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TWO FRACTIONAL ORDER LANGEVIN EQUATION WITH NEW CHAOTIC DYNAMICS

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Abstract. In the present paper, we introduce a two-order nonlinear fractional sequential Langevin equation using the derivatives of Atangana-Baleanu and Caputo-Fabrizio. The existence of solutions is proven using a fixed point theorem under a weak topology, and an illustrative example is then given. Furthermore, we present new fractional versions of the Adams-Bashforth three-step approach for the Atangana-Baleanu and Caputo derivatives. New nonlinear chaotic dynamics are performed by numerical simulations.

1. Introduction

Fractional calculus has several applications in biology, mechanics, physics, viscoelasticity, electromagnetic waves, fractional Brownian motions, image processing, and engineering. Numerous books and essays in the literature cover a wide spectrum of fractional calculus problems, see [2, 22, 33].
Unfortunately, the fundamental prestigious Caputo and Riemann-Liouville features have such a critical flaw, even though their kernel is non-local, it remains singular. This issue has an impact on the modeling of real-world problems. To address the aforementioned obstacles, Caputo and Fabrizio proposed a new differential operator with non-singular kernel, see for instance the papers [12, 13, 21]. On the other, some researchers have used these derivatives to handle specific challenges, see [3, 5, 21]. Regrettably, various concerns have been raised in opposition to this novel approach, leading them to conclude that this operator cannot be a derivative

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of fractional order but can be viewed as a regulatory parameter, see [35]. For these reasons, based on the Mittag Leffler function, Atangana and Baleanu devised a new fractional operator, see [4, 26].

Nowadays, the most common differential equations observed in engineering and applied research are of second order. They take the form of $\ddot{x} = f(t, x, \dot{x})$.
Among the important examples of second-order equations is the Newton equation: $m\ddot{x} = f(x)$, the RLC circuit equation in electrical engineering: $LC\ddot{x} + RC\,\dot{x} + x = v(t)$, as well as the forced harmonic oscillator: $m\ddot{x} + b\dot{x} + kx = f(t)$.
The ultimate focus of this paper is to thoroughly explore certain sophisticated fractional differential equations, which can typically produce chaotic behavior such as the Langevin equation. The relevance of the nonlinear Langevin problem arises from its implementation as a model of anomalous systems. Indeed, it is well known that in many cases, the Langevin equation is the most convenient way to measure time changes in Brownian motion velocity, see [11, 18, 19, 23, 32, 34].
In this contribution, we study the existence of solutions for the nonlinear Langevin equation using a fixed point theorem under a weak topology, see [9, 20, 21]. The considered problem involves, in particular, two fractional orders with non-local multi-point boundary conditions. For more information, see [1, 8, 15, 17, 31].
So let us consider the following problem:

$$D^{\alpha}\left(D^{\beta} - \lambda(t)\right) y(t) = f\left(t, y(t), D^{\beta} y(t)\right), \quad t \in [0, T], \quad 0 < \alpha, \beta \leq 1, \tag{1}$$

with its conditions:

$$\begin{aligned} &y(0) = 0, \quad D^{\beta} y(0) = \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i), \\ &0 < \beta \leq 1, \quad \gamma > 0, \quad r \in \mathbb{N}^*, \quad \xi_i \in [0, T], \end{aligned} \tag{2}$$

where D^{α} and D^{β} are fractional differential operators of order $0 < \alpha, \beta \leq 1$, J^{γ} is the Rieman Liouville fractional integral operator of order $\gamma > 0$ and $\lambda : [0, T] \to \mathbb{R}$ is a given continuous function. Two different approaches are used: the first one is of Caputo-Fabrizio and the second one is of Atangana Baleanu.
Then, inspired by [7, 25, 27, 28], we propose new three-step Adams-Bashforth fractional methods for Caputo and Atangana Baleanu fractional derivatives. Finally, we apply the three-step Adams-Bashforth fractional methods to obtain new nonlinear chaotic dynamics.

