http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 2, Pages 438–448 (2023)
DOI:10.31801/cfsuasmas.1124351
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: June 1, 2022; Accepted: November 22, 2022

ON STATISTICAL LIMIT POINTS WITH RESPECT TO POWER SERIES METHODS AND MODULUS FUNCTIONS

Canan SÜMBÜL[1], Cemal BELEN[2] and Mustafa YILDIRIM[1,3]

[1]Cumhuriyet University, Faculty of Arts and Sciences, Department of Mathematics, 58140 Sivas, TÜRKİYE

[2]Department of Mathematics Education, Ordu University, 52200, Ordu, TÜRKİYE

Abstract. In this study, we define a new type of statistical limit point using the notions of statistical convergence with respect to the J_p power series method and then we present some examples to show the relations between these points and ordinary limit points. After that we also study statistical limit points of a sequence with the help of a modulus function in the sense of the J_p power series method. Namely, we define f-J_p- statistical limit and cluster points of the real sequences and compare the set of these limit points with the set of ordinary points.

1. Introduction

The concept of statistical convergence was initially introduced by Fast [9]. The important properties of statistical convergence were established by Salat [15] and Fridy [11]. Fridy [12] introduced the concepts of statistical limit points and statistical cluster points of real sequences and compared them with ordinary limit points.

By using the modulus functions, Aizpuru et al. [1] introduced the concept of f -statistical convergence which depends on the other new concept of f-density of subsets of natural numbers (where f is a modulus function). Listán-García [13] gave the definition of f-statistical limit points and cluster points with respect to a modulus function f and proved some relations including the properties of the sets of f-statistical limit points and f-cluster points.

Ünver and Orhan [18] discussed the idea of statistical convergence via power series methods and they defined a new concept so-called P-statistical convergence.

2020 *Mathematics Subject Classification.* 40G15, 40F05, 40C15, 40G10.

Keywords. J_p-statistical limit points, J_p-statistical cluster points, f-J_p- statistical limit points.

[1] cnnhhs@gmail.com-Corresponding author; 0000-0002-8905-1247

[2] cbelen52@gmail.com; 0000-0002-8832-1524

[3] yildirim@cumhuriyet.edu.tr; 00000-0002-8880-5457.

Many authors used P-statistical convergence to obtain new results. In [3], Bayram gave some criteria for statistical convergence according to the power series methods. In [4], Bayram and Yıldız gave various Korovkin-type approximation theorems for linear operators defined on derivatives of functions, using statistical convergence according to power series methods. The recent results including P-statistical convergence can be seen in [2], [5], [8], [16], [17], [19].

In [6], the authors of this work developed the idea of statistical convergence by using J_p power series summability methods and a modulus function.

In this study, we introduce and study on the concepts of statistical limit points and statistical cluster points determined by the power series methods. Later on these notions are strenghtened via an unbounded modulus function. Some detailed examples are also presented to obtain strict inclusion relations.

Now recall some basic notions used in this paper.

Let $\mathbb{N}_0$ be the set of nonnegative integers, $E \subset \mathbb{N}_0$ and $E(n) = \{k \leq n : k \in E\}$. By $|E(n)|$, denote the cardinality of the set of $E(n)$. If limit

$$\lim_{n\to\infty} \frac{|E(n)|}{n+1}$$

exists, then E is said to have natural density and it is denoted by $\delta(E)$ [10]. Any number sequence $x = (x_k)$ is statistical convergent to L if for every $\varepsilon > 0$, $\delta(E_\varepsilon) = 0$, where $E_\varepsilon = \{k \in \mathbb{N}_0 : |x_k - L| \geq \varepsilon\}$. In this case we write $st\text{-}\lim x = L$.

Let $\left(x_{k(j)}\right)$be any subsequence of $x = (x_k)$ and $K = \{k(j) : j \in \mathbb{N}_0\}$, then $\left(x_{k(j)}\right)$ is denoted by $\{x\}_K$. $\{x\}_K$ is called a thin subsequence of x if $\delta(K) = 0$. If $\delta(K) \neq 0$, $\{x\}_K$ is called a nonthin subsequence of x [12]. We know that L is ordinary limit point of x if there exists a subsequence of x that converges to L. The definition of statistical limit point is given below. Following Fridy [12], we say that the number λ is a statistical limit point of the sequence x if there exists a nonthin subsequence of x that converges to λ. For any real sequence x, Λ_x denotes the set of statistical limit points of x, and L_x denotes the set of ordinary limit points of x. Also if $\delta(\{k \in \mathbb{N} : |x - \gamma| < \varepsilon\}) \neq 0$ for every $\varepsilon > 0$, the number γ is called a statistical cluster point of the number sequence x [12]. Γ_x denotes the set of all statistical cluster points of x. For every sequence x, we have $\Lambda_x \subset \Gamma_x \subset L_x$.

