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Pseudoparallel Invariant Submanifolds of a Para-Sasakian Manifold

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Abstract. In this paper, invariant submanifolds of a para-Sasakian manifold have been studied. Some special submanifolds such as pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel submanifolds of a para-Sasakian manifold have been considered. The necessary and sufficient conditions for an invariant submanifold to be totally geodesic under some conditions have been given.

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1. Introduction

Invariant submanifolds of a paracontact metric manifold are a very important concept for geometry. This concept allows us to grasp some important topics and problems in many areas of mathematics, such as applied mathematics. To give an important example, invariant submanifolds are used to discuss the properties of non-linear autonomous system [6].

One of the important concepts such as invariant submanifold is that a submanifold is a totally geodesic. If every geodesic in a submanifold is geodesic in ambient space, this submanifold is called a totally geodesic submanifold. Kon showed when invariant submanifolds of a Sasakian manifold would become totally geodesic submanifold in [10]. In [9], the answer to the question of when each submanifold of a Kenmotsu manifold will be totally geodesic is given. Similarly, the totally geodesic of a trans-Sasakian invariant submanifold is discussed in [12]. Also [14] deal with the totally geodesic invariant submanifolds of the (k,μ) −contact metric manifold.

As can be seen from many of the studies mentioned above, totally geodesic submanifolds, which are also the simplest submanifolds, play a very important role in an important theory such as the theory of relativity, and many geometers have submitted various important scientific works in this field [3, 7, 8, 11, 13, 16, 17].

On the other hand, Para-Sasakian and Lorentzian para-Sasakian manifolds are also studied on connections other than the Levi-Civita connection. For example, O. Bahadır investigated para-Sasakian and Lorentzian para-Sasakian manifolds on the quarter-symmetric non-metric connection in [4, 5].

Many geometers working on the theory of manifolds have studied para-Sasakian manifolds and investigated some important properties of these manifolds. Invariant submanifolds are very important for a para-Sasakian manifold.

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In this study, submanifolds that have not been studied before for para-Sasakian manifolds are discussed by making use of many studies mentioned above. Invariant submanifolds are used for submanifolds of Para-Sasakian manifolds. For invariant submanifolds of Para-Sasakian manifolds, the concepts of pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel submanifold are defined. Then, necessary and sufficient conditions are obtained for an invariant submanifold of the para-Sasakian manifold to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci generalized pseudoparallel.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\bar{M}^{2n+1}$ has an almost paracontact structure (ϕ, ξ, η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2 W = W - \eta(W)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.$$

If an almost paracontact manifold is endowed with a semi-Riemannian metric tensor g such that

$$g(\phi W, \phi Z) = g(W, Z) - \eta(W)\eta(Z),$$

for all vector fields W, Z on $\bar{M}^{2n+1}$, then $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be almost paracontact metric manifold. It is clear that

$$g(\xi, W) = \eta(W).$$

The fundamental 2-form Φ of an almost paracontact metric manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ is defined by

$$\Phi(W, Z) = g(W, \phi Z).$$

If $d\eta = \Phi$, then almost paracontact metric manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ is called paracontact metric manifold. If a paracontact metric structure is normal, this structure is called para-Sasakian. So equivalently, if the structure (ϕ, ξ, η, g) satisfies the equations

$$d\eta = 0\,, \; \bar{\nabla}_W \xi = \phi W, \tag{2.1}$$

$$\left(\bar{\nabla}_W \phi\right) Z = -g(W, Z)\xi - \eta(Z) W + 2\eta(W)\eta(Z)\xi,$$

the manifold $\bar{M}^{2n+1}$ is called para-Sasakian manifold or P-Sasakian manifold, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}^{2n+1}$. If the relation

$$\left(\bar{\nabla}_W \eta\right) Z = -g(W, Z) + \eta(W)\eta(Z)$$

is satisfied specifically, the para-Sasakian manifold is called the special para-Sasakian manifold or the Sp-Sasakian manifold.

