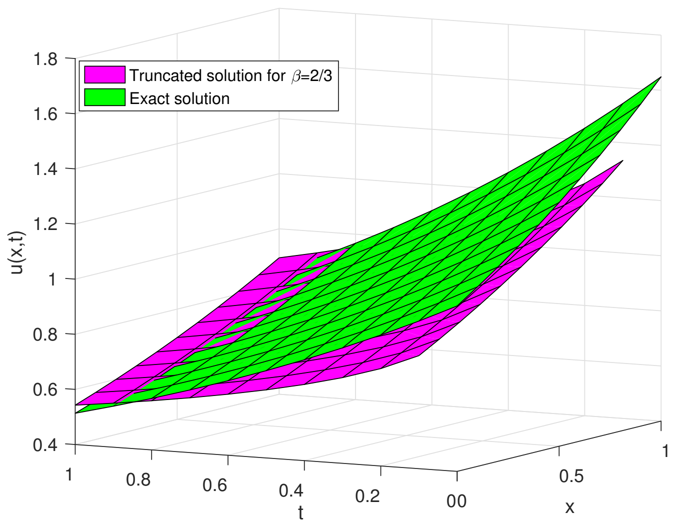

Figure 4.4: The graphs of analytical solution and truncated solutions of order 10 at $\beta = 2/3$ for example 4.2.

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