2. J_p-Statistical Limit Points

Let $(p_k)_{k\in\mathbb{N}_0}$ be a sequence of nonnegative integers with $p_0 > 0$,

$$P_n = \sum_{k=0}^{n} p_k \to \infty \quad (n \to \infty)$$

and

$$p(t) = \sum_{k=0}^{\infty} p_k t^k < \infty$$

for $0 < t < 1$. For any real sequence $x = (x_k)$, assume that

$$p_x(t) = \sum_{k=0}^{\infty} p_k t^k x_k \text{ convergent for } 0 < t < 1.$$

Then we say that (x_k) is J_p-convergent to L or J_p-summable to L if

$$\lim_{t \to 1^-} \frac{p_x(t)}{p(t)} = L.$$

In this case we write $x_k \to L\ (J_p)$. The condition $p(t) \to \infty\ (t \to 1^-)$ assures that J_p-method is regular (see, [7]). So we only consider regular J_p methods.

The ideas of natural density and statistical convergence are extended to power series methods by Unver and Orhan [18].

Let $E \subset \mathbb{N}_0$ be any set. If the limit

$$\delta_{J_p}(E) = \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E} p_k t^k$$

exits, then $\delta_{J_p}(E)$ is called J_p-density of E. From the definition it is clear that if $\delta_{J_p}(E)$ exists, then $0 \leq \delta_{J_p}(E) \leq 1$ and $\delta_{J_p}(E) = 1 - \delta_{J_p}(\mathbb{N}_0 \backslash E)$. If E is finite, then $\delta_{J_p}(E) = 0$. Note that J_p-density and natural density of any $E \subset \mathbb{N}_0$ need not to be equal to each other. For instance, let $(p_k) = (1, 0, 1, 0, ...)$. Then $p(t) = \sum_{k=0}^{\infty} t^{2k} = 1/\left(1 - t^2\right)$ for $0 < t < 1$. Now if $E = \{2k + 1 : k \in \mathbb{N}_0\}$, then $\delta_{J_p}(E) = 0$ but $\delta(E) = 1/2$ (see [18]).

A real sequence $x = (x_k)$ is called J_p-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E_\varepsilon} p_k t^k = 0.$$

where, $E_\varepsilon = \{k \in \mathbb{N}_0 : |x_k - L| \geq \varepsilon\}$ .That is, for every $\varepsilon > 0$, $\delta_{J_p}(E_\varepsilon) = 0$. In this case, we write $st_{J_p}\text{-}\lim x = L$. The set of all J_p-statistically convergent sequences is denoted by st_{J_p}.

The following example shows that a sequence x can be J_p statistical convergent even if x is not convergent or statistical convergent.

Example 1. *Let $(p_k) = (1, 0, 1, 0, ...)$ and $(x_k) = (0, 1, 0, 1, ...)$. Observe that (x_k) is neither convergent nor statistically convergent. But, since*

$$\delta_{J_p}(\{k \in \mathbb{N}_0 : |x_k| \geq \varepsilon\}) = \delta_{J_p}(\{2k + 1 : k \in \mathbb{N}_0\}) = 0$$

for each $\varepsilon > 0$, we have $st_{J_p}\text{-}\lim x = 0$.

Definition 1. *If $\delta_{J_p}(K) = 0$, $\{x\}_K$ is called J_p-thin subsequence and if $\delta_{J_p}(K) \neq 0$, $\{x\}_K$ is called a J_p-nonthin subsequence of x. If there exists a J_p-nonthin subsequence of the real sequence x that converges to λ, then the number λ is said to be a J_p-statistical limit point of x.*

For any sequence x , $\Lambda_x^{J_p}$ denote the set of J_p-statistical limit points of x. It is clear that $\Lambda_x^{J_p} \subset L_x$ for any sequence x. The following example shows that the inclusion is strict.

