http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 944–953 (2022)
DOI:10.31801/cfsuasmas.1123739
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: January 07, 2022; Accepted: June 8, 2022

IDEAL CONVERGENCE OF A SEQUENCE OF CHEBYSHEV RADII OF SETS

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Abstract. In this paper, we investigate the diameters, Chebyshev radii, Chebyshev self-radii and inner radii of a sequence of sets in the normed spaces. We prove that if a sequence of sets is $\mathcal{I}$-Hausdorff convergent to a set, the sequence of Chebyshev radii of that sequence is $\mathcal{I}$-convergent. Similar relations are showed for the sequence of diameters, Chebyshev self-radii and inner radii of that sequence.

1. Introduction

The concept of statistical convergence, which is a generalization of the ordinary convergence of sequences, was first introduced by Fast [3] and Stainhaus [13], independently. Fridy [4, 5] contributed greatly to the development of the theory of statistical convergence. In 2000, Kostyrko et al [7] introduced ideal convergence, which is a generalization of statistical convergence. Recently the ideal convergence theory continues to be popularly studied (see [9,10]). On the other hand, Hausdorff convergence of a sequence of sets, which is defined by the Hausdorff distance, corresponds to the uniform convergence of the sequence of distance (see [2, 6, 8]). The theory of statistical convergence and the theory of ideal convergence were combined with the theory of convergence of sequences of sets by Nuray and Rhoades [11] and by Talo and Sever [14], respectively.

In [12], Papini and Wu examined Kuratowski convergence and Hausdorff convergence of sequences of sets in Banach spaces. They showed that if a sequence of sets is Hausdorff convergent then the sequences of diameters, Chebyshev radii, Chebyshev self-radii, and inner radii, respectively, of this sequence are convergent.

2020 *Mathematics Subject Classification.* 40A05, 40A35, 54A20.
Keywords. Chebyshev radius, Hausdorff convergence, $\mathcal{I}$-convergence, $\mathcal{I}$-Hausdorff convergence.
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In this study, by generalizing some of the results in [12], we show that if a sequence $(A_n)_{n\in\mathbb{N}}$ of sets is $\mathcal{I}$-Hausdorff convergent to a set A then the sequence of Chebyshev radii of A_n's is $\mathcal{I}$-convergent to the Chebyshev radius of A. We give similar relations for diameter, relative Chebyshev radius, Chebyshev self-radius and inner radius.

2. Preliminaries

Let $(X, \|\cdot\|)$ be normed space. We denote the family of all nonempty closed subsets, the family of all nonempty closed and bounded subsets and the family of all nonempty closed, convex and bounded subsets of X by $\mathrm{Cl}(X)$, $\mathcal{B}(X)$ and $\mathcal{C}(X)$, respectively.

The *distance* $d(x, A)$ from a point $x \in X$ to a subset A of X is defined to be

$$d(x, A) = \inf_{a\in A} \|x - a\|.$$

The set A is said to be *bounded* if $\mathrm{diam}(A) < \infty$, where *diameter* $\mathrm{diam}(A)$ of a nonempty set A in a normed space $(X, \|\cdot\|)$ is defined by

$$\mathrm{diam}(A) = \sup_{a_1,a_2\in A} \|a_1 - a_2\|.$$

The open ball with centre $x \in X$ and radius $\delta > 0$ is the set

$$S(x, \delta) = \{y \in X : \|x - y\| < \delta\}.$$

Hausdorff distance of sets $A, B \subseteq X$ is defined as

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where $h(A, B) = \sup_{a\in A} d(a, B)$, or equivalently

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq B^{\varepsilon} \text{ and } B \subseteq A^{\varepsilon}\}$$

where $A^{\varepsilon} = \bigcup_{a\in A}\{x \in X : \|x - a\| < \varepsilon\} = \{x \in X : d(x, A) < \varepsilon\}$ is the ε-enlargement of A.