The remaining part of the paper is organized into sections. Section 2 provides an overview of some of the fundamental concepts of fractional differentiation and fixed-point theory. In Section 3, we assert the existence of at least one solution to the problem as an outcome of the study. Section 4 discusses the numerical approximation method for fractional derivatives. Section 5 investigates numerical experiments with chaotic fractional differential equations to illustrate the utility of the proposed technique. Finally, we conclude with Section 6.

2. Preliminaries

The following section introduces some fractional calculus notions and concepts, see [4, 9, 13, 20, 23] .

Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$ is defined as*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \quad a < t \leq b,$$

where $\Gamma(\alpha) := \int_0^\infty e^{-s} s^{\alpha-1} ds.$

Definition 2. *The Liouville-Caputo fractional derivative of order $\alpha \in (0,1)$, for a differentiable function f, is defined by*

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s) \frac{1}{(t-s)^\alpha} ds.$$

Definition 3. *The Laplace transform for the Liouville-Caputo fractional derivative of order α is:*

$$\mathcal{L}\left[D^\alpha f(t)\right](s) = s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1}\{f(0)\}.$$

Definition 4 ([13])**.** *The Caputo-Fabrizio derivative of order $\alpha \in]0,1[$, for $T > 0, f \in H^1(0,T)$, is given by*

$${}^{CF} D^\alpha f(t) = \frac{1}{2} \frac{M(\alpha)(2-\alpha)}{1-\alpha} \int_0^t f'(s) \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right] ds,$$

where $M(\alpha)$ is a normalizing function depending on α such that $M(0) = M(1) = 1$.

Definition 5 ([13])**.** *The Laplace transform for Caputo-Fabrizio derivative is defined as*

$$\mathcal{L}\left\{{}^{CF} D^\alpha f(t)\right\}(s) = \frac{1}{2} \frac{M(\alpha)(2-\alpha)}{1-\alpha} \frac{s\ \mathcal{L}\{f(t)\}(s) - f(0)}{s + \frac{\alpha}{1-\alpha}}.$$

Definition 6 ([23])**.** *The Caputo Fabrizio integral operator of order α is given in the following way:*

$${}^{CF} J^\alpha f(t) = \frac{2(1-\alpha)}{M(\alpha)(2-\alpha)} f(t) + \frac{2\alpha}{M(\alpha)(2-\alpha)} \int_0^t f(s) ds.$$

Definition 7 ([4])**.** *The Atangana Baleanu fractional derivative in Caputo sense, for $T > 0, f \in H^1[0,T]$, $\alpha \in]0,1[$, is given as:*

$${}^{ABC} D^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha \left[-\alpha \frac{(t-s)^\alpha}{1-\alpha}\right] ds.$$

The Atangana Baleanu fractional derivative in Riemann-Liouville sense is given as:

$$^{ABR}D_t^{\alpha}f(t)=\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right]ds,$$

where E_{α} is Mittag-Leffler function, given by

$$E_{\alpha}(u)=\sum_{k=0}^{\infty}\frac{u^k}{\Gamma(\alpha k+1)},\quad \alpha>0,\quad \alpha\in\mathbb{R},\quad u\in\mathbb{R},$$

where $B(\alpha)$ has the same properties as $M(\alpha)$ in Caputo-Fabrizio case.

Definition 8 ([4]). *The fractional integral associated to the Atangana-Baleanu fractional derivative is defined as:*

$$^{AB}J^{\alpha}f(t)=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-s)^{\alpha-1}ds.$$

Definition 9. *The Laplace transform of Atangana-Baleanu fractional derivative in Caputo sense, is defined by:*

$$\mathcal{L}\left\{^{ABC}D^{\alpha}f(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{f(t)\}(s)-s^{\alpha-1}f(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Definition 10 ([4]). *The Laplace transform of Atangana-Baleanu fractional derivative in Riemann-Liouville sense is given as:*

$$\mathcal{L}\left\{^{ABR}D^{\alpha}f(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{f(t)\}(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Definition 11. *Let E and F be two Banach spaces. The operator $f:E\to F$ is weakly sequentially continuous if, for each sequence $(y_n)_n$ with $y_n\to y$, we have $fy_n\to fy$.*

