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$\left(\frac{G'}{G}\right)$-expansion method to seek traveling wave solutions for some Fractional Nonlinear PDEs arising in natural sciences

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Abstract

The$(\frac{G'}{G})$-expansion method with the aid of symbolic computational system can be used to obtain the traveling wave solutions (hyperbolic, trigonometric and rational solutions) for nonlinear time-fractional evolution equations arising in mathematical physics and biology. In this work, we will process the analytical solutions of the time-fractional classical Boussinesq equation, the time-fractional Murray equation, and the space-time fractional Phi-four equation. With the fact that the method which we will propose in this paper is also a standard, direct and computerized method, the exact solutions for these equations are obtained.

Keywords: Exact solutions Traveling wave solutions Boussines equation Murray equation Phi-four equation.
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1. Introduction

The analytic treatment of nonlinear evolution equations have attracted attention of many mathematicians and physicists. Many authors are interested in the research of the exact solutions[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], because the exact solutions of nonlinear evolution equations are the key tool to understand the various physical phenomena that govern the real world today in many fields such as plasma physics, fluid

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physics, quantum field theory, biophysics, chemical kinematics, geochemistry, electricity and mathematical biology.

In recent years, with the assist of many computation packages like ***Mathematica 11***, many powerful analytical methods to construct exact solutions for nonlinear partial differential equations have been proposed, the tanh function method [5, 7, 14], the extended tanh method [14, 15], the sine-cosine method [14], the rational method, extended rational expansion method [12], the Lie group method [16, 17], Weierstrass elliptic function method [18], Exp-function method [19], the residual powers series (RPS) method [20], the modified simple equation (MSE) method [1], the generalized Kudryashov method [10], Cole-Hopf transformation constrictive method [21], He's semi-inverse method [22], the auxiliary differential equation method [8], the Backlund transformation method [23], the Hirota's bi linear transformation method [9] and others. More recently, the $(\frac{G'}{G})$-expansion method [11, 15, 24, 25, 26, 27, 28, 29, 30, 31, 32] has been proposed to obtain traveling wave solutions. This method is firstly proposed by Wang et al. [30] for which the traveling wave solutions of the nonlinear evolution equations are obtained [33].
By using this well known method, our main objective of this paper is to investigate the exact solutions for the following three generalized fractional nonlinear evolution equations, first the generalized time-fractional classical Boussinesq equation in the form:

$$D_t^{2\alpha}u + \epsilon u_{4x} + \gamma u_{2x} + \beta u u_{2x} + \beta\left(u_x\right)^2 = 0,$$

where $\epsilon, \gamma, \beta \in \mathbb{R}^*$, $\frac{1}{2} < \alpha \leq 1$ and $D_t^{2\alpha}u := \frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}$ in the sense of Caputo derivative.

In the particular case $\alpha = 1$ see [8, 9, 12, 16, 17, 22, 23, 34, 35, 36]: this very famous nonlinear evolution equation was developed to describe the motion of water with small amplitude and long wave [22].
The second considerable equation is the generalized time-fractional Murray equation in the form:

$$D_t^{\alpha}u - u_{2x} - a u u_x - bu + cu^2 = 0,$$

where $a, b, c \in \mathbb{R}^*$, $D_t^{\alpha}u := \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$ in the sense of Caputo derivative and $0 < \alpha \leq 1$.
For the case $\alpha = 1$ see [3, 4, 21, 37, 38, 39, 40], it is an important equation which was first suggested to describe the propagation of genes, and it was established later that this equation also generalizes a great number of well known nonlinear second-order evolution equations describing various processes in biology [38, 39].
Finally, we consider the Generalized space-time fractional Phi-four equation in the form:

$$D_t^{2\alpha}u - pD_x^{2\beta}u - qu + ru^3 = 0,$$

where $p, q, r \in \mathbb{R}_+^*$, $D_t^{2\alpha}u := \frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}, D_t^{2\beta}u := \frac{\partial^{2\beta}u(x,t)}{\partial t^{2\beta}}$ in the sense of Caputo derivative and $\frac{1}{2} < \alpha \leq 1$, $\frac{1}{2} < \beta \leq 1$.
Many authors treat the case $\alpha = \beta = 1$ in several contexts [1, 10, 14, 18, 19, 41, 42, 43, 44], the PHI-four equation is considered as a particular form of the Klein-Gordon equation that model phenomenon in particle physics where kink and anti-kink solitary waves interact [14].

1.1. Caputo derivative

The Caputo derivative of order α is defined by the formula [45, 46, 47, 48, 49]:

$$D_*^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, & \text{if } \ m-1 < \alpha < m \\ \frac{d^m}{dt^m}f(t), & \text{if } \ \alpha = m, \end{cases} \tag{1}$$

where $m \in \mathbb{N}^*$ and $\Gamma(.)$ denotes the Gamma function
defined by $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad x > 0.$

The important properties of the Caputo derivative that will be used in this paper are :

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \tag{2}$$

$$D_t^\alpha \Big[f(g(t)) \Big] = f'_g(g(t)) D_t^\alpha g(t) = D_g^\alpha f(g(t)) \Big[g'_t(t) \Big]^\alpha, \tag{3}$$

$$d^\alpha h(t) = \Gamma(1+\alpha) d\, h(t). \tag{4}$$

2. Description of the $\left(\frac{G'}{G}\right)$-expansion method

The general nonlinear Time-Fractional evolution equation, say in two independent variables x and t, is given by

$$P\Big(u, D_t^\alpha u, u_x, D_t^{2\alpha} u, u_{xx}, D_t^\alpha u_x, \dots\Big) = 0, \quad 0 < \alpha \le 1, \tag{5}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial of u and its partial fractional derivatives, in which the nonlinear terms and the highest order derivatives are included. To find the traveling wave solution of Eq. (5) by $\left(\frac{G'}{G}\right)$-expansion method, we follow the following steps

- **Step 1:** To obtain exact traveling wave solution, the following fractional complex transformation [50, 51, 52, 53] has been applied

$$u(x,t) = U(\xi), \quad \xi = kx - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}, \text{ or } \xi = \frac{k x^\beta}{\Gamma(1+\beta)} - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}, \tag{6}$$

  where $k, \omega \in \mathbb{R}^*$ are constants to be determined latter. Then, the Eq (5) is reduced to the following nonlinear ordinary differential equation

$$P\Big(U, -\omega U', kU, \omega^2 U'', k^2 U'', -\omega k U', \dots\Big) = 0, \tag{7}$$

  where $U^{(i)} = U_{i\xi}$.

- **Step 2:** Assuming that the solution of Eq. (7) can be expressed as a finite power series of the form:

$$U(\xi) = \sum_{n=0}^{N} a_n \left(\frac{G'(\xi)}{G(\xi)} \right)^n, \tag{8}$$

  where $a_0, a_1, \dots a_N$ $(a_N \neq 0)$ are constants to be determined later.