Lemma 2.1. *A para-Sasakian manifold provides the following relations:*

$$S(W, \xi) = -(n-1)\eta(W), \tag{2.2}$$

$$Q\xi = -(n-1)\xi,$$

$$\bar{R}(W, Z)\xi = \eta(W) Z - \eta(Z) W,$$

$$\bar{R}(\xi, W) Z = \eta(Z) W - g(W, Z)\xi,$$

$$\bar{R}(\xi, W)\xi = W - \eta(W)\xi, \tag{2.3}$$

$$\eta\left(\bar{R}(W, Z) T\right) = g(W, T)\eta(Z) - g(Z, T)\eta(W),$$

$$S(\phi W, \phi Z) = S(W, Z) + (n-1)\eta(W)\eta(Z),$$

for any vector fields W, Z on $\bar{M}^{2n+1}$, where $\bar{\nabla}$ is the Levi-Civita connection, $\bar{R}$ and S denote the Riemannian curvature tensor and Ricci tensor of $\bar{M}^{2n+1}$, respectively.

Now, let M be an immersed submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\bar{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\bar{\nabla}_W Z = \nabla_W Z + h(W, Z)$$

and

$$\bar{\nabla}_W V = -A_V W + \nabla^{\perp}_W V,$$

for all $W, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$, h and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(A_V W, Z) = g(h(W, Z), V).$$

The covariant derivative of h is defined by

$$(\bar{\nabla}_W h)(Z, T) = \nabla^{\perp}_W h(Z, T) - h(\nabla_W Z, T) - h(Z, \nabla_W T), \tag{2.4}$$

for all $W, Z, T \in \Gamma(TM)$. If

$$\bar{\nabla} h = 0,$$

then the submanifold M is said to be its second fundamental form is parallel.

If S and g are linearly dependent, that is,

$$S\,(W, Z) = \lambda g\,(W, Z)$$

with λ a constant, the para-Sasakian manifold is called the Einstein manifold.

By R, we denote the Riemannian curvature tensor of submanifold M, we have the following Gauss equation

$$\bar{R}(W, Z)T = R(W, Z)T + A_{h(W,T)}Z - A_{h(Z,T)}W + (\bar{\nabla}_W h)(Z, T) - (\bar{\nabla}_Z h)(W, T).$$

For a $(0, k)$-type tensor field T, $k \geq 1$ and a $(0, 2)$-type tensor field A on a Riemannian manifold (M, g), $Q(A, B)$-tensor field is defined by

$$\begin{aligned} Q(A, B)(W_1, W_2, ..., W_k; W, Z) = \ & -B((W \Lambda_A Z)W_1, W_2, ..., W_k)... \\ & -B(W_1, W_2, ... W_{k-1}, (W \Lambda_A Z)W_k), \end{aligned}$$

for all $W_1, W_2, ..., W_k, W, Z \in \Gamma(T\bar{M})$, where

$$(W \wedge_A Z)T = A(Z, T)W - A(W, T)Z.$$

Definition 2.2. A submanifold of a Riemannian manifold (M, g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci-generalized pseudoparallel if

$$\bar{R} \cdot h \text{ and } Q(g, h),$$
$$\bar{R} \cdot \bar{\nabla} h \text{ and } Q(g, \bar{\nabla} h),$$
$$\bar{R} \cdot h \text{ and } Q(S, h),$$
$$\bar{R} \cdot \bar{\nabla} h \text{ and } Q(S, \bar{\nabla} h)$$

are linearly dependent, respectively [1, 2, 15].

Equivalently, these can be expressed by the following relations;

$$\bar{R} \cdot h = \lambda_1 Q(g, h), \tag{2.5}$$

$$\bar{R} \cdot \bar{\nabla} h = \lambda_2 Q(g, \bar{\nabla} h), \tag{2.6}$$

$$\bar{R} \cdot h = \lambda_3 Q(S, h), \tag{2.7}$$

$$\bar{R} \cdot \bar{\nabla} h = \lambda_4 Q(S, \bar{\nabla} h), \tag{2.8}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are, respectively, functions defined on $M_1 = \{x \in M : h(x) \neq g(x)\}$, $M_2 = \{x \in M : \bar{\nabla} h(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq h(x)\}$ and $M_4 = \{x \in M : S(x) \neq \bar{\nabla} h(x)\}$.

Particularly, if $\lambda_1 = 0$, then the submanifold is said to be semiparallel, if $\lambda_2 = 0$, the submanifold is said to be 2-semiparallel.

Thus, we have the following lemma.