Definition 12. *Let E be a Banach space with a norm $\|\cdot\|_E$. A mapping $\Psi: E\longrightarrow E$ is called D-Lipschitz, if there exists a continuous nondecreasing function $\mathfrak{W}:\ R^+\longrightarrow R^+$ satisfying*

$$\|\Psi x-\Psi y\|_E\leq\mathfrak{W}(\|x-y\|_{\infty}),$$

for all $x,y\in E$ with $\mathfrak{W}(0)=0$. The function $\mathfrak{W}$ is called a D-function of Ψ on E. Particularly, once $\mathfrak{W}(r)=kr$ for a given $k>0$ is a Lipschitz mapping with a Lipschitzian constant k. In addition, if $k<1$ is a contraction on E with a contraction constant k.

Remark 1. *Any Lipitzian correspondence is automatically D-Lipschitz, but the reverse may not be true. If $\mathfrak{W}$ is not necessarily increasing and satisfies $\mathfrak{W}(r)<r$ for $r>0$, then Ψ is called a nonlinear contraction on E.*

Remark 2. *Note that any weakly sequentially continuous nonlinear contraction is ω-condensing.*

Corollary 1. *Let Ω be a nonempty, convex, and closed set in a Banach space E. Assume that $\Psi : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous and condensing map in Ω. If $\Psi(\Omega)$ is bounded, then, Ψ has at least a fixed point.*

Corollary 2. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. Assume that $\Phi : \Omega \longrightarrow \Omega$ is weakly sequentially continuous. If $\Phi(\Omega)$ is relatively weakly compact, then Φ has at least a fixed point in Ω.*

Theorem 1 ([9])**.** *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. Suppose that $\Phi : \Omega \longrightarrow E$ and $\Psi : E \longrightarrow E$ are two weakly sequentially continuous mappings such that:*

(i) *Φ is weakly compact,*
(ii) *Ψ is a nonlinear contraction,*
(iii) *$(y = \Psi x + \Phi y, x \in \Omega) \Longrightarrow y \in \Omega$.*

Then, there exists $y \in \Omega$ such that $y = \Psi y + \Phi y$.

Theorem 2 (Eberlein-Smulian)**.** *Let $\mathcal{B}$ be a weakly closed subset of the Banach space E. Then the following assertions are equivalent:*

* *$\mathcal{B}$ is weakly compact.*
* *$\mathcal{B}$ is weakly sequentially compact.*

Lemma 1. *Let $T > 0, f \in H^1(0,T), \alpha \in]0,1[$. Then the solution of the problem* (1)-(2)*, for Atangana Baleanu fractional derivative in Caputo sense, is*

$$\begin{aligned} y(t) =& \mathcal{A}_1 \left[\begin{array}{l} \displaystyle\int_0^t \left(\frac{(t-u)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} + \frac{(1-\alpha)(t-u)^{\beta-1}}{\alpha\Gamma(\beta)} + \frac{(1-\beta)(t-u)^{\alpha-1}}{\beta\Gamma(\alpha)} \right) f(u)du \\ \displaystyle + \frac{(1-\beta)(1-\alpha)f(t)}{\beta\alpha} \end{array} \right] \\ &+ \mathcal{B}_1 \left[\int_0^t \frac{(t-u)}{\Gamma(B)}^{\beta-1} \lambda(u)y(u)du + \lambda(0)\left(\frac{1-\beta}{\beta} + \frac{t^\beta}{\Gamma(\beta+1)} \right) \sum_{i=0}^r \delta_i J^\gamma y(\xi_i) \right]. \end{aligned} \tag{3}$$