  1. Let $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{9}$$

     where λ, μ are constants to be discuss later.
     The general solutions of (9) can be written in the forms as follow

$$G(\xi) = \begin{cases} e^{\frac{1}{2}(-\lambda)\xi} \left(A_2 \sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu}\right) + A_1 \cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right) \right), & \lambda^2 - 4\mu > 0, \\ e^{\frac{1}{2}(-\lambda)\xi} \left(A_2 \sin\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right) + A_1 \cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right) \right), & \lambda^2 - 4\mu < 0, \\ (A_2\xi + A_1)\, e^{\frac{1}{2}(-\lambda)\xi}, \quad \lambda^2 - 4\mu = 0, \end{cases} \tag{10}$$

it yields

$$\frac{G'}{G}=\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right]-\frac{\lambda}{2}, & \lambda^2-4\mu>0,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)-A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right]-\frac{\lambda}{2}, & \lambda^2-4\mu<0,\\ \frac{A_2}{A_2\xi+A_1}-\frac{\lambda}{2}, \quad \lambda^2-4\mu=0, \end{cases} \tag{11}$$

where A_1 and A_2 are arbitrary constants.

2. Let $G=G(\xi)$ satisfies the Riccati equation in the form

$$G'=\mu G^2+\nu G+\lambda, \tag{12}$$

where μ,ν,λ are constants and $\mu\neq 0, \nu\neq 0$.
The general solutions of (12) (when $\lambda=0$) can be written in the form as follow

$$G(\xi)=-\frac{A\nu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}, \tag{13}$$

it yields

$$\frac{G'}{G}=-\frac{\nu}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}, \tag{14}$$

where A is arbitrary constant $(A\neq 0)$.

- **Step 3:** The degree N of the power series (8) is determined by considering the homogeneous balance between the nonlinear term in Eq. (7) and the highest-order derivative. Suppose the degree of $U(\xi)$ is N, then the degree of the other expression can be evaluated as follows:

$$D(U^{(p)})=N+p,\ D\Big(U^p(U^{(q)})^s\Big)=Np+s(N+q). \tag{15}$$

- **Step 4:** Substituting Eq.(8) using Eq. (9) into Eq. (7). Then collecting the coefficients of like powers of $\left(\frac{G'}{G}\right)^n, (n=0,1,2,\ldots,N)$. A set of nonlinear algebraic equations is obtained, by equating each coefficient to zero. The resulting algebraic system is solved with the help of ***Mathematica 11*** to get the values of unknown constants $a_0,a_1,\ldots a_N$ and k,ω.

- **Step 5:** Since the general solution of (9) has been well known for us, then substituting a_n,k,ω and (14) into (8), we have three types of the Exact traveling wave solutions of the time-fractional nonlinear evolution equation (5).

3. The time-fractional classical Boussinesq equation

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+\gamma\frac{\partial^2 u(x,t)}{\partial x^2}+\beta u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+\beta\left(\frac{\partial u(x,t)}{\partial x}\right)^2+\epsilon\frac{\partial^4 u(x,t)}{\partial x^4}=0, \tag{16}$$

where $\epsilon,\gamma,\beta\in\mathbb{R}^*$, and $\frac{1}{2}<\alpha\leq 1$.
Using the fractional complex transformation $u(x,t)=U(\xi),\ \ \xi=kx-\frac{\omega t^\alpha}{\Gamma(\alpha+1)}$, the (TFCBE) (16) is converted to the (NLODE)

$$\left(\omega^2+\gamma k^2\right)U''+\beta k^2\mathrm{UU}''+\beta k^2\left(U'\right)^2+\epsilon k^4U^{(4)}=0. \tag{17}$$

Balancing $U^{(4)}$ with UU'' in (17) gives $N+4=2N+2$, hence $N=2$. We then suppose that (17) has the following formal solutions:

$$U(\xi)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right)+a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2. \tag{18}$$

Substituting Equation (18) with using (9) into Equation (17) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand sides of Equation (17) are converted into a polynomial in $\left(\frac{G'}{G}\right)$. Setting each coefficient of each term to zero, we derive a set of algebraic equations for a_0, a_1, a_2, k, ω.

$$\begin{aligned}
&a_1\lambda^3k^4\mu\epsilon+14a_2\lambda^2k^4\mu^2\epsilon+8a_1\lambda k^4\mu^2\epsilon+16a_2k^4\mu^3\epsilon+a_0a_1\beta\lambda k^2\mu+a_1^2\beta k^2\mu^2\\
&+2a_0a_2\beta k^2\mu^2+a_1\gamma\lambda k^2\mu+2a_2\gamma k^2\mu^2+a_1\lambda\mu\omega^2+2a_2\mu^2\omega^2=0,\\
&a_1\lambda^4k^4\epsilon+30a_2\lambda^3k^4\mu\epsilon+22a_1\lambda^2k^4\mu\epsilon+120a_2\lambda k^4\mu^2\epsilon+16a_1k^4\mu^2\epsilon+a_0a_1\beta\lambda^2k^2\\
&+3a_1^2\beta\lambda k^2\mu+6a_0a_2\beta\lambda k^2\mu+6a_1a_2\beta k^2\mu^2+2a_0a_1\beta k^2\mu+a_1\gamma+k^2\lambda^2+6a_2\gamma\lambda k^2\mu\\
&+2a_1\gamma k^2\mu+a_1\lambda^2\omega^2+6a_2\lambda\mu\omega^2+2a_1\mu\omega^2=0,\\
&16a_2\lambda^4k^4\epsilon+15a_1\lambda^3k^4\epsilon+232a_2\lambda^2k^4\mu\epsilon+60a_1\lambda k^4\mu\epsilon+136a_2k^4\mu^2\epsilon+2a_1^2\beta\lambda^2k^2\\
&+4a_0a_2\beta\lambda^2k^2+15a_1a_2\beta\lambda k^2\mu+3a_0a_1\beta\lambda k^2+6a_2^2\beta k^2\mu^2+4a_1^2\beta k^2\mu+8a_0a_2\beta k^2\mu\\
&+4a_2\gamma\lambda^2k^2+3a_1\gamma\lambda k^2+8a_2\gamma k^2\mu+4a_2\lambda^2\omega^2+3a_1\lambda\omega^2+8a_2\mu\omega^2=0,\\
&130a_2\lambda^3k^4\epsilon+50a_1\lambda^2k^4\epsilon+440a_2\lambda k^4\mu\epsilon+40a_1k^4\mu\epsilon+9a_1a_2\beta\lambda^2k^2+14a_2^2\beta\lambda k^2\mu+5a_1^2\beta\lambda k^2\\
&+10a_0a_2\beta\lambda k^2+18a_1a_2\beta k^2\mu+2a_0a_1\beta k^2+10a_2\gamma\lambda k^2++2a_1\gamma k^2+10a_2\lambda\omega^2+2a_1\omega^2=0,\\
&330a_2\lambda^2k^4\epsilon+60a_1\lambda k^4\epsilon+240a_2k^4\mu\epsilon+8a_2^2\beta\lambda^2k^2+21a_1a_2\beta\lambda k^2\\
&+16a_2^2\beta k^2\mu+3a_1^2\beta k^2+6a_0a_2\beta k^2+6a_2\gamma k^2+6a_2\omega^2=0,\\
&336a_2\lambda k^4\epsilon++24a_1k^4\epsilon+18a_2^2\beta\lambda k^2+12a_1a_2\beta k^2=0,\\
&120a_2k^4\epsilon+10a_2^2\beta k^2=0.
\end{aligned} \tag{19}$$