Example 2. *Let* (p_k) *and* (x_k) *be in 1 and*

$$E_1 = \{k : k = 2n+1,\ n \in \mathbb{N}_0\ \} \text{ and } E_2 = \{k : k = 2n,\ n \in \mathbb{N}_0\}.$$

In this case we have

$$\begin{aligned} p(t) &= \sum_{k=0}^{\infty} p_k t^k = \sum_{k=0}^{\infty} p_{2k+1} t^{2k+1} + \sum_{k=0}^{\infty} p_{2k} t^{2k} \\ &= \sum_{k=0}^{\infty} 0.t^{2k} + \sum_{k=0}^{\infty} 1.t^{2k+1} \\ &= \frac{t}{1-t^2},\ |t| < 1, \end{aligned}$$

$$\begin{aligned} \delta_{J_p}(E_1) &= \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E_1} p_k t^k = \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} 1.t^{2k+1} = 1, \\ \delta_{J_p}(E_2) &= \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E_2} p_k t^k = \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} 0.t^{2k} = 0. \end{aligned}$$

Since $\{x\}_{E_1} \to 1$ *and* $\delta_{J_p}(E_1) \neq 0$, *we obtain that* $\Lambda_x^{J_p} = \{1\}$. *But, it is clear that* $L_x = \{0, 1\}$.

We write an example that shows Λ_x and $\Lambda_x^{J_p}$ are not same.

Remark 1. *The notions of statistical limit point and* J_p-*statistical limit point are not comparable. For instance, let* J_p*-method be determined by the sequence*

$$p_k = \begin{cases} 1 & \text{, if } k \text{ is square} \\ 0 & \text{, if } k \text{ is nonsquare} \end{cases}$$

and consider the sequence $x = (x_k)$ *defined by*

$$x_k = \begin{cases} 2 & \text{, if } k \text{ is square} \\ 1, & \text{, if } k \text{ is an odd nonsquare} \\ 0 & \text{, if } k \text{ is an even nonsquare} \end{cases}$$

Then we easily see that $\Lambda_x = \{0, 1\}$ *and* $\Lambda_x^{J_p} = \{2\}$.

We give an example below to show that $\Lambda_x^{J_p}$ and L_x can be very different.

Example 3. *Let* $\{r_k\}_{k=1}^{\infty}$ *be a sequence whose range is the set of all rational numbers and define*

$$x_k := \begin{cases} r_k & \text{, if } k = 2n \\ k & \text{, if } k = 2n+1 \end{cases},\ n \in \mathbb{N}_0$$

$$p_k := \begin{cases} 1 & \text{, if } k = 2n+1 \\ 0 & \text{,if } k = 2n \end{cases},\ n \in \mathbb{N}_0$$

Also let $E_1 = \{k : k = 2n+1,\ n \in \mathbb{N}_0\ \}$ *and* $E_2 = \{k : k = 2n,\ n \in \mathbb{N}_0\}$. *Since*

$$\delta_{J_p}(E_2) \quad = \lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in E_2} p_k t^k = \lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} 0.t^{2k} = 0,$$

we have $\Lambda_x^{J_p} = \emptyset$. *But* $L_x = \mathbb{R}$, *since* $\{r_k : k \in \mathbb{N}_0\}$ *is dense in* $\mathbb{R}$.

Definition 2. *If* $\delta_{J_p}(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$ *for every* $\varepsilon > 0$, *then* γ *is called* J_p-*statistical cluster point of the sequence* $x = (x_k)$.

We show the set of all J_p-statistical cluster points of x with $\Gamma_x^{J_p}$.

Theorem 1. *For every sequence* x, $\Gamma_x^{J_p} \subset L_x$.

Proof. Assume that $\gamma \in \Gamma_x^{J_p}$. For every $\varepsilon > 0$, $\delta_{J_p}(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$. So the set $A := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ is infinite. That means that there are infinitely many $x_k \in (\gamma - \varepsilon, \gamma + \varepsilon)$. From this we get $\gamma \in L_x$. □

Theorem 2. *For any number sequence* x, $\Lambda_x^{J_p} \subset \Gamma_x^{J_p}$.