Lemma 2. *Let $T > 0, f \in H^1(0,T), \alpha \in]0,1[$. Then the solution of* (1)-(2)*, for the case of Caputo Fabrizio derivative, is*

$$\begin{aligned} y(t) =& \mathcal{A}_2 \left[\begin{array}{l} \displaystyle\int_0^t \left((t-u) + \frac{(1-\alpha)}{\alpha} + \frac{(1-\beta)}{\beta} \right) F_y(u)du \\ \displaystyle + \frac{(1-\beta)(1-\alpha)F_y(t)}{\beta\alpha} \end{array} \right] \\ &+ \mathcal{B}_2 \left[\int_0^t \lambda(u)y(u)du + \lambda(0)\left(\frac{1-\beta}{\beta} + t \right) \sum_{i=0}^r \delta_i J^\gamma y(\xi_i) \right]. \end{aligned} \tag{4}$$

Proof of Lemmas 1 and 2 : For computational purposes, we include the following quantity:

$$g(t) \quad := \quad D^\beta y(t) - \lambda(t)y(t),$$

$$\mathcal{A}_1 := \frac{\beta\alpha}{B(\alpha)\ \Big(B(\beta)-\lambda(t)(1-\beta)\Big)},$$

$$\mathcal{B}_1 := \frac{\beta}{B(\beta)-\lambda(t)+\beta},$$

$$\mathcal{A}_2 := \frac{\beta\alpha}{4M(\alpha)\ \Big(\alpha-2\Big)\Big(M(\beta)(\beta-2)-2\lambda(t)(1-\beta)\Big)},$$

$$\mathcal{B}_2 := \frac{\beta}{-2\ M(\beta)(\beta-2)-2\lambda(t)(1-\beta)}.$$

(Proof of Lemma 1) From the property of Laplace transform, we have

$$\mathcal{L}\left\{D^{\alpha}g(t)\right\}(s) = \frac{\frac{B(\alpha)}{1-\alpha}s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\mathcal{L}(g(t))(s) + \frac{\frac{B(\alpha)}{1-\alpha}s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}g(0) = \mathcal{L}\left\{f(t)\right\},$$

thus,

$$\mathcal{L}\left\{g(t)\right\}(s) = \frac{s^{\alpha}+\frac{\alpha}{1-\alpha}}{\frac{B(\alpha)}{1-\alpha}s^{\alpha}}\mathcal{L}\left\{f(t)\right\}(s) + \frac{g(0)}{s}.$$

Then, we have

$$\mathcal{L}\left\{D^{\beta}y(t)\right\}(s) = \frac{s^{\alpha}+\frac{\alpha}{1-\alpha}}{\frac{B(\alpha)}{1-\alpha}s^{\alpha}}\mathcal{L}(f(t))(s) + \frac{g(0)}{s} + \mathcal{L}(\lambda(t)y(t))(s) + \frac{y(0)}{s}.$$

Hence, it yields that

$$\begin{aligned}\mathcal{L}\left\{y(t)\right\}(s) =& \frac{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)\left(s^{\beta}+\frac{\beta}{1-\beta}\right)}{\frac{B(\alpha)}{1-\alpha}\frac{B(\beta)}{1-\beta}s^{\alpha+\beta}}\ \mathcal{L}(f(t))(s) + \frac{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\ g(0)}{\frac{B(\beta)}{1-\beta}s^{\beta+1}} \\ &+ \frac{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\ \mathcal{L}(\lambda(t)y(t))(s)}{\frac{B(\beta)}{1-\beta}s^{\beta}}.\end{aligned} \tag{5}$$

Substituting the conditions (2) in (5) and thanks to the properties of inverse Laplace transform, we deduce (3), which ends the proof.
(Proof of Lemma 2) Using the same arguments as before, we can write

$$\mathcal{L}\left\{D^{\alpha}g(t)\right\}(s) = \frac{\frac{M(\alpha)(2-\alpha)}{2(1-\alpha)}s}{s+\frac{\alpha}{1-\alpha}}\mathcal{L}(g(t))(s) + \frac{\frac{M(\alpha)(2-\alpha)}{2(1-\alpha)}}{s+\frac{\alpha}{1-\alpha}}g(0) = \mathcal{L}\left\{f(t)\right\}.$$