The resulting algebraic system (19) is solved with the help of ***Mathematica 11*** to get the values of unknown constants a_0 ; a_1 ; a_2 ; k ; ω

$$\left\{a_0\to\frac{-\lambda^2k^4\epsilon-8k^4\mu\epsilon-\gamma k^2-\omega^2}{\beta k^2}, a_1\to-\frac{12k^2\lambda\epsilon}{\beta}, a_2\to-\frac{12k^2\epsilon}{\beta}\right\}, \tag{20}$$

where λ and μ are arbitrary constants.
By using Eqs. (20) , expression (18) can be written as:

$$U(\xi)=\frac{-\lambda^2k^4\epsilon-8k^4\mu\epsilon-\gamma k^2-\omega^2}{\beta k^2}-\frac{12k^2\lambda\epsilon}{\beta}\left(\frac{G'(\xi)}{G(\xi)}\right)-\frac{12k^2\epsilon}{\beta}\left(\frac{G'(\xi)}{G(\xi)}\right)^2. \tag{21}$$

Now, using Eqs. (11) into (21), we have three kinds of traveling wave solutions of Eq. (16) as follows:

- **case 1:** $\lambda^2-4\mu>0$, we get the hyperbolic function solutions of Eq. (16)

$$\begin{aligned}
u_1(x,t)&=U_1(\xi)\\
&=-\frac{3k^2\epsilon}{\beta}\left(\lambda^2-4\mu\right)\left[\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right]^2\\
&+\frac{2k^4\epsilon\left(\lambda^2-4\mu\right)-\gamma k^2-\omega^2}{\beta k^2},
\end{aligned} \tag{22}$$

where $\xi=kx-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}$.
In particular, if $\lambda=3, \mu=2, \epsilon=1, \gamma=-1, \beta=3, A_1=1, A_2=0, k=1$, $\omega=1, \alpha=0.6$, then (22) becomes

$$u_1(x,t)=\frac{2}{3}-\tanh^2\left(\frac{1}{2}x-\frac{t^{0.6}}{2\Gamma(1.6)}\right). \tag{23}$$

- **case 2:** $\lambda^2 - 4\mu < 0$, we get the trigonometric function solutions of Eq. (16)

$$\begin{aligned} u_2(x,t) &= U_2(\xi) \\ &= -\frac{3k^2\epsilon}{\beta}\left(4\mu-\lambda^2\right)\left[\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right]^2 \\ &- \frac{2k^4\epsilon\left(4\mu-\lambda^2\right)+\gamma k^2+\omega^2}{\beta k^2}, \end{aligned} \tag{24}$$

where $\xi = kx - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}$.
Also, if $\lambda = \sqrt{3}, \mu = 1, k = 1, \epsilon = 1, \gamma = -1, \beta = 3, A_1 = 0, A_2 = 1, k = 1,$
$\omega = 1, \alpha = 0.75$, then (24) becomes

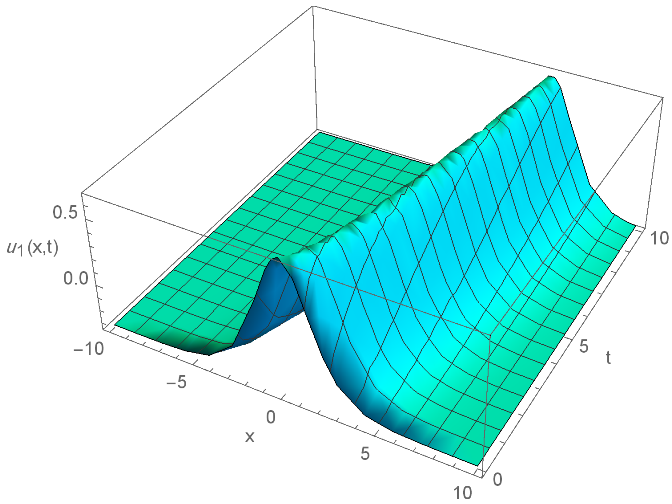
$$u_2(x,t) = -\cot^2\left(x - \frac{t^{0.75}}{\Gamma(1.75)}\right). \tag{25}$$

- **case 3:** $\lambda^2 - 4\mu = 0$, we get the rational function solutions of Eq. (16)

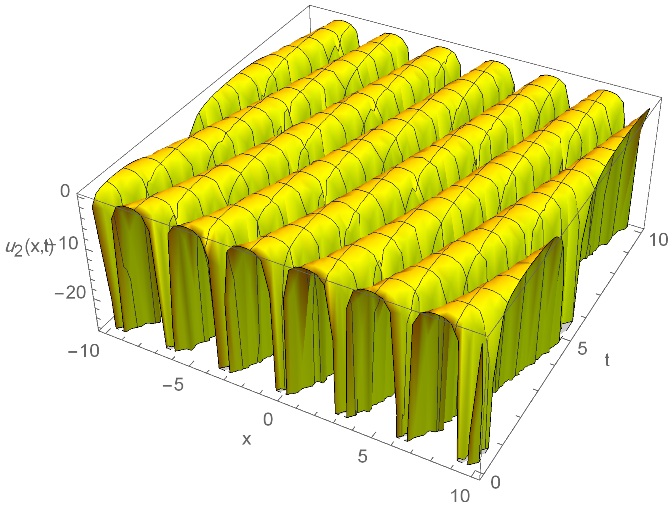
$$u_3(x,t) = U_3(\xi) = \frac{-\gamma k^2 - \omega^2}{\beta k^2} - \frac{12k^2\epsilon}{\beta}\left(\frac{A_2}{A_2\xi + A_1}\right)^2, \tag{26}$$

where $\xi = kx - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}$.
If $\lambda = 1, \mu = \frac{1}{4}, \epsilon = \frac{1}{4}, \gamma = -1, \beta = 3, A_1 = A_2 = 1, k = 2, \omega = 1, \alpha = 1$, then