Lemma 2.3. *Let M be an invariant submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}$. Then, the following equalities hold on $\bar{M}^{2n+1}$;*

$$h(\phi W, Z) = h(W, \phi Z) = \phi h(W, Z) \text{ and } h(W, \xi) = 0 \tag{2.9}$$

for all $W, Z \in \Gamma(TM)$.

3. Invariant Pseudoparallel Submanifolds of Para-Sasakian Manifold

Next, we will discuss the types of submanifolds given in the definition for the invariant submanifold M of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$.

Let M be an immersed submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, g, \eta)$. If

$$\phi(T_x M) \subseteq T_x M,$$

for each point $x \in M$, then M is said to be an invariant submanifold. We clearly know that all properties of an invariant submanifold inherit an ambient manifold as well.

In the rest of this paper, we will assume that M is an invariant submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$.

Proposition 3.1. *Let M be an invariant submanifold of a para-Sasakian manifold $\bar{M}$. The second fundamental form h of M is parallel if and only if M is totally geodesic.*

Proof. Let's assume that the h second fundamental form of M is parallel. So by definition, we can write

$$(\bar{\nabla}_W h)(Z, T) = 0,$$

for all $W, Z, T \in \Gamma(TM)$. That's mean, from (2.4)

$$\nabla^{\perp}_W h(Z, T) - h(\nabla_W Z, T) - h(Z, \nabla_W T) = 0.$$

Here, taking $T = \xi$, by virtue of (1) and (9), we obtain

$$-h(\nabla_W \xi, Z) = -h(-\phi W, Z) = \phi h(W, Z) = 0.$$

So by definition M is a totally geodesic. The converse is obvious. This proves our assertion. □

This proposition is important for later theorems.

Theorem 3.2. *Let M be an invariant pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then M is either totally geodesic or $\lambda_1 = 1$.*

Proof. Let's assume M is the invariant pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}$. From (2.5), we have

$$(\bar{R}(W, Z) \cdot h)(U, V) = \lambda_1 Q(g, h)(U, V; W, Z),$$

for all $W, Z, U, V \in \Gamma(TM)$. This leads to

$$\begin{aligned} &R^{\perp}(W, Z)h(U, V) - h(R(W, Z)U, V) - h(U, R(W, Z)V) \\ &= -\lambda_1\{h((W \wedge_g Z)U, V) + h(U, (W \wedge_g Z)V)\} \\ &= -\lambda_1\{h(g(Z, U)W - g(W, U)Z, V) \\ &+h(U, g(Z, V)W - g(W, V)Z)\} \end{aligned} \tag{3.1}$$

for all $W, Z, U, V \in \Gamma(TM)$. Taking $V = \xi$ in (3.1) and taking into account (2.9), we obtain

$$h(U, R(W, Z)\xi) = \lambda_1\{\eta(Z)h(U, W) - \eta(W)h(U, Z)\}.$$

Again taking $Z = \xi$ and considering proposition, we conclude that

$$\lambda_1 h(U, W) \quad = h(U, R(W, \xi)\xi) = h(U, \eta(W)\xi - W) = -h(U, W).$$

From here,

$$(\lambda_1 + 1)\, h(U, W) = 0$$

this proves our assertion. □

From the Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let M be an invariant pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then M is semiparallel if and only if M is totally geodesic.*

Theorem 3.4. *Let M be an invariant 2-pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then M is either totally geodesic or $\lambda_2 = -1$.*

Proof. Let's assume that, M is the invariant 2-pseudoparallel submanifold. So by definition, $\bar{R} \cdot \bar{\nabla} h$ and $Q(g, \bar{\nabla} h)$ are linearly dependent. Then, we have from (2.6)

$$(\bar{R}(W, Z) \cdot \bar{\nabla} h)(U, V, T) = \lambda_2 Q(g, \bar{\nabla} h)(U, V, T; W, Z),$$

for all $W, Z, U, V, T \in \Gamma(TM)$. This means that

$$\begin{aligned} R^{\perp}(W, Z)(\bar{\nabla}_U h)(V, T) - (\bar{\nabla}_{R(W,Z)U} h)(V, T) - (\bar{\nabla}_U h)(R(W, Z)V, T) - (\bar{\nabla}_U h)(V, R(W, Z)T) \\ = -\lambda_2 \{(\bar{\nabla}_{(W \wedge_g Z)U} h)(V, T) + (\bar{\nabla}_U h)((W \wedge_g Z)V, T) + (\bar{\nabla}_U h)(V, (W \wedge_g Z)T)\}. \end{aligned} \tag{3.2}$$