Proof. Assume that $\lambda \in \Lambda_x^{J_p}$. Then there exists $K = \{k(j) : j \in \mathbb{N}_0\}$ such that $\{x\}_K$ is a J_p-nonthin subsequence of x. Thus for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $k(j) > n_0$, $\left|x_{k(j)} - \lambda\right| < \varepsilon$ and $\delta_{J_p}(K) \neq 0$. Also it is clear that

$$\left\{k(j) \in K : \left|x_{k(j)} - \lambda\right| < \varepsilon\right\} \subset \{k \in \mathbb{N}_0 : |x_k - \lambda| < \varepsilon\}.$$

From this we have

$$0 \neq \delta_{J_p}\left\{k(j) \in K : \left|x_{k(j)} - \lambda\right| < \varepsilon\right\} \subset \delta_{J_p}\{k \in \mathbb{N} : |x_k - \lambda| < \varepsilon\}.$$

Thus $\lambda \in \Gamma_x^{J_p}$ and so $\Lambda_x^{J_p} \subset \Gamma_x^{J_p}$. □

The following example shows that the inclusion $\Lambda_x^{J_p} \subset \Gamma_x^{J_p}$ is strict.

Example 4. *Define the sequence* x *by*

$$x_k = \begin{cases} 0, & \text{if } k = 0 \\ \frac{1}{r} & \text{if } k = 2^{r-1}(2q+1). \end{cases}$$

Also let $(p_k) = (1, 1, 1, ...)$. *Then* $p(t) = 1/(1-t)$ *for* $|t| < 1$, *and*

$$\delta_{J_p}(\{k : x_k = 1\}) = \delta_{J_p}(\{k = 2n+1 : n \in \mathbb{N}_0)\}) = \lim_{t\to 1^-}(1-t)\sum_{k=0}^{\infty} t^{2k+1} = 2^{-1},$$

$$\delta_{J_p}(\{k : x_k = 1/2\}) = \delta_{J_p}(\{k = 4n+2 : n \in \mathbb{N}_0)\}) = \lim_{t\to 1^-}(1-t)\sum_{k=0}^{\infty} t^{4k+2} = 2^{-2},$$

$$\delta_{J_p}(\{k : x_k = 1/2\}) = \delta_{J_p}(\{k = 8n+4 : n \in \mathbb{N}_0)\}) = \lim_{t\to 1^-}(1-t)\sum_{k=0}^{\infty} t^{8k+4} = 2^{-3},$$

$$\vdots$$

Thus we have for each r that $\delta_{J_p}(\{k : x_k = 1/r\}) = 2^{-r} > 0$, whence $\frac{1}{r} \in \Lambda_x^{J_p}$. It can be seen by a similar method that

$$\delta_{J_p}\left(\left\{k : |x_k| < \frac{1}{r}\right\}\right) = \delta_{J_p}\left(\left\{k : 0 < x_k < \frac{1}{r}\right\}\right) = 2^{-r}.$$

Hence we get $0 \in \Gamma_x^{J_p}$ and so $\Gamma_x^{J_p} = \{0\} \cup \left\{\frac{1}{r}\right\}_{r=1}^{\infty}$. Now we claim that $0 \notin \Lambda_x^{J_p}$. For this, if the limit of the subsequence $\{x\}_K$ is zero then we show that $\delta_{J_p}(K) = 0$. For each r we have

$$\begin{aligned}\delta_{J_p}(K) &= \lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in K,\ x_k<1/r} p_k t^k + \lim_{t\to 1^-} \frac{1}{p(t)} \sum_{k\in K,\ x_k\geq 1/r} t^k \\ &\leq 2^{-r} + O(1).\end{aligned}$$

Since $r > 0$ is arbitrary, we conclude that $\delta_{J_p}(K) = 0$.

Theorem 3. *For any sequence x, the set $\Gamma_x^{J_p}$ is a closed point set.*

Proof. Assume that α is an accumulation point of $\Gamma_x^{J_p}$. Then for all $\varepsilon > 0$, $\Gamma_x^{J_p}$ contains some points

$$\gamma \in (\alpha - \varepsilon, \alpha + \varepsilon).$$

Choose ε' so that

$$(\alpha - \varepsilon', \alpha + \varepsilon') \subset (\alpha - \varepsilon, \alpha + \varepsilon).$$

Since $\gamma \in \Gamma_x^{J_p}$

$$\delta_{J_p}(\{k : x_k \in (\gamma - \varepsilon', \gamma + \varepsilon')\}) \neq 0.$$

From this

$$\delta_{J_p}(\{k : x_k \in (\alpha - \varepsilon, \alpha + \varepsilon)\}) \neq 0.$$

So we get $\alpha \in \Gamma_x^{J_p}$. □

If x and y are sequences such that $\delta_{J_p}(\{k : x_k \neq y_k\}) = 0$ then we say that $x_k = y_k$ for almost all k.