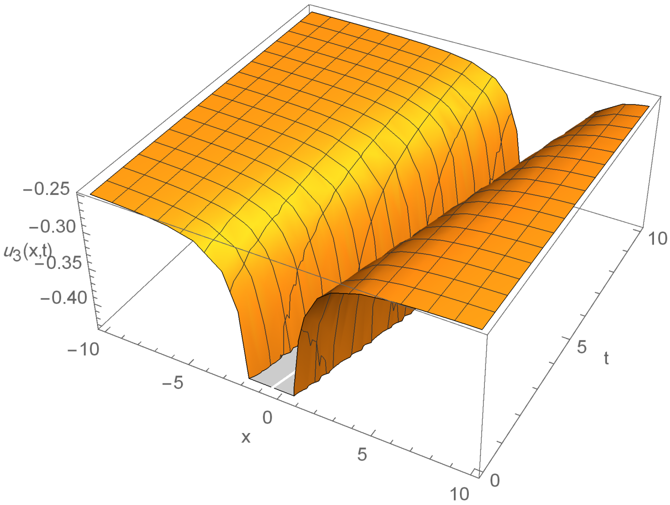
$$u_3(x,t) = \frac{1}{4} - \left(\frac{1}{1+2x-t}\right)^2. \tag{27}$$



(a) $u_1(x,t)\,(\lambda = 3, \mu = 2, \epsilon = 1,$
$A_1 = 1, A_2 = 0, k = 1, \omega = 1, \alpha = 0.6)$



(b) $u_2(x,t)\,(\lambda = \sqrt{3}, \mu = 1, \epsilon = 1,$
$A_1 = 0, A_2 = 1, k = 1, \omega = 1, \alpha = 0.75)$



(c) $u_3(x,t)\,(\lambda = 1, \mu = \frac{1}{4}, \epsilon = \frac{1}{4},$
$A_1 = A_2 = 1, k = 2, \omega = 1, \alpha = 1)$

Figure 1: 3DPlot of the exact solutions of Eq. (16) given by (23) , (25) and (27) respectively, for $(x,t) \in [-10, 10] \times [0, 10]$.

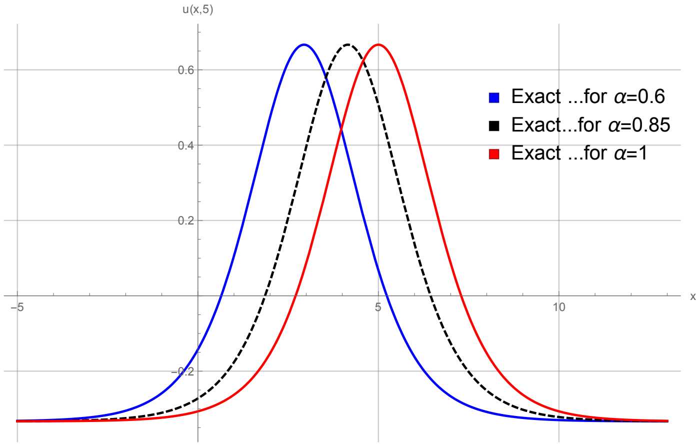


Figure 2: 2DPlot of the exact solutions of Eq. (16) given by (23), at $t = 5$ and $x \in [-5, 13]$, for $\alpha = 0.6, 0.85, 1$.

4. Time-fractional Murray equation

Let's consider the (TFME) equation of the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} - au(x,t)\frac{\partial u(x,t)}{\partial x} - bu(x,t) + cu(x,t)^2 = 0, \tag{28}$$

where $a, b, c \in \mathbb{R}^*$ and $0 < \alpha \leq 1$.
The fractional complex transformation $u(x,t) = U(\xi)$, $\xi = kx - \omega\frac{t^\alpha}{\Gamma(1+\alpha)}$, transform the Eq. (28) to the following ordinary differential equation:

$$-\omega U' - k^2 U'' - akUU' - bU + cU^2 = 0. \tag{29}$$

By the same procedure as illustrated above, we can determine the value of N by balancing U'' and UU' in Eq. (29). We find $N = 1$. We can suppose that the solutions of Eq. (29) is of the form:

$$U(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{30}$$

1. Let $G = G(\xi)$ satisfies (9).
   Proceeding as above, we obtain a set of algebraic equations for $a_0, a_1, k, \omega, \lambda, \mu$.

$$\begin{aligned}
&-a_0 b + a_0^2 c - a_1\lambda k^2\mu + aa_0a_1k\mu + a_1\mu\omega = 0,\\
&-a_1 b + 2a_0a_1c - a_1\lambda^2k^2 - 2a_1k^2\mu + aa_0a_1\lambda k + aa_1^2k\mu + a_1\lambda\omega = 0,\\
&a_1^2c - 3a_1\lambda k^2 + aa_1^2\lambda k + aa_0a_1k + a_1\omega = 0,\\
&aa_1^2k - 2a_1k^2 = 0.
\end{aligned} \tag{31}$$

   Solving the system with the help of ***Mathematica 11***, we get

$$\left\{a_0 \to \frac{ab + 2ck\lambda}{2ac}, a_1 \to \frac{2k}{a}, \omega \to \frac{-a^2bk - 4c^2k}{2ac}, \mu \to \frac{4c^2k^2\lambda^2 - a^2b^2}{16c^2k^2}\right\}. \tag{32}$$

   Where k and λ are arbitrary constants. using Eqs. (32) , the expression (30) can be written as:

$$U(\xi) = \frac{ab + 2ck\lambda}{2ac} + \frac{2k}{a}\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{33}$$

   With the fact that $\lambda^2 - 4\mu = \lambda^2 - \frac{4(4c^2k^2\lambda^2 - a^2b^2)}{16c^2k^2} = \frac{a^2b^2}{4c^2k^2} > 0$, and using (11) then, the solution of (28) is in the form:

$$U(\xi) = \frac{b}{2c} + \frac{b}{2c}\left[\frac{A_1\sinh\left(\frac{ab}{4ck}\xi\right) + A_2\cosh\left(\frac{ab}{4ck}\xi\right)}{A_2\sinh\left(\frac{ab}{4ck}\xi\right) + A_1\cosh\left(\frac{ab}{4ck}\xi\right)}\right], \tag{34}$$

where $\xi = kx + \frac{t^{\alpha}(a^2bk+4c^2k)}{(2ac)\Gamma(\alpha+1)}$.
After simplifying the expression (34), we get