Then, we have

$$\mathcal{L}\left\{y(t)\right\}(s) = \frac{\left(s+\frac{\alpha}{1-\alpha}\right)\left(s+\frac{\beta}{1-\beta}\right)}{\frac{M(\alpha)(2-\alpha)}{2(1-\alpha)}\frac{M(\beta)(2-\alpha)}{2(1-\alpha)}s}\ \mathcal{L}(f(t))(s) + \frac{\left(s+\frac{\beta}{1-\beta}\right)\ g(0)}{\frac{M(\beta)(2-\beta)}{2(1-\beta)}s^2}$$
$$+ \frac{\left(s+\frac{\beta}{1-\beta}\right)\ \mathcal{L}(\lambda(t)y(t))(s)}{\frac{M(\alpha)(2-\alpha)}{2(1-\alpha)}s}.$$

Replacing the conditions (2) in (2), we obtain (4), which completes the proof. □

3. Main Results

The next section addresses the existence of at least one solution to our problem by utilizing two different approaches. We apply a fixed point theorem of Krasnoselskii type. It is based on the sum of two sequentially weakly continuous mappings.
We consider the Banach space:

$$\mathfrak{E} = \left\{y \in \mathcal{C}([0,T],\mathbb{R}),\ D^{\beta}y \in \mathcal{C}([0,T],\mathbb{R})\right\},$$

equipped with norm

$$\|y\|_{\mathfrak{E}} = \sup_{t\in[0,T]}|y(t)| + \sup_{t\in[0,T]}\left|D^{\beta}y(t)\right|.$$

Certainly, $(\mathfrak{E}, \|.\|_{\mathfrak{E}})$ is a Banach space.
Let $\Omega_j := \{y \in \mathfrak{E}, \|y\|_{\mathfrak{E}} \leq \eta_j\}, \quad j = 1, 2.$
The assumptions below are required:

(H1): The function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a jointly continuous function.
(H2): There exist non negative function $h \in \mathcal{C}([0,T],\mathbb{R}^+)$ and a non negative non decreasing function $\mathfrak{W} : \mathbb{R}^+ \to \mathbb{R}^+$, for each $t \in [0,T]$, and for all $x_i,\ y_i \in \mathbb{R}, i = 1, 2$, such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq h(t)\ \mathfrak{W}(\|x-y\|_{\mathfrak{E}}).$$

For $x \in \Omega_j, j = 1, 2$, we have

$$|f(t,x_1,x_2)| \leq h(t)\ \mathfrak{W}(\eta_j).$$

To simplify, we consider the following formulas

$$F_y(t) := f\left(t, y(t), D^{\beta}y(t)\right),$$

$$k_1 := \|h\|_\infty \mathcal{A}_1 \left| \frac{(1-\beta)(1-\alpha)}{\beta\alpha} + \frac{T^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{(1-\alpha)T^{\beta}}{\alpha\Gamma(\beta+1)} + \frac{(1-\beta)T^{\alpha}}{\beta\Gamma(\alpha+1)} \right|,$$

$$k_2 := \|h\|_\infty \mathcal{A}_2 \left| \frac{(1-\beta)(1-\alpha)}{\beta\alpha} + \frac{T^2}{2} + \frac{(1-\alpha)T}{\alpha} + \frac{(1-\beta)T}{\beta} \right|,$$

$$k_3 := \|h\|_\infty \left| \frac{1-\alpha}{B(\alpha)} + \frac{\alpha T^{\alpha}}{B(\alpha)\Gamma(\alpha+1)} \right|,$$

$$k_4 := \|h\|_\infty \left| \frac{-\alpha T^2}{M(\alpha)(\alpha-2)} + \frac{2(\alpha-1)}{M(\alpha)(\alpha-2)} \right|,$$

$$p_1 := \mathcal{B}_1|\ \|\lambda\|_\infty \left(\frac{T^{\beta}}{\Gamma(\beta+1)} + \left| \frac{1-\beta}{\beta} + \frac{T^{\beta}}{\Gamma(\beta+1)} \right| \frac{r\delta\xi^{\gamma}}{\Gamma(\gamma+1)} \right),$$