for all $W, Z, U, V, T \in \Gamma(TM)$. In (3.2), taking $W = T = \xi$, we have

$$\begin{aligned} R^{\perp}(\xi, Z)(\bar{\nabla}_U h)(V, \xi) - (\bar{\nabla}_{R(\xi,Z)U} h)(V, \xi) - (\bar{\nabla}_U h)(R(\xi, Z)V, \xi) - (\bar{\nabla}_U h)(V, R(\xi, Z)\xi) \\ = -\lambda_2 \{\left(\bar{\nabla}_{(\xi \wedge_g Z)U} h\right)(V, \xi) + (\bar{\nabla}_U h)\left(\left(\xi \wedge_g Z\right)V, \xi\right) + (\bar{\nabla}_U h)\left(V, \left(\xi \wedge_g Z\right)\xi\right)\}. \end{aligned} \tag{3.3}$$

Now, let's calculate each of these expressions. From (2.1), (2.4) and (2.9), we obtain

$$\begin{aligned} R^{\perp}(\xi, Z)(\bar{\nabla}_U h)(V, \xi) &= R^{\perp}(\xi, Z)\{\nabla^{\perp}_U h(V, \xi) - h(\nabla_U V, \xi) - h(V, \nabla_U \xi)\} \\ &= R^{\perp}(\xi, Z)\{-h(V, \nabla_U \xi)\} \\ &= -R^{\perp}(\xi, Z)h(V, \phi U) \\ &= -R^{\perp}(\xi, Z)\phi h(V, U). \end{aligned} \tag{3.4}$$

Moreover, taking into account (2.1) and (2.9), we have

$$\begin{aligned} (\bar{\nabla}_{R(\xi,Z)U} h)(V, \xi) &= \nabla^{\perp}_{R(\xi,Z)U} h(V, \xi) - h(\nabla_{R(\xi,Z)U} V, \xi) - h(\nabla_{R(\xi,Z)U} \xi, V) \\ &= -h(\phi R(\xi, Z)U, V) \\ &= -\phi h(R(\xi, Z)U, V) \\ &= -\phi h\left(\eta\left(U\right)Z - g\left(\xi, U\right)\xi, V\right) \\ &= -\phi \eta\left(U\right) h\left(Z, V\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\bar{\nabla}_U h)(R(\xi, Z)V, \xi) &= \nabla^{\perp}_U h(R(\xi, Z)V, \xi) - h(\nabla_U R(\xi, Z)V, \xi) - h(R(\xi, Z)V, \nabla_U \xi) \\ &= -h(\phi U, R(\xi, Z)V) \\ &= -\phi h(U, R(\xi, Z)V) \\ &= -\phi h\left(U, \eta\left(V\right)Z - g\left(Z, V\right)\xi\right) \\ &= -\phi \eta\left(V\right) Z h\left(U, Z\right), \end{aligned} \tag{3.6}$$

$$\begin{aligned} (\bar{\nabla}_U h)(V, R(\xi, Z)\xi) &= (\bar{\nabla}_U h)(V, Z - \eta(Z)\xi) \\ &= (\bar{\nabla}_U h)(V, Z) - (\bar{\nabla}_U h)(V, \eta(Z)\xi) \\ &= (\bar{\nabla}_U h)(V, Z) - \nabla^{\perp}_U h(V, \eta(Z)\xi) + h(\nabla_U V, \eta(Z)\xi) + h(V, \nabla_U \eta(Z)\xi) \\ &= (\bar{\nabla}_U h)(V, Z) + h\left(V, \nabla_U \eta(Z)\xi\right) \\ &= (\bar{\nabla}_U h)(V, Z) + \eta(Z)\phi h(V, U), \end{aligned} \tag{3.7}$$

$$\begin{aligned} (\bar{\nabla}_{(\xi \wedge_g Z)U} h)(V, \xi) &= \nabla^{\perp}_{(\xi \wedge_g Z)U} h(V, \xi) - h(\nabla_{(\xi \wedge_g Z)U} V, \xi) - h(V, \nabla_{(\xi \wedge_g Z)U} \xi) \\ &= -h(V, \nabla_{g(Z,U)\xi - \eta(U)Z} \xi) \\ &= -h(V, \phi(g(Z, U)\xi - \eta(U)Z)) \\ &= -h(V, -\phi \eta(U)Z) = \eta(U)\phi h(V, Z), \end{aligned} \tag{3.8}$$