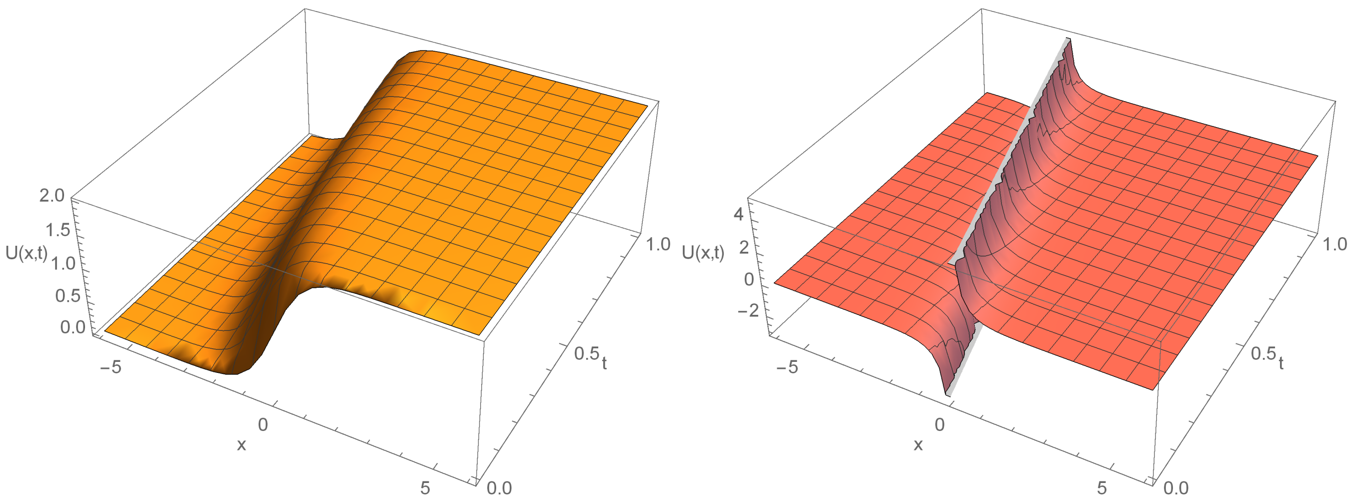
$$u(x,t) = \frac{b}{2c} + \frac{b}{2c}\left[\frac{A_1 \sinh\left(\frac{ab}{4c}x + \frac{(a^2b^2+4bc^2)}{8c^2\Gamma(\alpha+1)}t^{\alpha}\right) + A_2 \cosh\left(\frac{ab}{4c}x + \frac{(a^2b^2+4bc^2)}{8c^2\Gamma(\alpha+1)}t^{\alpha}\right)}{A_1 \cosh\left(\frac{ab}{4c}x + \frac{(a^2b^2+4bc^2)}{8c^2\Gamma(\alpha+1)}t^{\alpha}\right) + A_2 \sinh\left(\frac{ab}{4c}x + \frac{(a^2b^2+4bc^2)}{8c^2\Gamma(\alpha+1)}t^{\alpha}\right)}\right]. \tag{35}$$

In particular case: if we take $a = b = 2, c = 1, A_1 \neq 0, A_2 = 0, \alpha = 0.5$, the solution (35) becomes

$$u(x,t) = 1 + \tanh\left(x + \frac{3t^{0.5}}{\Gamma(1.5)}\right). \tag{36}$$

And if: $a = b = 2, c = 1, A_1 = 0, A_2 \neq 0, \alpha = 1$, the solution (35) becomes

$$u(x,t) = 1 + \coth(x + 3t). \tag{37}$$



(a) $u(x,t)$ $(a = b = 2, c = 1, A_2 = 0, A_1 \neq 0, \alpha = 0.5)$ (b) $u(x,t)$ $(a = b = 2, c = 1, A_1 = 0, A_2 \neq 0, \alpha = 1)$

Figure 3: 3DPlot of the exact solutions of Eq. (28) given by (36) and (37) respectively, for $(x,t) \in [-6,6] \times [0,1]$.

2. Let $G = G(\xi)$ satisfies (12).

   Substituting Eq. (30) with using (12) into Eq. (29), collecting the coefficients of $\Big(G(\xi)\Big)^i, (i = 0, \pm1, \pm2, \pm3)$ and set it to zero, yields a set of algebraic equations for $a_0, a_1, k, \lambda, \omega$. These systems are:

$$\begin{aligned}
&-a_1b\nu - a_0b + 2a_1^2c\lambda\mu + a_1^2c\nu^2 + 2a_0a_1c\nu + a_0^2c - 2a_1\lambda k^2\mu\nu = 0,\\
&-a_1b\mu + 2a_1^2c\mu\nu + 2a_0a_1c\mu - 2a_1k^2\lambda\mu^2 - a_1k^2\mu\nu^2 - aa_1^2k\lambda\mu^2\\
&-aa_1^2k\mu\nu^2 - aa_0a_1k\mu\nu - a_1\mu\nu\omega = 0,\\
&a_1^2c\mu^2 - 3a_1k^2\mu^2\nu - 2aa_1^2k\mu^2\nu - aa_0a_1k\mu^2 - a_1\mu^2\omega = 0,\\
&-2a_1k^2\mu^3 - aa_1^2k\mu^3 = 0,\\
&-a_1b\lambda + 2a_1^2c\lambda\nu + 2a_0a_1c\lambda - 2a_1k^2\lambda^2\mu - a_1k^2\lambda\nu^2 + aa_1^2k\lambda^2\mu + aa_1^2k\lambda\nu^2\\
&+aa_0a_1k\lambda\nu + a_1\lambda\nu\omega = 0,\\
&a_1^2c\lambda^2 - 3a_1k^2\lambda^2\nu + a_1\lambda^2\omega + 2aa_1^2k\lambda^2\nu + aa_0a_1k\lambda^2 = 0,\\
&aa_1^2k\lambda^3 - 2a_1k^2\lambda^3 = 0.
\end{aligned} \tag{38}$$

   We obtain the roots of Eqs. (38) with the aid of ***Mathematica 11*** as

$$\left\{a_0 \to 0, a_1 \to \frac{b}{c\nu}, k \to -\frac{ab}{2c\nu}, \omega \to \frac{b\left(a^2b+4c^2\right)}{4c^2\nu}, \lambda \to 0\right\},$$
$$\left\{a_0 \to \frac{b}{c}, a_1 \to -\frac{b}{c\nu}, k \to \frac{ab}{2c\nu}, \omega \to \frac{-a^2b^2-4bc^2}{4c^2\nu}, \lambda \to 0\right\}. \tag{39}$$

Now, substituting (39) and using Eq. (14) into (30), we obtain two exact traveling wave solutions of Eq. (28) as follows:

$$U_1(\xi) = -\frac{b}{A\mu c\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-c}, \tag{40}$$

where $\xi = -\frac{ab}{2c\nu}x - \frac{b\left(a^2b+4c^2\right)}{\Gamma(\alpha+1)(4c^2\nu)}t^{\alpha}$.
If $(a=2, b=c=1, \nu=1, \mu=1, A=1, \alpha=0.6)$, then (40) becomes