$$p_2 = \mathcal{B}_2|\ \|\lambda\|_\infty \left(\frac{T^2}{2} + \left| \frac{1-\beta}{\beta} + T \right| \frac{r\delta\xi^{\gamma}}{\Gamma(\gamma+1)} \right),$$

$$p_3 = p_4 = \|\lambda\|_\infty \left(1 + \frac{r\delta\xi^{\gamma}}{\Gamma(\gamma+1)} \right),$$

and

$$1-\rho_1 \neq 0,\ 1-\rho_2 \neq 0, \quad \kappa_1 := k_1+k_3,\ \kappa_2 := k_2+k_4, \quad \rho_1 := p_1+p_3,\ \rho_2 := p_2+p_4.$$

$$\delta := \max\{\delta_i, i=\overline{1,r}\},\ \xi := \max_{\xi_i\in[0,T]}\{\xi_i, i=\overline{1,r}\}$$

Our main results are given by the following theorem:

Theorem 3. *Assume that (H1) and (H2) are satisfied and suppose that* $\frac{\kappa_j}{(1-\rho_j)} \leq \frac{\eta_j}{\mathfrak{W}(\eta_j)},\ j=1,2.$
Then problem (1)-(2) *has at least a solution* y, $\|y\|_{\mathfrak{E}} \leq \eta_j,\ j=1,2.$

Proof. Let's introduce the applications $\mathcal{H}_j : \mathfrak{E} \to \mathfrak{E}, j=1,2,$ by

$$\begin{aligned} &\mathcal{H}_1 y(t) \\ &= \mathcal{A}_1 \left[\int_0^t \left(\frac{(t-u)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} + \frac{(1-\alpha)(t-u)^{\beta-1}}{\alpha\Gamma(\beta)} + \frac{(1-\beta)(t-u)^{\alpha-1}}{\beta\Gamma(\alpha)} \right) F_y(u)du \right. \\ &\left. + \frac{(1-\beta)(1-\alpha)F_y(t)}{\beta\alpha} \right] \qquad (6) \\ &+ \mathcal{B}_1 \left[\int_0^t \frac{(t-u)}{\Gamma(B)}^{\beta-1} \lambda(u)y(u)du + \lambda(0)\left(\frac{1-\beta}{\beta} + \frac{t^{\beta}}{\Gamma(\beta+1)}\right) \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i) \right], \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{H}_2 y(t) =& \mathcal{A}_2 \left[\begin{array}{l} \displaystyle\int_0^t \left((t-u) + \frac{(1-\alpha)}{\alpha} + \frac{(1-\beta)}{\beta} \right) F_y(u) du \\ \displaystyle + \frac{(1-\beta)(1-\alpha) F_y(t)}{\beta\alpha} \end{array} \right] \\
&+ \mathcal{B}_2 \left[\int_0^t \lambda(u) y(u) du + \lambda(0) \left(\frac{1-\beta}{\beta} + t \right) \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i) \right].
\end{aligned} \tag{7}
$$

Obviously, the establishment of the existence of solutions for (1)-(2) is equivalent to studying the existence of solutions of equation (6) (for Atangana Baleanu derivative), or the existence of solution of equation (7) (for Caputo Fabrizio derivative). For this aim, let us define the operators:

$$\Psi_j := (\Psi_{j,1}, \Psi_{j,2}) \text{ and } \Phi_j := (\Phi_{j,1}, \Phi_{j,2}), \quad j = 1, 2,$$

such that

$$\Psi_{j,\, i} : \mathfrak{E} \to \mathfrak{E} \ \text{ and } \ \Phi_{j,\, i} : \Omega_j \to \mathfrak{E}, \quad i, j = 1, 2,$$

by

$$
\Psi_{1,1} y(t) = \mathcal{A}_1 \left[\begin{array}{l} \displaystyle\int_0^t \left(\frac{(t-u)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)} + \frac{(1-\alpha)(t-u)^{\beta-1}}{\alpha\Gamma(\beta)} + \frac{(1-\beta)(t-u)^{\alpha-1}}{\beta\Gamma(\alpha)} \right) F_y(u) du \\ \displaystyle + \frac{(1-\beta)(1-\alpha) F_y(t)}{\beta\alpha} \end{array} \right],
$$