Briefly, we recall some of basic notations in the theory of $\mathcal{I}-$convergence and we refer readers to [7,8] for more details. A family $\mathcal{I} \subseteq 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ is said to be an *ideal* in $\mathbb{N}$ if $\emptyset \in \mathcal{I}$, and $A \cup B \in \mathcal{I}$ for each $A, B \in \mathcal{I}$, and $B \in \mathcal{I}$ for each $A \in \mathcal{I}$ such that $B \subseteq A$ (see [8]). An ideal is called *proper* if $\mathbb{N} \notin \mathcal{I}$, and a proper ideal is called *admissible* if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Obviously, an admissible ideal includes all finite subset of $\mathbb{N}$ (see [7]).

The definition of ideal convergence for real numbers is as follows: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Let $\mathcal{I}$ be any ideal on $\mathbb{N}$. If for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\} \in \mathcal{I}$$

then (x_n) is said to be ideal convergent (briefly, $\mathcal{I}$-convergent) to x_0. Then we write $\mathcal{I} - \lim x_n = x_0$ (see [7]).

Define $\mathcal{I}_f = \{A \subset \mathbb{N} : \text{the set } A \text{ has finite number of elements}\}$. Then $\mathcal{I}_f$-convergence and classical convergence is equivalent to each other. Similarly, if we denote $\mathcal{I}_d = \{A \subset \mathbb{N} : \text{the set } A \text{ has natural density zero}\}$, then $\mathcal{I}_d$-convergence and statistical convergence is equivalent to each other. We note that the ideals $\mathcal{I}_f$ and $\mathcal{I}_d$ are admissible.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. We say that the sequence (A_n) is *$\mathcal{I}$-Hausdorff convergent* to the set A if

$$\left\{ n \in \mathbb{N} : \sup_{x \in X} |d(x, A_n) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}$$

for every $\varepsilon > 0$, or if $\mathcal{I} - \lim H(A_n, A) = 0$, i.e., for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : H(A_n, A) \geq \varepsilon\} \in \mathcal{I}$$

or equivalently

$$\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon \text{ or } h(A, A_n) \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $A_n \xrightarrow{\mathcal{I}\text{-}H} A$ (see [14]).

Now, we list some definitions of radii and centers associated with these radii (see [1, 12, 15]). Let A be a bounded subset of X and $Y \subseteq X$.

$R(x, A) = \sup_{a \in A} \|a - x\| \quad (x \in X)$	
$R_Y(A) = \inf_{y \in Y} R(y, A)$	: Relative Chebyshev radius of A in Y
$= \inf_{y \in Y} \sup_{a \in A} \|a - y\|$	
$R(A) = R_X(A)$	: Chebyshev radius of A
$R_A(A)$	: Chebyshev self-radius of A
$R'(A) = \sup_{a \in A} \inf_{x \notin A} \|x - a\|$	: Inner radius of A
$Z_Y(A) = \{y \in Y : R(y, A) = R_Y(A)\}$	: Relative Chebyshev center set of A in Y
$Z(A) = \{x \in X : R(x, A) = R(A)\}$	: Chebyshev center set of A
$Z_A(A) = \{a \in A : R(a, A) = R_A(A)\}$	: Chebyshev self center set of A
$Z'(A) = \{a \in A : R(a, A) = R'(A)\}$	: Inner center set of A

Example 1. *Consider the normed space $(\mathbb{R}^2, \|\cdot\|_1)$ where $\|\cdot\|_1$ is the ℓ_1 norm (aka the taxicab norm). Let A be a square whose vertices are on the points $(-1, -1)$, $(-1, 1)$, $(1, -1)$ and $(1, 1)$, and let $Y = \{(x, y) \in \mathbb{R}^2 : x = 3\}$. We have the following results:*

$$\begin{aligned} R(A) &= 2 & Z(A) &= \{(0,0)\} \\ R_A(A) &= 3 & Z_A(A) &= \{(-1,0), (1,0), (0,-1), (0,1)\} \\ R'(A) &= 0 & Z'(A) &= \emptyset \\ R_Y(A) &= 5 & Z_Y(A) &= \{(3,0)\} \end{aligned}$$

Lemma 1. *Let $A \in \mathcal{B}(X)$, $Y \subseteq X$ and $\varepsilon > 0$. Then the following is provided:*