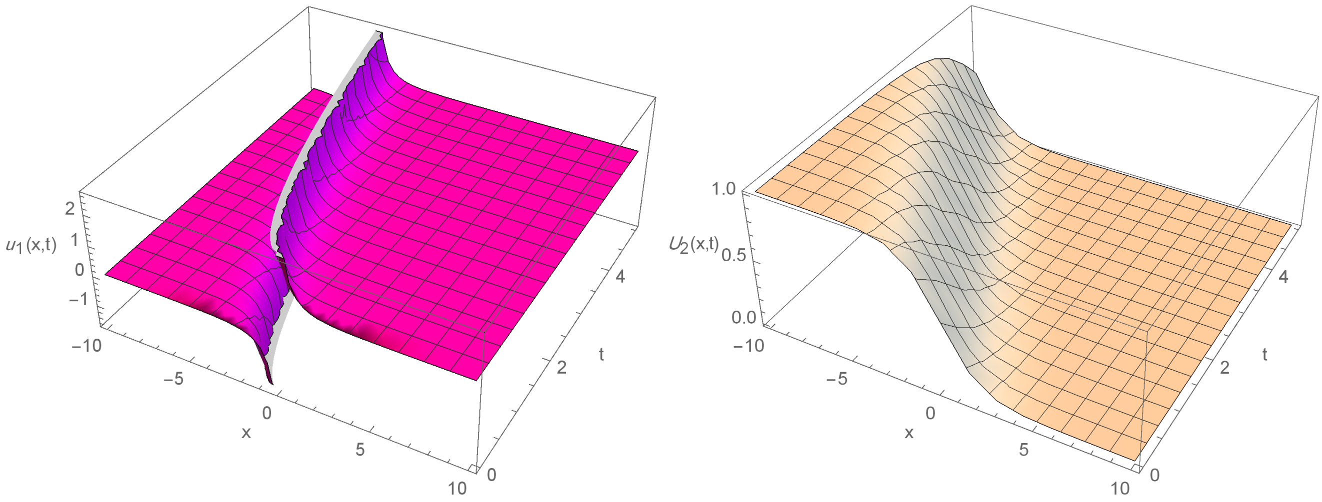
$$u_1(x,t) = -\frac{1}{\cosh\left(-x-\frac{2t^{0.6}}{\Gamma(1.6)}\right)+\sinh\left(-x-\frac{2t^{0.6}}{\Gamma(1.6)}\right)-1}, \tag{41}$$

or

$$U_2(\xi) = \frac{A\mu b\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)}{A\mu c\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-c}, \tag{42}$$

where $\xi = \frac{ab}{2c\nu}x + \frac{b\left(a^2b+4c^2\right)}{\Gamma(\alpha+1)(4c^2\nu)}t^{\alpha}$.
If $(a=2, b=c=1, \nu=1, \mu=1, A=1, \alpha=1)$, then (42) becomes

$$u_2(x,t) = \frac{\sinh(x+2t)+\cosh(x+2t)}{\sinh(x+2t)+\cosh(x+2t)-1}. \tag{43}$$



(a) $u_1(x,t)$ $(a=2, b=c=1\nu=\mu=1, A=1, \alpha=0.6)$ (b) $u_2(x,t)$ $(a=2, b=c=1, \nu=\mu=1, A=1, \alpha=1)$

Figure 4: 3DPlot of the exact solutions of Eq. (28) given by (41) and (43) respectively, for $(x,t) \in [-10,10]\times[0,5]$.

5. The space-time fractional Phi-four equation

we study the following (STFPFE) in the form

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - p\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - qu(x,t) + ru(x,t)^3 = 0, \tag{44}$$

where $p, q, r \in \mathbb{R}_+^*$ and $\frac{1}{2} < \alpha \leq 1, \frac{1}{2} < \beta \leq 1$.
We see that the following fractional complex transformation $u(x,t) = U(\xi)$,
$\xi = \frac{kx^\beta}{\Gamma(1+\beta)} - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}$, permit us converting the (STFGPFE) (44) to the (NLODE)

$$\left(\omega^2 - k^2 p\right) U'' - qU + rU^3 = 0. \tag{45}$$

Now, balancing the terms U'' and U^3 gives $N + 2 = 3N$, so that $N = 1$ and thus we write

$$U(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right). \tag{46}$$

1. Let $G = G(\xi)$ satisfies (9).
By following the same steps as above, Substituting Equation (46) with using (9) into Equation (45) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, Equating the coefficients of these terms to be zero, we obtain a set of algebraic equations for $a_0, a_1, , k, \omega, \lambda, \mu$ as follow:

$$\begin{aligned}
&- a_1 k^2 \lambda \mu p + a_1 \lambda \mu \omega^2 + a_0 q + a_0^3 r = 0, \\
&a_1 \lambda^2 k^2 (-p) - 2 a_1 k^2 \mu p + a_1 \lambda^2 \omega^2 + 2 a_1 \mu \omega^2 - a_1 q + 3 a_0^2 a_1 r = 0, \\
&- 3 a_1 \lambda k^2 p + 3 a_1 \lambda \omega^2 + 3 a_0 a_1^2 r = 0, \\
&- 2 a_1 k^2 p + a_1^3 r + 2 a_1 \omega^2 = 0.
\end{aligned} \tag{47}$$

On solving the above set of algebraic equations, we have

$$\begin{aligned}
&\left\{a_0 \to 0, a_1 \to -\frac{\sqrt{2}\sqrt{k^2 p - \omega^2}}{\sqrt{r}}, \lambda \to 0, \mu \to \frac{q}{2\left(\omega^2 - k^2 p\right)}\right\}, \\
&\left\{a_0 \to 0, a_1 \to \frac{\sqrt{2}\sqrt{k^2 p - \omega^2}}{\sqrt{r}}, \lambda \to 0, \mu \to \frac{q}{2\left(\omega^2 - k^2 p\right)}\right\}.
\end{aligned} \tag{48}$$

By substituting (48) in (46) and using (11), because $\lambda^2 - 4\mu = \frac{2q}{k^2 p - \omega^2}$, we distinguish two cases:

- if $k^2 p - \omega^2 > 0$, we derive two solutions

$$U_{11}(\xi) = \sqrt{\frac{q}{r}} \left[\frac{A_1 \sinh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right) + A_2 \cosh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right)}{A_2 \sinh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right) + A_1 \cosh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right)}\right], \tag{49}$$

or

$$U_{12}(\xi) = -\sqrt{\frac{q}{r}} \left[\frac{A_1 \sinh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right) + A_2 \cosh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right)}{A_2 \sinh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right) + A_1 \cosh\left(\xi\sqrt{\frac{q}{2(k^2p-\omega^2)}}\right)}\right], \tag{50}$$

where $\xi = \frac{kx^\beta}{\Gamma(\beta+1)} - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$.
In particular, if $p = 1, q = 2 = r, k = 5, \omega = 4, A_1 \neq 0, A_2 = 0$,
$\alpha = 0.6, \beta = 0.8$, (50) becomes

$$u_{12}(x,t) = -\tanh\left(\frac{5x^{0.8}}{3\Gamma(1.8)} - \frac{4t^{0.6}}{3\Gamma(1.6)}\right). \tag{51}$$