Theorem 4. *If x and y are sequences such that $x_k = y_k$ for almost all k, then $\Lambda_x^{J_p} = \Lambda_y^{J_p}$ and $\Gamma_x^{J_p} = \Gamma_y^{J_p}$.*

Proof. Let $\delta_{J_p}(\{k : x_k \neq y_k\}) = 0$ and $\lambda \in \Lambda_x^{J_p}$. Then there exists a J_p-nonthin subsequence $\{x\}_K$ of x which is convergent to λ. Since $\delta_{J_p}(\{k \in K : x_k \neq y_k\}) = 0$, $\delta_{J_p}(\{k : k \in K \text{ and } x_k = y_k\}) \neq 0$. From this if we take $K' = \{k \in \mathbb{N} : x_k = y_k\}$, then $\{y\}_{K'}$is a J_p-nonthin subsequence of $\{y\}_K$ which is convergent to λ. Thus $\lambda \in \Lambda_y^{J_p}$and so we get $\Lambda_x^{J_p} \subset \Lambda_y^{J_p}$. Likewise, it can be shown that $\Lambda_y^{J_p} \subset \Lambda_x^{J_p}$. Hence we get $\Lambda_x^{J_p} = \Lambda_y^{J_p}$. Now let $\gamma \in \Gamma_x^{J_p}$ and show that $\Gamma_x^{J_p} = \Gamma_y^{J_p}$. For every $\varepsilon > 0$, $\delta_{J_p}(\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}) \neq 0$. Define the sets $E' := \{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$, $E'' := \{k \in \mathbb{N} : x_k \neq y_k \text{ and } |x_k - \gamma| < \varepsilon\}$, $E''' := \{k \in \mathbb{N} : x_k = y_k \text{ and } |x_k - \gamma| < \varepsilon\}$. Since

$$\frac{1}{p(t)} \sum_{k\in E'} p_k t^k = \frac{1}{p(t)} \sum_{k\in E''} p_k t^k + \frac{1}{p(t)} \sum_{k\in E'''} p_k t^k$$

we get

$$0 \neq \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E'} p_k t^k = \frac{1}{p(t)} \sum_{k \in E'''} p_k t^k$$

i.e., for every $\varepsilon > 0$, $\delta_{J_p}(\{k \in \mathbb{N} : |y_k - \gamma| < \varepsilon\}) \neq 0$. Hence $\gamma \in \Gamma_y^{J_p}$. The inclusion $\Gamma_y^{J_p} \subset \Gamma_x^{J_p}$ can be shown similarly. □

The following result can be obtained by a similar way to the Theorem 2 of [12].

Theorem 5. *If x is a number sequence then there exists a sequence y such that $L_y = \Gamma_x^{J_p}$ and $y_k = x_k$ for almost all k ; moreover, the range of y is a subset of the range of x.*

Note that L_x is always closed set while $\Lambda_x^{J_p}$ is not (see Example 4). Hence the conclusion of Theorem 4 is not valid if we replace $\Gamma_x^{J_p}$ with $\Lambda_x^{J_p}$.

Following the line of Fridy (see [12], Section 3), we can prove the J_p-statistical analogues of some of the well-known completeness theorems of real numbers theorems.

Theorem 6. *Let x be a real sequence and $M = \{k : x_k \leq x_{k+1}\}$. If $\delta_{J_p}(M) = 1$ and x is bounded on M then x is J_p-statistically convergent.*

Theorem 7. *If x contains a bounded J_p-nonthin subsequence then x has a J_p-statistical cluster point.*

This theorem leads naturally to the following corollary.

Corollary 1. *If x is a bounded sequence then x has a J_p-statistical cluster point.*

Theorem 8. *If x is a bounded sequence, then x has a J_p-nonthin subsequence $\{x\}_K$ such that $\{x_k : k \in \mathbb{N} \backslash K\} \cup \Gamma_x^{J_p}$ is compact set.*

3. f-J_p-Statistical Limit Points

In this section, we aim to examine the f-J_p-statistical version of cluster points and limit points and relate them to classical limit points.