$$
\Psi_{2,1} y(t) = \mathcal{A}_2 \left[\int_0^t \left((t-u) + \frac{(1-\alpha)}{\alpha} + \frac{(1-\beta)}{\beta} \right) F_y(u) du + \frac{(1-\beta)(1-\alpha) F_y(t)}{\beta\alpha} \right],
$$

$$
\Phi_{1,1} y(t) = \mathcal{B}_1 \left[\int_0^t \frac{(t-u)^{\beta-1}}{\Gamma(\beta)} \lambda(u) y(u) du + \lambda(0) \left(\frac{1-\beta}{\beta} + \frac{t^{\beta}}{\Gamma(\beta+1)} \right) \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i) \right],
$$

$$
\Phi_{2,1} y(t) = \mathcal{B}_2 \left[\int_0^t \lambda(u) y(u) du + \lambda(0) \left(\frac{1-\beta}{\beta} + t \right) \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i) \right],
$$

$$
\Psi_{1,2} y(t) = \frac{\alpha}{B(\alpha)} \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} F_y(u) du + \frac{(1-\alpha)\, F_y(t)}{B(\alpha)},
$$

$$
\Psi_{2,2} y(t) = \frac{-2\alpha}{M(\alpha)(\alpha-2)} \int_0^t F_y(u) du + \frac{2(\alpha-1) F_y(t)}{M(\alpha)(\alpha-2)},
$$

$$
\Phi_{1,2} y(t) = \Phi_{2,2} y(t) = \lambda(t) y(t) + \lambda(0) \sum_{i=0}^{r} \delta_i J^{\gamma} y(\xi_i),
$$

where

$$\mathcal{H}_j = \Psi_{j,1} + \Phi_{j,1}, \quad D^{\beta} \mathcal{H}_j = \Psi_{j,2} + \Phi_{j,2}, \quad j = 1, 2.$$

Firstly, we need to prove that $\Psi_1,\ \Phi_1$ are two weakly sequential continuous mappings. Let $y_n \in \Omega_j$ be a sequence with $y_n \to y$, for some $y \in \mathfrak{E}$.
By (H_1) and (H_2), for $j = 1, 2$, we can write

$$|\Psi_{j,1}y_n(t) - \Psi_{j,1}y(t)| \leq k_j\ \mathfrak{W}(\ \|y_n - y\|_{\mathfrak{E}}\ ),$$

and

$$|\Psi_{j,2}y_n(t) - \Psi_{j,2}y(t)| \leq k_{j+2}\ \mathfrak{W}(\ \|y_n - y\|_{\mathfrak{E}}\ ).$$

Thus, we can write

$$\|\Psi_j y_n - \Psi_j y\|_{\mathfrak{E}} \leq \kappa_j \mathfrak{W}(\ \|y_n - y\|_{\mathfrak{E}}). \tag{8}$$

With the same arguments as before, we have

$$|\Phi_{j,1}y_n(t) - \Phi_{j,1}y(t)| \leq p_j \|y_n - y\|_\infty$$

and

$$|\Phi_{j,2}y_n(t) - \Phi_{j,2}y_n(t)| \leq p_{j+2}\|y_n - y\|_\infty.$$

Therefore,

$$\|\Phi_j y_n - \Phi_j y\|_{\mathfrak{E}} \leq \rho_j \|y_n - y\|_{\mathfrak{E}}. \tag{9}$$

Since $\|y_n - y\|_{\mathfrak{E}} \to 0$, the right hand sides of (8) and (9) tend to zero, then Ψ_j and Φ_j are weakly sequentially continuous mapping.