$$\begin{aligned}(\bar{\nabla}_U h)((\xi \wedge_g Z)V, \xi) &= \nabla^{\perp}_U h((\xi \wedge_g Z)V, \xi) - h(\nabla_U(\xi \wedge_g Z)V, \xi) - h((\xi \wedge_g Z)V, \nabla_U \xi)\\ &= -h(g(Z, V)\xi - \eta(V)Z, \phi U)\\ &= \eta(V)\phi h(Z, U),\end{aligned} \tag{3.9}$$

$$\begin{aligned}(\bar{\nabla}_U h)(V, (\xi \wedge_g Z)\xi) &= (\bar{\nabla}_U h)(V, \eta(Z)\xi - Z)\\ &= (\bar{\nabla}_U h)(V, \eta(Z)\xi) - (\bar{\nabla}_U h)(V, Z)\\ &= \nabla^{\perp}_U h(V, \eta(Z)\xi) - h(\nabla_U V, \eta(Z)\xi) - h(V, \nabla_U \eta(Z)\xi) - (\bar{\nabla}_U h)(V, Z)\\ &= -h(V, U\eta(Z)\xi + \eta(Z)\nabla_U \xi) - (\bar{\nabla}_U h)(V, Z)\\ &= -\eta(Z)h(V, \phi U) - (\bar{\nabla}_U h)(V, Z)\\ &= -\eta(Z)\phi h(V, U) - (\bar{\nabla}_U h)(V, Z).\end{aligned} \tag{3.10}$$

Consequently, if we put (3.4), (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) in (3.3), we reach at

$$\begin{aligned}-R^{\perp}(\xi, Z)\phi h(V, U) + \phi\eta(U)h(Z, V) + \eta(V)\phi h(Z, U) - \left(\bar{\nabla}_U h\right)(V, Z) - \eta(Z)\phi h(V, U)\\ = -\lambda_2\{\eta(U)\phi h(V, Z) + \eta(V)\phi h(Z, U) - \eta(Z)\phi h(V, U) - (\bar{\nabla}_U h)(V, Z)\}.\end{aligned} \tag{3.11}$$

If ξ is taken of V in (3.11), considering (2.3) and (2.9), we get

$$\phi h(Z, U) - (\bar{\nabla}_U h)(\xi, Z) = \lambda_2\{-\phi h(Z, U) - (\bar{\nabla}_U h)(\xi, Z)\}, \tag{3.12}$$

where

$$\begin{aligned}(\bar{\nabla}_U h)(\xi, Z) &= \nabla^{\perp}_U h(Z, \xi) - h(\nabla_U Z, \xi) - h(Z, \nabla_U \xi)\\ &= -h(Z, \phi U) = -\phi h(Z, U).\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we conclude that

$$\phi h(Z, U) + \phi h(Z, U) = -\lambda_2\,[2\phi h(Z, U)] = 0,$$

that is,

$$(\lambda_2 + 1)\,h(Z, U) = 0$$

which proves our assertions. □

From Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let M be an invariant pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then, M is 2-semiparallel if and only if M is totally geodesic.*

Theorem 3.6. *Let M be an invariant Ricci-generalized pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then, M is either totally geodesic or $\lambda_3 = \frac{1}{n-1}$.*

Proof. Let's assume M Ricci generalized pseudoparallel submanifold. Then, from (2.7), we have

$$\begin{aligned}(\bar{R}(W, Z)\cdot h)(U, V) &= \lambda_3 Q(S, h)(U, V; W, Z)\\ &= -\lambda_3\left\{h((W \wedge_S Z)U, V) + h(U, (W \wedge_S Z)V)\right\},\end{aligned}$$

for all $W, Z, U, V \in \Gamma(TM)$. This means that,

$$\begin{aligned}R^{\perp}(W, Z)h(U, V) - h(R(W, Z)U, V) - h(U, R(W, Z)V)\\ = -\lambda_3\{h(S(Z, U)W - S(W, U)Z, V) + h(S(V, Z)W - S(W, V)Z, U)\}.\end{aligned}$$