Any function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties is called a modulus function;

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$,
3. f is increasing,
4. f, is continuous from the right at zero [14].

$f(x) = \frac{x}{1+x}$ and $f(x) = x^p$ for $0 < p \leq 1$ are examples of modulus functions. A modulus function can be bounded or unbounded.

Definition 3. *Let f be an unbounded modulus function and $E \subset \mathbb{N}_0$. If the limit*

$$\delta_{J_p}^{f}(E) := \lim_{t \to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k \in E} p_k t^k\right)$$

exists, then $\delta_{J_p}^{f}(E)$ is called f-J_p-density of E [6].

Definition 4. *Let $x = (x_n)$ be a real sequence and f be a unbounded modulus function. If a set $K \subset \mathbb{N}_0$ has the propert $\delta_{J_p}^{f}(K) = 0$, then $\{x\}_K$ is called f-J_p-thin subsequence of x. If $\delta_{J_p}^{f}(K) \neq 0$, then $\{x\}_K$ is called f-J_p-nonthin subsequence of x.*

Definition 5. *Let $x = (x_n)$ be a real sequence. If x has a f-J_p-nonthin subsequence that converges to ℓ, then ℓ is called f-J_p-statistical limit point of x. The set of all f-J_p-statistical limit point of x is denoted by $\Lambda_x^{f-J_p}$.*

Definition 6. *Let $x = (x_n)$ be a real sequence. If $\delta_{J_p}^{f}(\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}) \neq 0$ for each $\varepsilon > 0$, then γ is called an f-J_p-statistical cluster point of x. The set of all f-J_p-statistical cluster point of x is denoted by $\Gamma_x^{f-J_p}$.*

It is known that $\delta_{J_p}^{f}(A) = 0$ means $\delta_{J_p}(A) = 0$ for any unbounded modulus function f and for any $A \subseteq \mathbb{N}$ [6].

Theorem 9. *x is a sequence in $\mathbb{R}$, then the followings hold:*
i) $\Lambda_x^{J_p} \subset \Lambda_x^{f-J_p}$
ii) $\Gamma_x^{J_p} \subset \Gamma_x^{f-J_p}$
iii) $\Lambda_x^{f-J_p} \subset \Gamma_x^{f-J_p}$
iv) $\Gamma_x^{f-J_p} \subset L_x$.

Proof. (i) - (ii) Since f-J_p-density zero sets are J_p-density zero, it is clear that $\Lambda_x^{J_p} \subset \Lambda_x^{f-J_p}$ and $\Gamma_x^{J_p} \subset \Gamma_x^{f-J_p}$.
(iii) To show $\Lambda_x^{f-J_p} \subset \Gamma_x^{f-J_p}$ assume that $\gamma \in \Lambda_x^{f-J_p}$. In this case there exists $K \subseteq \mathbb{N}$ such that $\delta_{J_p}^{f}(K) \neq 0$ and $\lim_{k \in K} x_k = \gamma$. For every $\varepsilon > 0$, $A = \{n \in K : |x_n - \gamma| \geq \varepsilon\}$ is finite which implies $\delta_{J_p}^{f}(K \backslash A) \geq \delta_{J_p}^{f}(K) - \delta_{J_p}^{f}(A) = \delta_{J_p}^{f}(K) \neq 0$. Since f is increasing and $K \backslash A \subseteq \{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}$,

$$\delta_{J_p}^{f}(\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}) \geq \delta_{J_p}^{f}(K \backslash A) \neq 0$$

hence $\gamma \in \Gamma_x^{f-J_p}$.
(iv) We show that $\Gamma_x^{f-J_p} \subseteq L_x$. Assume that $\gamma \in \Gamma_x^{f-J_p}$. For each $j \in \mathbb{N}$, we have

$$\delta_{J_p}^{f}\left(\left\{n \in \mathbb{N} : |x_n - \gamma| < \frac{1}{j}\right\}\right) \neq 0.$$

Thus if we say $A_j = \left\{n \in \mathbb{N} : |x_n - \gamma| < \frac{1}{j}\right\}$, then $A_j \subset \mathbb{N}$ and for each $j \in \mathbb{N}$, $A_{j+1} \subset A_j$. We can now take an increasing sequence of indices $n_1 < n_2 < \cdots$ with each $n_j \in A_j$. If $k \geq j$, $j \in \mathbb{N}$, then $|x_{n_k} - \gamma| < \frac{1}{k} \leq \frac{1}{j}$. So (x_{n_k}) is a subsequence of x that converges to γ, therefore $x \in L_x$. □

Note that all inclusions in Theorem 9 are strict. For instance for (i), we present the following example.