(i) $\operatorname{diam}(A^{\varepsilon}) \leq \operatorname{diam}(A) + 2\varepsilon$

(ii) $R(x, A^{\varepsilon}) \leq R(x, A) + \varepsilon$ *for every* $x \in X$

(iii) $R_Y(A^{\varepsilon}) \leq R_Y(A) + \varepsilon$

(iv) $R(A^{\varepsilon}) \leq R(A) + \varepsilon$

(v) $R_{A^{\varepsilon}}(A^{\varepsilon}) \leq R_A(A) + \varepsilon$

Proof. (i)

$$\alpha_1, \alpha_2 \in A^{\varepsilon} \Longrightarrow \exists a_1, a_2 \in A \text{ such that } \|\alpha_1 - a_1\| < \varepsilon \text{ and } \|\alpha_2 - a_2\| < \varepsilon$$

Then, for every $\alpha_1, \alpha_2 \in A^{\varepsilon}$ we have

$$\begin{aligned} \|\alpha_1 - \alpha_2\| &\leq \|\alpha_1 - a_1\| + \|a_1 - a_2\| + \|\alpha_2 - a_2\| \\ &< \|a_1 - a_2\| + 2\varepsilon \\ &\leq \sup_{a_1, a_2 \in A} \|a_1 - a_2\| + 2\varepsilon \\ &= \operatorname{diam}(A) + 2\varepsilon \end{aligned}$$

and so

$$\operatorname{diam}(A^{\varepsilon}) = \sup_{\alpha_1, \alpha_2 \in A^{\varepsilon}} \|\alpha_1 - \alpha_2\| \leq \operatorname{diam}(A) + 2\varepsilon.$$

(ii)

$$\alpha \in A^{\varepsilon} \Longrightarrow \exists a \in A \text{ such that } \|\alpha - a\| < \varepsilon$$

Let $x \in X$. For every $\alpha \in A^{\varepsilon}$ we have

$$\begin{aligned} \|\alpha - x\| &\leq \|\alpha - a\| + \|a - x\| \\ &< \|a - x\| + \varepsilon \\ &\leq \sup_{a \in A} \|a - x\| + \varepsilon \\ &= R(x, A) + \varepsilon \end{aligned}$$

and so

$$R(x, A^{\varepsilon}) = \sup_{\alpha \in A^{\varepsilon}} \|\alpha - x\| \leq R(x, A) + \varepsilon.$$

(iii) From (ii), we have $R(y, A^{\varepsilon}) \leq R(y, A) + \varepsilon$ for every $y \in Y$. Then we get

$$\begin{aligned} \inf_{y \in Y} R(y, A^{\varepsilon}) &\leq \inf_{y \in Y} R(y, A) + \varepsilon \\ R_Y(A^{\varepsilon}) &\leq R_Y(A) + \varepsilon. \end{aligned}$$

(iv) It is easily obtained by taking $Y = X$ in (iii).

(v) From (ii), we have

$$R(a, A^{\varepsilon}) \leq R(a, A) + \varepsilon$$

for every $a \in A$, and so

$$\inf_{a \in A} R(a, A^{\varepsilon}) \leq \inf_{a \in A} R(a, A) + \varepsilon.$$

From the fact that

$$\inf_{\alpha \in A^{\varepsilon}} R(\alpha, A^{\varepsilon}) \leq \inf_{a \in A} R(a, A^{\varepsilon}),$$

we get

$$\begin{aligned} \inf_{\alpha \in A^{\varepsilon}} R(\alpha, A^{\varepsilon}) &\leq \inf_{a \in A} R(a, A) + \varepsilon \\ R_{A^{\varepsilon}}(A^{\varepsilon}) &\leq R_A(A) + \varepsilon. \end{aligned}$$

□

We cannot give similar results above for the inner radius, i.e., the inequality $R'(A^{\varepsilon}) \leq R'(A) + \varepsilon$ may not be satisfied. Such as, if we take $\varepsilon = \frac{3}{2}$ in Example 1, we get

$$R'(A^{\varepsilon}) = \frac{5}{2} \nleq R'(A) + \varepsilon = 0 + \frac{3}{2}.$$