Here taking $W = V = \xi$, we obtainand by using (2.2) and (2.9)

$$-h(U, Z - \eta(Z)\xi) = -\lambda_3\,[-h(U, -(n-1)Z)]\,.$$

Then, we can infer

$$(n-1)\,\lambda_3 h(U, Z) = h(U, Z)\,.$$

This proves our assertion. □

Theorem 3.7. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a para-Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Then, M is either totally geodesic or $\lambda_4 = \frac{1}{n-1}$.*

Proof. Let us assume that M is a 2-Ricci-generalized pseudoparallel submanifold. Then, from (2.8), we have

$$(\bar{R}(W,Z)\cdot\bar{\nabla}h)(U,V,T)=\lambda_4 Q(S,\bar{\nabla}h)(U,V,T;W,Z),$$

for all $W,Z,U,V,T\in\Gamma(TM)$. This implies that

$$\begin{aligned}R^{\perp}(W,Z)(\bar{\nabla}_U h)(V,T)&-(\bar{\nabla}_{R(W,Z)U}h)(V,T)-(\bar{\nabla}_U h)(R(W,Z)V,T)-(\bar{\nabla}_U h)(V,R(W,Z)T)\\&=-\lambda_4\{(\bar{\nabla}_{(W\wedge_S Z)U}h)(V,T)+(\bar{\nabla}_U h)((W\wedge_S Z)V,T)+(\bar{\nabla}_U h)(V,(W\wedge_S Z)T)\}.\end{aligned}$$

Here, taking $W=V=\xi$, we have

$$\begin{aligned}R^{\perp}(\xi,Z)(\bar{\nabla}_U h)(\xi,T)&-(\bar{\nabla}_{R(\xi,Z)U}h)(\xi,T)-(\bar{\nabla}_U h)(R(\xi,Z)\xi,T)-(\bar{\nabla}_U h)(\xi,R(\xi,Z)T)\\&=-\lambda_4\{(\bar{\nabla}_{(\xi\wedge_S Z)U}h)(\xi,T)+(\bar{\nabla}_U h)((\xi\wedge_S Z)\xi,T)+(\bar{\nabla}_U h)(\xi,(\xi\wedge_S Z)T)\}.\end{aligned}\tag{3.14}$$

Now, let's calculate each of these expressions. Also, taking into account (2.1) and (2.9), we arrive at

$$\begin{aligned}R^{\perp}(\xi,Z)(\bar{\nabla}_U h)(\xi,T)&=R^{\perp}(\xi,Z)\{\nabla^{\perp}_U h(\xi,T)-h(\nabla_U T,\xi)-h(T,\nabla_U\xi)\}\\&=-R^{\perp}(\xi,Z)h(\phi U,T)\\&=-R^{\perp}(\xi,Z)\phi h(U,T).\end{aligned}\tag{3.15}$$

On the other hand, by using (2.1) and (2.9), we have

$$\begin{aligned}(\bar{\nabla}_{R(\xi,Z)U}h)(\xi,T)&=\nabla^{\perp}_{R(\xi,Z)U}h(\xi,T)-h(\nabla_{R(\xi,Z)U}\xi,T)\quad -h(\xi,\nabla_{R(\xi,Z)U}T)\\&=-h(\phi R(\xi,Z)U,T)\\&=-\phi h(R(\xi,Z)U,T)\\&=-\phi h\left(\eta(U)Z-g(Z,U)\xi,T\right)\\&=-\eta(U)\,\phi h(Z,T),\end{aligned}\tag{3.16}$$

$$\begin{aligned}(\bar{\nabla}_U h)(R(\xi,Z)\xi,T)&=(\bar{\nabla}_U h)(Z-\eta(Z)\xi,T)\\&=(\bar{\nabla}_U h)(Z,T)-(\bar{\nabla}_U h)(\eta(Z)\xi,T)\\&=(\bar{\nabla}_U h)(Z,T)-\nabla^{\perp}_U h(\eta(Z)\xi,T)+h(\nabla_U\eta(Z)\xi,T)+h(\eta(Z)\xi,\nabla_U T)\\&=(\bar{\nabla}_U h)(Z,T)+h(U\eta(Z)\xi+\eta(Z)\nabla_U\xi,T)\\&=(\bar{\nabla}_U h)(Z,T)+h(\phi U,T)\eta(Z)\\&=(\bar{\nabla}_U h)(Z,T)+\eta(Z)\phi h(U,T),\end{aligned}\tag{3.17}$$