Example 5. *Let (x_k) and (p_k) are defined as*

$$x_k := \begin{cases} 1 & , k = 2j \\ 2 & , k = 2j+1 \end{cases} , j = 0, 1, 2, ...$$

$$p_k = \begin{cases} \dfrac{1}{k} & , k = 2j+1 \\ 1 & , k = 2j \end{cases} , j = 0, 1, 2, ...$$

In this case,

$$p(t) = \sum_{k=0}^{\infty} p_k t^k = \frac{1}{2} \ln\left(\frac{1+t}{1-t}\right) + \frac{1}{1-t^2}$$

for $0 < t < 1$ and then we get $\delta_{J_p}(E_1) = 1$ and $\delta_{J_p}(E_2) = 0$ for the sets $E_1 := \{2j : j \in \mathbb{N}_0\}$ and $E_2 := \{2j+1 : j \in \mathbb{N}_0\}$. Hence $\Lambda_x^{J_p} = \{2\}$. Indeed $\{x\}_{\mathbb{N}/E_1}$ is the subsequence of x which converges to 2 and $\delta_{J_p}(\mathbb{N} \backslash E_1) \neq 0$. For the modulus function $f(x) = \log(x+1)$, observe that

$$\delta_{J_p}^f(E_1) = \lim_{t \to 1^-} \frac{1}{f(p(t))} f\left(\sum_{k \in E_1} p_k t^k\right) = 1.$$

So we obtain that $\Lambda_x^{f-J_p} = \{1, 2\}$. Thus we see that the inclusion $\Lambda_x^{J_p} \subset \Lambda_x^{f-J_p}$ is strict.

Definition 7. *If there exists a bounded set B such that $\delta_{J_p}^f(\{n \in \mathbb{N} : x_n \notin B\}) = 0$, then $x = (x_n)$ is called f-J_p-statistical bounded sequence.*

Theorem 10. *If $x = (x_n)$ and $y = (y_n)$ are sequences in $\mathbb{R}$ such that $\delta_{J_p}^f(\{n \in \mathbb{N} : x_n \neq y_n\}) = 0$, then $\Lambda_x^{f-J_p} = \Lambda_y^{f-J_p}$ and $\Gamma_x^{f-J_p} = \Gamma_y^{f-J_p}$.*

Proof. Let $\alpha \in \Lambda_x^{f-J_p}$. In this case there exists a set $B \subset \mathbb{N}$ such that

$$\lim_{n \in B} x_n = \alpha$$

where $\delta_{J_p}^f(B) \neq 0$ and $|B|$ is infinite. We get the set $A = \{n \in \mathbb{N} : x_n \neq y_n\}$ for which $\delta_{J_p}^f(A) = 0$. Now take the sequence $(y_n)_{n \in B \backslash A}$, that convergence to α and $(y_n)_{n \in B \backslash A}$ is f-J_p-nonthin subsequence of y. Indeed, if $\delta_{J_p}^f(B \backslash A) = 0$, then

$$\delta_{J_p}^f(A \cup B) = \delta_{J_p}^f(A \cup (B \backslash A)) \leq \delta_{J_p}^f(A) + \delta_{J_p}^f(B \backslash A) = 0,$$

but $B \subset A \cup B$ and B doesn't have null f-J_p-density. Thus we get $\alpha \in \Lambda_y^{f-J_p}$. The other side of the equation can be shown in a similar way. Now take $\gamma \in \Gamma_x^{f-J_p}$. For $\varepsilon > 0$, $\delta_{J_p}^{f}\left(\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}\right) \neq 0$. Consider the sets

$$B_\varepsilon = \{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\} \text{ and } C_\varepsilon = \{n \in \mathbb{N} : |y_n - \gamma| < \varepsilon\}$$

for given $\varepsilon > 0$. We get $B_\varepsilon \backslash A \subseteq C_\varepsilon$ and so

$$\delta_{J_p}^{f}(C_\varepsilon) \geq \delta_{J_p}^{f}(B_\varepsilon \backslash A) \geq \delta_{J_p}^{f}(B_\varepsilon) - \delta_{J_p}^{f}(A) = \delta_{J_p}^{f}(B_\varepsilon) \neq 0.$$

Thus we get $\gamma \in \Gamma_y^{f-J_p}$. The other side of the equation can be shown similarly. □

From this theorem, the following result is obtained.