- If $k^2 p - \omega^2 < 0$, we obtain two solutions

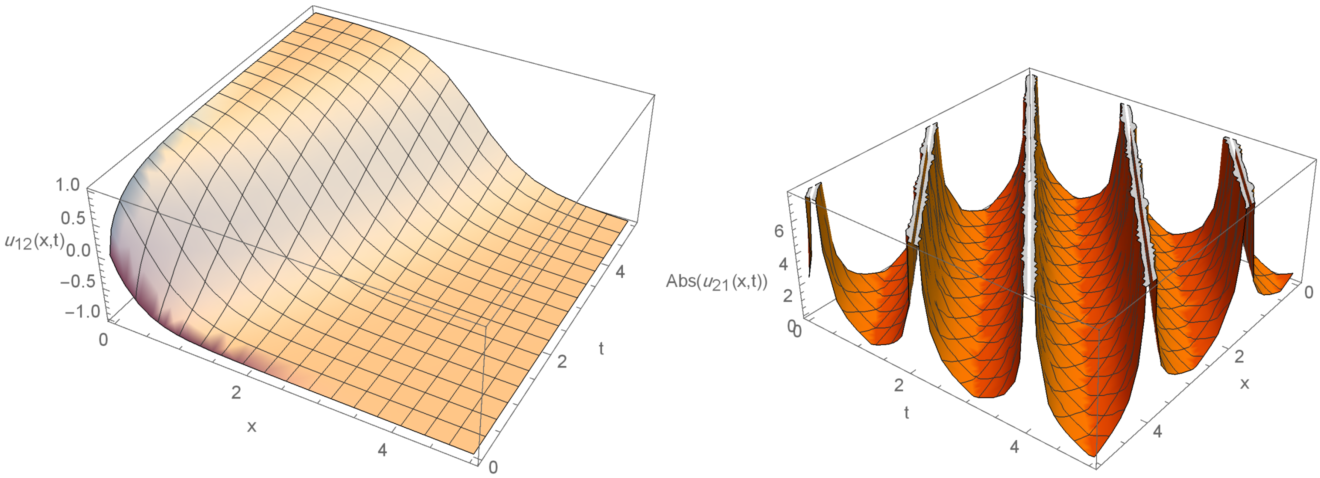
$$U_{21}(\xi) = i\sqrt{\frac{q}{r}} \left[\frac{A_2 \cos\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right) - A_1 \sin\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right)}{A_2 \sin\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right) + A_1 \cos\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right)}\right], \tag{52}$$

or

$$U_{22}(\xi) = -i\sqrt{\frac{q}{r}} \left[\frac{A_2 \cos\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right) - A_1 \sin\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right)}{A_2 \sin\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right) + A_1 \cos\left(\xi\sqrt{\frac{q}{2(\omega^2-k^2p)}}\right)}\right], \tag{53}$$

where $\xi = \frac{kx^{\beta}}{\Gamma(\beta+1)} - \frac{\omega t^{\alpha}}{\Gamma(\alpha+1)}$.
Also, if $p = 1, q = 2 = r, k = 4, \omega = 5, A_1 = 0, A_2 \neq 0, \alpha = \beta = 1$, the equation (53) becomes

$$u_{21}(x,t) = i \cot\left(\frac{4x}{3} - \frac{5t}{3}\right). \tag{54}$$



(a) $u_{12}(x,t)$ $(p = 1, q = r = 2, k = 5, \omega = 4,$ $A_1 \neq 0, A_2 = 0, \alpha = 0.6, \beta = 0.8)$

(b) $\|u_{21}(x,t)\|$ $(p = 1, q = r = 2, k = 4, \omega = 5,$ $A_1 = 0, A_2 \neq 0, \alpha = \beta = 1)$

Figure 5: 3DPlot of the exact solutions of Eq. (44) given by (51) and (54) respectively, for $(x,t) \in [0,5]\times[0,5]$.

2. Let $G = G(\xi)$ satisfies (12).

Proceeding as above, substituting (46) with using (12) into (45) and equating the coefficients of $\left(G\right)^{i}$, $(i = 0, i = \pm 1, i = \pm 2, i = \pm 3)$ to zero, yields a set of simultaneous algebraic equations among $a_0, a_1, k, \lambda, \omega$.

$$\begin{aligned}
&-2a_1k^2\lambda\mu\nu p + 2a_1\lambda\mu\nu\omega^2 - a_1\nu q - a_0 q + 6a_1^3\lambda\mu\nu r + 6a_1^2a_0\lambda\mu r + a_1^3\nu^3 r \\
&+ 3a_1^2a_0\nu^2 r + 3a_1a_0^2\nu r + a_0^3 r = 0, \\
&-2a_1k^2\lambda\mu^2 p - a_1k^2\mu\nu^2 p + 2a_1\lambda\mu^2\omega^2 + a_1\mu\nu^2\omega^2 - a_1\mu q + 3a_1^3\lambda\mu^2 r \\
&+ 3a_1^3\mu\nu^2 r + 6a_0a_1^2\mu\nu r + 3a_0^2a_1\mu r = 0, \\
&-3a_1k^2\mu^2\nu p + 3a_1\mu^2\nu\omega^2 + 3a_1^3\mu^2\nu r + 3a_0a_1^2\mu^2 r = 0, \\
&-2a_1k^2\mu^3 p + 2a_1\mu^3\omega^2 + a_1^3\mu^3 r = 0, \\
&2a_1\lambda^2\mu\omega^2 - 2a_1k^2\lambda^2\mu p - a_1k^2\lambda\nu^2 p + a_1\lambda\nu^2\omega^2 - a_1\lambda q + 3a_1^3\lambda^2\mu r \\
&+ 3a_1^3\lambda\nu^2 r + 6a_0a_1^2\lambda\nu r + 3a_0^2a_1\lambda r = 0, \\
&3a_1\lambda^2\nu\omega^2 - 3a_1k^2\lambda^2\nu p + 3a_1^3\lambda^2\nu r + 3a_0a_1^2\lambda^2 r = 0, \\
&2a_1\lambda^3\omega^2 - 2a_1k^2\lambda^3 p + a_1^3\lambda^3 r = 0.
\end{aligned} \tag{55}$$