$$\begin{aligned}(\bar{\nabla}_U h)(\xi,R(\xi,Z)T)&=\nabla^{\perp}_U h(\xi,R(\xi,Z)T)-h(\nabla_U\xi,R(\xi,Z)T)-h(\xi,\nabla_U R(\xi,Z)T)\\&=-h(\phi U,R(\xi,Z)T)\\&=-\phi h(U,R(\xi,Z)T)\\&=-\phi h\left(U,\eta(T)Z-g(Z,T)\xi\right)\\&=-\eta(T)\,\phi h(U,T).\end{aligned}\tag{3.18}$$

Now, let's calculate the right side of (3.14). Making use of (2.1), (2.2) and (2.9), we have

$$\begin{aligned}(\bar{\nabla}_{(\xi\wedge_S Z)U}h)(\xi,T)&=\nabla^{\perp}_{(\xi\wedge_S Z)U}h(\xi,T)-h(\nabla_{(\xi\wedge_S Z)U}\xi,T)-h(\xi,\nabla_{(\xi\wedge_S Z)U}T)\\&=-h(\phi(S(Z,U)\xi-S(\xi,U)Z),T)\\&=\phi h(S(U,\xi)Z,T)=-(n-1)\,\eta(U)\,\phi h(Z,T),\end{aligned}\tag{3.19}$$

$$\begin{aligned}(\bar{\nabla}_U h)((\xi\wedge_S Z)\xi,T)&=(\bar{\nabla}_U h)(S(Z,\xi)\xi-S(\xi,\xi)Z,T)\\&=(\bar{\nabla}_U h)\left(S\left(Z,\xi\right)\xi,T\right)-(\bar{\nabla}_U h)\left(S\left(\xi,\xi\right)Z,T\right)\\&=(\bar{\nabla}_U h)\left(-(n-1)\,\eta(Z)\,\xi,T\right)-(\bar{\nabla}_U h)\left(-(n-1)\,Z,T\right)\\&=-(n-1)\,(\bar{\nabla}_U h)\left(\eta(Z)\,\xi,T\right)+(n-1)\,(\bar{\nabla}_U h)(Z,T)\\&=(n-1)\,h\Big(\eta(Z)\,\bar{\nabla}_U\xi,T\Big)+(n-1)\,(\bar{\nabla}_U h)(Z,T)\\&=(n-1)\,\eta(Z)\,\phi h(U,T)+(n-1)\,(\bar{\nabla}_U h)(Z,T),\end{aligned}\tag{3.20}$$

Finally,

$$\begin{aligned}(\bar{\nabla}_U h)(\xi,(\xi\wedge_S Z)T) &= (\bar{\nabla}_U h)(\xi, S(Z,T)\xi - S(\xi,T)Z)\\ &= (\bar{\nabla}_U h)(\xi, S(Z,T)\xi) + (n-1)\,(\bar{\nabla}_U h)(\xi,\eta(T)Z)\\ &= \nabla^{\perp}_U h(\xi, S(Z,T)\xi) - h(\nabla_U\xi, S(Z,T)\xi) - h(\xi,\nabla_U S(Z,T)\xi)\\ &+ (n-1)\,\{\nabla^{\perp}_U h(\xi,\eta(T)Z) - h(\nabla_U\xi,\eta(T)Z) - h(\xi,\nabla_U\eta(T)Z)\}\\ &= (n-1)\,\{-h(\phi U,\eta(T)Z)\}\\ &= -\,(n-1)\,\eta(T)\phi h(Z,U).\end{aligned} \tag{3.21}$$

By substituting (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) into (3.14), we reach at

$$\begin{aligned}-R^{\perp}(\xi,Z)\phi h(T,U) + \eta(U)\,\phi h(Z,T) - (\bar{\nabla}_U h)(Z,T) - \eta(Z)\phi h(U,T) + \eta(T)\,\phi h(U,Z)\\ = -\lambda_4\{-\,(n-1)\,\eta(U)\phi h(Z,T) + (n-1)\,\eta(Z)\,h(U,T)\\ + (n-1)\,(\bar{\nabla}_U h)(Z,T) - (n-1)\,\eta(T)\,\phi h(U,Z)\}.\end{aligned} \tag{3.22}$$