Also, we cannot say a general upper bound for the difference $R'(A^{\varepsilon}) - R'(A)$. For example, in the Euclidean space $\mathbb{R}^2$, let the set A be a spiral with $r = \theta$ ($0 \leq \theta \leq 2n\pi$, $n \in \mathbb{N}$) polar equation. Let's take $\varepsilon > \pi$. Then we have $R'(A) = 0$ and $R'(A^{\varepsilon}) \geq (2n-1)\pi$. Thus the difference $R'(A^{\varepsilon}) - R'(A)$ depends not only on ε but also on n.

3. Main Results

For a sequence of closed and bounded sets, we show that $\mathcal{I}$-Hausdorff convergence implies $\mathcal{I}$-convergence of the sequence of Chebyshev radii (diameters, relative Chebyshev radii and Chebyshev self-radii, respectively) of this sequence. If the sets are convex as an additional condition, this proposition is also true for the sequence of inner radii.

Proposition 1. *Let $A, A_n \in \mathcal{B}(X)$ ($n \in \mathbb{N}$) and $Y \subseteq X$. If $A_n \xrightarrow{\mathcal{I}-H} A$ then the following hold:*

(i) $\mathcal{I} - \lim \operatorname{diam}(A_n) = \operatorname{diam}(A)$
(ii) $\mathcal{I} - \lim R_Y(A_n) = R_Y(A)$
(iii) $\mathcal{I} - \lim R(A_n) = R(A)$
(iv) $\mathcal{I} - \lim R_{A_n}(A_n) = R_A(A)$

Proof. (i) Let $\varepsilon > 0$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon) := \left\{ n \in \mathbb{N} : H(A_n, A) \geq \frac{\varepsilon}{3} \right\} \in \mathcal{I}.$$

For every $n \in \mathbb{N} \setminus L(\varepsilon)$ we have

$$A \subseteq A_n^{\varepsilon/3} \text{ and } A_n \subseteq A^{\varepsilon/3}.$$

Then

$$\begin{aligned} A \quad \subseteq \quad A_n^{\varepsilon/3} &\Longrightarrow \operatorname{diam}(A) \leq \operatorname{diam}\left(A_n^{\varepsilon/3}\right) \leq \operatorname{diam}(A_n) + \frac{2\varepsilon}{3} \\ &\Longrightarrow \operatorname{diam}(A) - \operatorname{diam}(A_n) \leq \frac{2\varepsilon}{3} \\ A_n \quad \subseteq \quad A^{\varepsilon/3} &\Longrightarrow \operatorname{diam}(A_n) \leq \operatorname{diam}\left(A^{\varepsilon/3}\right) \leq \operatorname{diam}(A) + \frac{2\varepsilon}{3} \\ &\Longrightarrow \operatorname{diam}(A_n) - \operatorname{diam}(A) \leq \frac{2\varepsilon}{3} \end{aligned}$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$. Hence we get

$$\begin{aligned} \{n \in \mathbb{N} : |\operatorname{diam}(A_n) - \operatorname{diam}(A)| \geq \varepsilon\} \quad &\subseteq \quad L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |\operatorname{diam}(A_n) - \operatorname{diam}(A)| \geq \varepsilon\} \quad &\in \quad \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$. Consequently, we obtain $\mathcal{I} - \lim \operatorname{diam}(A_n) = \operatorname{diam}(A)$.