After solving these algebraic system, we get four sets of values of arbitrary constants:

$$\begin{aligned}
&\left\{a_0 \to \frac{\sqrt{q}}{\sqrt{r}}, a_1 \to -\frac{2\sqrt{q}}{\nu\sqrt{r}}, \lambda \to 0, k \to -\frac{\sqrt{\nu^2\omega^2 + 2q}}{\nu\sqrt{p}}\right\}, \\
&\left\{a_0 \to \frac{\sqrt{q}}{\sqrt{r}}, a_1 \to -\frac{2\sqrt{q}}{\nu\sqrt{r}}, \lambda \to 0, k \to \frac{\sqrt{\nu^2\omega^2 + 2q}}{\nu\sqrt{p}}\right\}, \\
&\left\{a_0 \to -\frac{\sqrt{q}}{\sqrt{r}}, a_1 \to \frac{2\sqrt{q}}{\nu\sqrt{r}}, \lambda \to 0, k \to -\frac{\sqrt{\nu^2\omega^2 + 2q}}{\nu\sqrt{p}}\right\}, \\
&\left\{a_0 \to -\frac{\sqrt{q}}{\sqrt{r}}, a_1 \to \frac{2\sqrt{q}}{\nu\sqrt{r}}, \lambda \to 0, k \to \frac{\sqrt{\nu^2\omega^2 + 2q}}{\nu\sqrt{p}}\right\}.
\end{aligned} \tag{56}$$

By substituting of a_0, a_1, k from (56) with using (14) into (46), we derive four exact traveling wave solutions of Eq.(44)

$$u_1(x,t) = U_1(\xi) = \frac{\sqrt{q}}{\sqrt{r}}\left[\frac{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)+1}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}\right], \tag{57}$$

where $\xi = -\frac{\sqrt{\nu^2\omega^2+2q}}{\Gamma(\beta+1)\left(\nu\sqrt{p}\right)}x^\beta - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$.
In particular, $p = 6, q = r = 4, A = 1, \mu = -1, \nu = 2, \omega = 2, \beta = 0.9, \alpha = 0.75$

$$u_1(x,t) = -\frac{1-\cosh\left(-\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)-\sinh\left(-\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)}{\cosh\left(-\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)+\sinh\left(-\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)+1}, \tag{58}$$

$$u_2(x,t) = U_2(\xi) = \frac{\sqrt{q}}{\sqrt{r}}\left[\frac{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)+1}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}\right], \tag{59}$$

where $\xi = \frac{\sqrt{\nu^2\omega^2+2q}}{\Gamma(\beta+1)\left(\nu\sqrt{p}\right)}x^\beta - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$.

$$u_3(x,t) = U_3(\xi) = -\frac{\sqrt{q}}{\sqrt{r}}\left[\frac{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)+1}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}\right], \tag{60}$$

where $\xi = -\frac{\sqrt{\nu^2\omega^2+2q}}{\Gamma(\beta+1)\left(\nu\sqrt{p}\right)}x^\beta - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$. Also

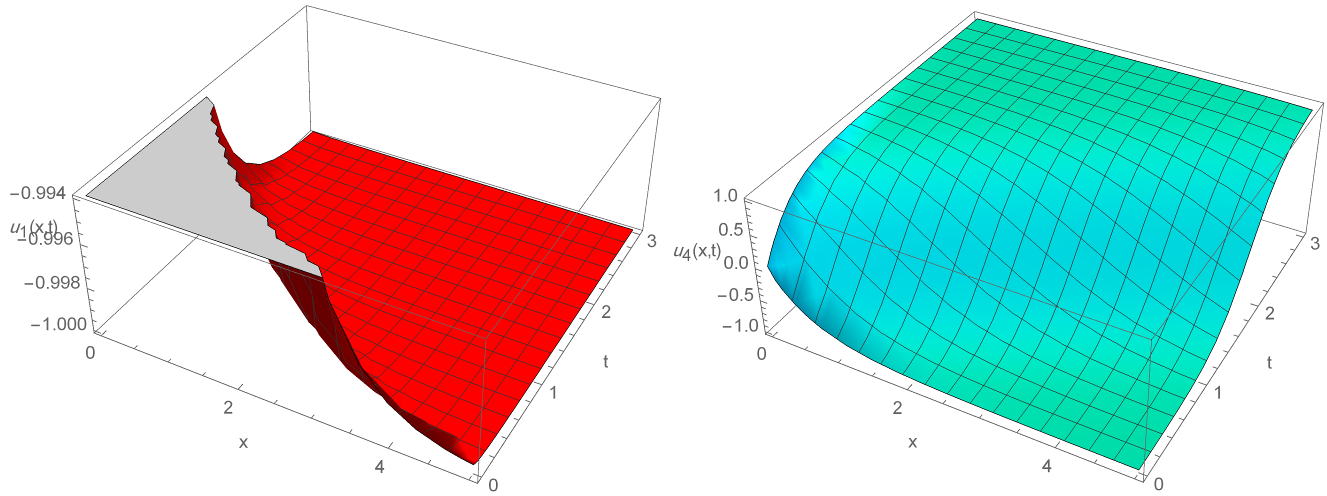
$$u_4(x,t) = U_4(\xi) = -\frac{\sqrt{q}}{\sqrt{r}}\left[\frac{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)+1}{A\mu\Big(\sinh(\nu\,\xi)+\cosh(\nu\,\xi)\Big)-1}\right], \tag{61}$$

where $\xi = \frac{\sqrt{\nu^2\omega^2+2q}}{\Gamma(\beta+1)\left(\nu\sqrt{p}\right)}x^\beta - \frac{\omega t^\alpha}{\Gamma(\alpha+1)}$.
In particular $p = 6, q = r = 4, A = 1, \mu = -1, \nu = 2, \omega = 2, \beta = 0.9, \alpha = 0.75$

$$u_4(x,t) = U_4(\xi) = \frac{1-\cosh\left(\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)-\sinh\left(\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)}{\cosh\left(\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)+\sinh\left(\frac{2x^{0.9}}{\Gamma(1.9)}-\frac{4t^{0.75}}{\Gamma(1.75)}\right)+1}. \tag{62}$$

6. Conclusion

In this work we found the solutions of three important nonlinear time-fractional evolution equations, TFCBE, TFME and STFPFE.
We have used the $(\frac{G'}{G})$-expansion method to derive three types exact solutions (hyperbolic, trigonometric and rational solutions).
The availability of computer systems like ***Mathematica 11*** facilitates the tedious algebraic calculations and plots of surfaces of solutions. The method which we have proposed in this paper is also a standard, direct and computerizable method, which allows us to do complicated and tedious algebraic calculation.
In general, $(\frac{G'}{G})$-expansion method is a very effective and powerful mathematical tool, can be further applied to solve various types of nonlinear fractional partial differential equations and also can be extended to physical mathematics, engineering and other nonlinear sciences.
We hope that these solutions will explain some nonlinear physical phenomena.



(a) $u_1(x,t)$ $(p=6, q=r=4, A=1, \mu=-1, \nu=2, \omega=2, \beta=0.9, \alpha=0.75)$

(b) $u_4(x,t)$ $(p=6, q=r=4, A=1, \mu=-1, \nu=2, \omega=2, \beta=0.9, \alpha=0.75)$

Figure 6: 3DPlot of the exact solutions of Eq. (44) given by (58) and (62) respectively, for $(x,t) \in [0,5]\times[0,3]$.

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