Here, if taking $T=\xi$ in (3.22), then

$$-\,(n-1)\,\lambda_4\{(\bar{\nabla}_U h)(Z,\xi) - \phi h(U,Z)\} \quad = -(\bar{\nabla}_U h)(Z,\xi) + \phi h(U,Z).$$

We conclude that

$$h(U,Z) = (n-1)\,\lambda_4 h(U,Z) = 0$$

which proves our assertion. □

4. Conclusion

In this study, submanifolds that have not been studied before for para-Sasakian manifolds are discussed by making use of many studies mentioned above. Invariant submanifolds are used for submanifolds of Para-Sasakian manifolds. For invariant submanifolds of Para-Sasakian manifolds, the concepts of pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel submanifold are defined. Then, necessary and sufficient conditions are obtained for an invariant submanifold of the para-Sasakian manifold to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci generalized pseudoparallel.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

References

[1] Arslan, K., Lumiste, U., Murathan, C., Özgür, C., *2-semiparallel surfaces in space forms, Two particular cases*, Proc. Estonian Acad. Sci. Phys. Math., **49**(3)(2000), 139–148.

[2] Asperti, A.C., Lobos, G.A., Mercuri, F., *Pseudoparallel immersion in space forms*, 10th School on Differential Geometry Mat. Contemp, **17**(1998), 59–70.

[3] Atçeken, M., Yıldırım, Ü., Dirik, S., *Semiparallel submanifolds of a normal paracontact metric manifold*, Hacettepe J. of Math. and Stat., **48**(2019), 501–509.

[4] Bahadır, O., *Lorentzian para-Sasakian manifold with quarter-symmetric non-metricconnection*, Journal of Dynamical Systems and Geometric Theories, **14**(1)(2016), 17–33.

[5] Bahadır, O., *P-Sasakian manifold with quarter-symmetric non-metric connection*, Universal Journal of Applied Mathematics, **6**(4)(2018), 123–133.

[6] Guojing, Z., Jianguo, W., *Invariant submanifolds and modes of non-linear autonomous system*, Appl. Math. Mech., **19**(1998), 587–693.

[7] Hui, S.K., Mishra, V.N., Pal, T.,Vandana, *Some classes of invariant submanifolds of* $(LCS)_n$ *−manifolds*, Italian J. of Pure and Appl. Math., **39**(2018, 359–372.

[8] Hui, S.K, Uddin, S., Alkhaldi, A.H., Mandal, P., *Invariant submanifolds of generalized Sasakian-space forms*, International Journal of Geometric Methods in Modern Physics, **15**(2018), 1–21.

[9] Kobayashi, M., *Semi invariant submanifolds of a certain class of almost contact metric manifolds*, Tensor (N.S), **43**(1986), 28–36.

[10] Kon, M., *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep., **27**(1973), 330–336.

[11] Montano, B.C., Terlizzi, L.D., Tripathi, M.M., *Invariant submanifolds of contact* $(k,\eta)-$*manifolds*, Glasgow Math. J., **50**(2008), 499–507.

[12] Sarkar, A., Sen, M., *On invariant submanifolds trans-Sasakian manifolds*, Proc. Estonian Acad. Sci., **61**(2012), 29–37.

[13] Shukla, S.S., *On* $\phi-$*Symmetric para-Sasakian manifolds*, Int. Journal of Math. Analysis, **16**(2010), 761–769.

[14] Siddesha, M., Bagewadi, C.S., *On some clases an ınvariant submanifolds of* $(k,\mu)-$*contact manifold*, J. of inforatics and mathematical sciences, **9**(1)(2017), 13–26.

[15] Sular, S., Kenmotsu Manifolds and Some of Their Submanifolds, Ph.D. Thesis, Balikesir University, Institute of Science and Technology, 2009.

[16] Sular, S., Özgür, C., Murathan, C., *Pseudoparallel anti-invariant submanifolds of Kenmotsu manifolds*, Hacettepe J. of Math. and Stat., **39**(2010), 535–543.

[17] Venkatesha, Naik, D.M., *Certain results on K-paracontact and para-Sasakian Manifolds*, J. Geom., **108**(2017), 939–952.