(ii) Let Y be any subset of X. From the triangle inequality, we have

$$\|a_n - y\| - \|a - y\| \quad \leq \quad \|a_n - a\| \tag{1}$$

$$\|a - y\| - \|a_n - y\| \quad \leq \quad \|a_n - a\| \tag{2}$$

where $y \in Y, a_n \in A_n$ and $a \in A$. Then, from (1)

$$\begin{aligned} \inf_{a \in A} (\|a_n - y\| - \|a - y\|) \quad &\leq \quad \inf_{a \in A} \|a_n - a\| \\ \|a_n - y\| - \sup_{a \in A} \|a - y\| \quad &\leq \quad \inf_{a \in A} \|a_n - a\| \\ \sup_{a_n \in A_n} \|a_n - y\| - \sup_{a \in A} \|a - y\| \quad &\leq \quad \sup_{a_n \in A_n} \inf_{a \in A} \|a_n - a\| \end{aligned}$$

$$\begin{aligned} R_Y(A_n) - R_Y(A) \quad &= \inf_{y \in Y} \sup_{a_n \in A_n} \|a_n - y\| - \inf_{y \in Y} \sup_{a \in A} \|a - y\| \\ &\leq \sup_{a_n \in A_n} \inf_{a \in A} \|a_n - a\| = h(A_n, A) \end{aligned} \tag{3}$$

and similarly, from (2)

$$\begin{aligned} R_Y(A) - R_Y(A_n) \quad &= \inf_{y \in Y} \sup_{a \in A} \|a - y\| - \inf_{y \in Y} \sup_{a_n \in A_n} \|a_n - y\| \\ &\leq \sup_{a \in A} \inf_{a_n \in A_n} \|a_n - a\| = h(A, A_n). \end{aligned} \tag{4}$$

Take $\varepsilon > 0$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$, we have

$$L(\varepsilon) := \{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon \text{ or } h(A, A_n) \geq \varepsilon\} \in \mathcal{I}.$$

From (3) and (4), we get

$$R_Y(A_n) - R_Y(A) \leq h(A_n, A) < \varepsilon,$$
$$R_Y(A) - R_Y(A_n) \leq h(A, A_n) < \varepsilon$$

and so

$$|R_Y(A_n) - R_Y(A)| < \varepsilon$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$. Hence we get

$$\begin{aligned} \{n \in \mathbb{N} : |R_Y(A_n) - R_Y(A)| \geq \varepsilon\} &\subseteq L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |R_Y(A_n) - R_Y(A)| \geq \varepsilon\} &\in \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$. This means that $\mathcal{I} - \lim R_Y(A_n) = R_Y(A)$.

(iii) It is the special case of (ii), with $Y = X$.

(iv) Let $\varepsilon > 0$. From $A_n \xrightarrow{\mathcal{I}-H} A$ we have

$$L(\varepsilon) := \left\{n \in \mathbb{N} : h(A_n, A) \geq \frac{\varepsilon}{2} \text{ or } h(A, A_n) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

If $a_0 \in Z_A(A)$ then $a_0 \in A$ and

$$R(a_0, A) = \sup_{a \in A} \|a - a_0\| = R_A(A). \tag{5}$$

Take $n \in \mathbb{N} \setminus L(\varepsilon)$. From $h(A, A_n) < \frac{\varepsilon}{2}$ we have

$$\sup_{a \in A} d(a, A_n) < \frac{\varepsilon}{2}. \tag{6}$$

From the closeness of A_n there exists an $a_n^{(1)} \in A_n$ such that

$$\left\|a_0 - a_n^{(1)}\right\| < \frac{\varepsilon}{2}. \tag{7}$$

Also, there exists an $a_n^{(2)} \in A_n$ such that

$$\sup_{a_n \in A_n} \left\|a_n - a_n^{(1)}\right\| = \left\|a_n^{(2)} - a_n^{(1)}\right\|. \tag{8}$$

From $h(A_n, A) < \frac{\varepsilon}{2}$ we get

$$d\left(a_n^{(2)}, A\right) \leq \sup_{a_n \in A_n} d(a_n, A) < \frac{\varepsilon}{2} \tag{9}$$

and so

$$\left\|a_0 - a_n^{(2)}\right\| < R_A(A) + \frac{\varepsilon}{2}. \tag{10}$$

From (7) and (10) we obtain

$$\begin{aligned} R_{A_n}(A_n) \leq \left\|a_n^{(1)} - a_n^{(2)}\right\| &\leq \left\|a_n^{(1)} - a_0\right\| + \left\|a_0 - a_n^{(2)}\right\| \\ &< R_A(A) + \varepsilon \end{aligned}$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$.
Similarly, it can be shown that

$$R_A(A) < R_{A_n}(A_n) + \varepsilon$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$.
Consequently, we get

$$\begin{aligned} \{n \in \mathbb{N} : |R_{A_n}(A_n) - R_A(A)| \geq \varepsilon\} &\subseteq L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |R_{A_n}(A_n) - R_A(A)| \geq \varepsilon\} &\in \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$, and so $\mathcal{I} - \lim R_{A_n}(A_n) = R_A(A)$.