Corollary 2. *Let $x = (x_n)$ be a f-J_p-statistical bounded real sequence. Then $\Gamma_x^{f-J_p}$ is bounded.*

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

References

[1] Aizpuru, A., Listán-García, M. C., Rambla-Barreno F., Density by moduli and statistical convergence, *Quaestiones Mathematicae*, 37 (2014), 525-530. https://doi.org/10.2989/16073606.2014.981683

[2] Arif, A., Yurdakadim, T., Approximation results on nonlinear operators by P_p-statistical convergence, *Advanced Studies: Euro-Tbilisi Mathematical Journal,* 15(3) (2022), 1-10. DOI: 10.32513/asetmj/19322008220

[3] Bayram, N. Ş., Criteria for statistical convergence with respect to power series methods, *Positivity.* 25, (2021), 1097-1105. https://doi.org/10.1007/s11117-020-00801-6

[4] Bayram, N. Ş., Yıldız, S., Approximation by statistical convergence with respect to power series methods, *Hacet. J. Math. Stat.,* 51(4) (2022), 1108-1120. DOI: 10.15672/hujms.1022072

[5] Bayram, N. Ş., P-strong convergence with respect to an Orlicz function, *Turk J Math.,* 46 (2022), 832-838. https://doi.org/10.55730/1300-0098.3126

[6] Belen, C., Yıldırım, M., Sümbül, C., On statistical and strong convergence with respect to a modulus function and a power series method, *Filomat*, 34(12) (2020), 3981-3993. https://doi.org/10.2298/FIL2012981B

[7] Boos, J., Classical and modern methods in summability, Oxford University Press, Oxford, 2000.

[8] Demirci, K., Dirik, F., Yıldız, S., Approximation via equi-statistical convergence in the sense of power series method, *RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.*, 116(65) (2022). https://doi.org/10.1007/s13398-021-01191-4

[9] Fast, H., Sur la convergence statistique, *Colloq. Math.,* 2 (1951), 241-244.

[10] Freedman, A. R., Sember, J. J., Densities and summability, *Pacific J. Math.,* 95(2) (1981), 293-305.

[11] Fridy, J. A., On statistical convergence, *Analysis* 5 (1985), 301-313.

[12] Fridy, J.A., Statistical limit points, *Proc. Amer. Math. Soc.*, 118(8) (1993), 1187-1193.

[13] Listán-García, M. C., f-statistical convergence, completeness and f-cluster points, *Bull. Belg. Math. Soc. Simon Stevin,* 23(2) (2016), 235-245.

[14] Nakano, H., Concave modulars, *J. Math. Soc. Japan*, 5 (1953), 29-49.

[15] Šalát, T., On statistically convergent sequences of real numbers, *Math. Slovaca*, 30(2) (1980), 139-150.

[16] Söylemez, D., A Korovkin type approximation theorem for Balázs Type Bleimann, Butzer and Hahn Operators via power series statistical convergence, *Math. Slovaca*, 72(1) (2022), 153-164. https://doi.org/10.1515/ms-2022-0011

[17] Sümbül, C., Belen, C., Yıldırım, M., Properties of J_p-statistical convergence, *Cumhuriyet Sci. J.*, 43(2) (2022), 294-298. https://doi.org/10.17776/csj.1064559

[18] Ünver, M., Orhan, C., Statistical convergence with respect to power series methods and applications to approximation theory, *Numer. Func. Anal Opt.,* 40(5) (2019), 535-547. https://doi.org/10.1080/01630563.2018.1561467

[19] Ünver, M., Bayram, N. Ş., On statistical convergence with respect to power series methods, *Positivity* (2022), 26-55. https://doi.org/10.1007/s11117-022-00921-1