□

Lemma 2. *(see [12, Lemma 1]) Let $A, B \in \mathcal{C}(X)$. If $R'(A) > 0$ and $H(A, B) < \dfrac{R'(A)}{2}$ then*

$$R'(B) \geq R'(A) - H(A, B) > 0.$$

As a result of the above lemma we can give the following corollary.

Corollary 1. *Let $A \in \mathcal{C}(X)$ and $\varepsilon > 0$. If $R'(A^{\varepsilon}) > 2\varepsilon$ then*

$$R'(A) \geq R'(A^{\varepsilon}) - \varepsilon$$

(That is, $R'(A^{\varepsilon}) \leq R'(A) + \varepsilon$). Of course, for the condition here to be satisfied, $R'(A) > \varepsilon$ must be.

Proposition 2. *Let $A, A_n \in \mathcal{C}(X)$ $(n \in \mathbb{N})$. If $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ then*

$$\mathcal{I} - \lim R'(A_n) = R'(A).$$

Proof. First let's assume that $R'(A) = 0$. Suppose that $\mathcal{I} - \lim R'(A_n) \neq 0$. Then there is an $\varepsilon_0 > 0$ such that

$$K(\varepsilon_0) := \{n \in \mathbb{N} : R'(A_n) \geq \varepsilon_0\} \notin \mathcal{I}.$$

From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon_0) := \left\{n \in \mathbb{N} : H(A_n, A) \geq \frac{\varepsilon_0}{2}\right\} \in \mathcal{I}.$$

Then $(\mathbb{N} \setminus L(\varepsilon_0)) \cap K(\varepsilon_0) \neq \emptyset$ and so we have

$$H(A_n, A) < \frac{\varepsilon_0}{2} \leq \frac{1}{2} R'(A_n)$$

for every $n \in (\mathbb{N} \setminus L(\varepsilon_0)) \cap K(\varepsilon_0)$. From Lemma 2, we get

$$R'(A) > 0$$

and this is a contradiction. Therefore, $\mathcal{I}-\lim R'(A_n)=0=R'(A)$ holds.
Now let's assume that $R'(A)>0$. Let $0<\varepsilon<\dfrac{R'(A)}{3}$. From $A_n \xrightarrow{\mathcal{I}-H} A$ we have

$$L(\varepsilon):=\{n\in\mathbb{N}:H(A_n,A)\geq\varepsilon\}\in\mathcal{I}.$$

Then we have

$$H(A_n,A)<\varepsilon<\frac{R'(A)}{3}<\frac{R'(A)}{2}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. From Lemma 2, we get

$$R'(A_n)\geq R'(A)-H(A_n,A)>R'(A)-\varepsilon \tag{11}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. We also have

$$H(A_n,A)<\varepsilon<\frac{1}{2}(R'(A)-\varepsilon)<\frac{R'(A_n)}{2}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. Again from Lemma 2, we get

$$R'(A)\geq R'(A_n)-H(A,A_n)>R'(A_n)-\varepsilon \tag{12}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. From (11) and (12) we obtain

$$\begin{aligned}\{n\in\mathbb{N}:|R'(A_n)-R'(A)|\geq\varepsilon\} &\subseteq L(\varepsilon)\in\mathcal{I}\\ \{n\in\mathbb{N}:|R'(A_n)-R'(A)|\geq\varepsilon\} &\in \mathcal{I}\end{aligned}$$

for every $\varepsilon>0$, and so $\mathcal{I}-\lim R'(A_n)=R'(A)$. □

Declaration of Competing Interests The author declare that there are no conflicts of interest regarding the publication of this paper.

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