Secondly, we show that $\Phi_j(\Omega_j)$ is relatively weakly compact.
Step 1: Let $y \in \Omega_j\ j = 1, 2, t \in [0, T]$. We prove that $\Phi_j(\Omega_j)$ are bounded.
By (H_2), we get

$$|\Phi_{j,1}y(t)| \leq \eta_j\ p_j \quad \text{and} \quad |\Phi_{j,2}y(t)| \leq \eta_j\ p_{j+2},$$

so that

$$\|\Phi_j y\|_{\mathfrak{E}} \leq \eta_j\ \rho_j. \tag{10}$$

It follows that $\Phi_j(\Omega_j)$ are bounded.
Step 2: Let $y \in \Omega_j\ j = 1, 2$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we will show that Φ_j are equicontinuous.

By application of (H_1), for $j=1$, we have

$$
\begin{aligned}
&|\Phi_{1,1}y(t_2)-\Phi_{1,1}y(t_1)| \\
&\leq \frac{|\mathcal{B}_1|}{\Gamma(\beta)}\int_0^{t_1}\left|(t_2-u)^{\beta-1}-(t_1-u)^{\beta-1}\right| \ |\lambda(u)y(u)|\,du \\
&\quad+\frac{|\mathcal{B}_1|}{\Gamma(\beta)}\left[\ \int_{t_1}^{t_2}\left|(t_2-u)^{\beta-1}\right| \ |\lambda(u)y(u)|\,du \ +\frac{|t_2^B-t_1^B|}{\Gamma(B+1)}\sum_{i=0}^{r}\delta_i J^\gamma y(\xi_i)\ \right] \\
&\leq \frac{\eta_1 \ |\mathcal{B}_1| \ \|\lambda\|_\infty}{\Gamma(\beta)}\left[\int_0^{t_1}\left|(t_2-u)^{\beta-1}-(t_1-u)^{\beta-1}\right|+\ \int_{t_1}^{t_2}\left|(t_2-u)^{\beta-1}\right| \ du\right] \\
&\quad+\frac{|\mathcal{B}_1| \ \delta \ r \ \eta_1 \ \xi^\gamma \ |t_2^B-t_1^B|}{\Gamma(B+1)\Gamma(\gamma+1)}.
\end{aligned}
$$

Also, we have

$$|\Phi_{1,2}y(t_2)-\Phi_{1,2}y(t_1)|\leq \ |\lambda(t_2)-\lambda(t_1)| \ |y(t_2)-y(t_1)|.$$

Consequently,

$$
\begin{aligned}
|\Phi_1y(t_2)-\Phi_1y(t_1)| &\leq \frac{\eta_1 \ |\mathcal{B}_1| \ \|\lambda\|_\infty}{\Gamma(\beta)}\left[\begin{array}{l}\displaystyle\int_0^{t_1}\left|(t_2-u)^{\beta-1}-(t_1-u)^{\beta-1}\right|du \\ \displaystyle+\ \int_{t_1}^{t_2}\left|(t_2-u)^{\beta-1}\right| \ du\end{array}\right] \\
&\quad+\frac{\delta \ r \ \eta_1 \ \xi^\gamma \ |t_2^B-t_1^B|}{\Gamma(B+1)\Gamma(\gamma+1)}+|\lambda(t_2)-\lambda(t_1)| \ |y(t_2)-y(t_1)|.
\end{aligned}
\tag{11}
$$

In the same way as the previous part, for $j=2$, we get

$$
\begin{aligned}
|\Phi_{2,1}y(t_2)-\Phi_{2,1}y(t_1)| &\leq \ |\mathcal{B}_2|\left[\ \int_{t_1}^{t_2}|\lambda(u)y(u)|\,du \ +|t_2-t_1|\sum_{i=0}^{r}\delta_i J^\gamma y(\xi_i)\ \right] \\
&\leq \eta_2 \ |\mathcal{B}_2|\,|t_2-t_1|\left[\ \|\lambda\|_\infty+\ \delta \ r \ \frac{\xi^\gamma}{\Gamma(\gamma+1)}\right]
\end{aligned}
$$

and

$$|\Phi_{2,2}y(t_2)-\Phi_{2,2}y(t_1)|\leq \ |\lambda(t_2)-\lambda(t_1)| \ |y(t_2)-y(t_1)|.$$

These imply that

$$
\begin{aligned}
|\Phi_2y(t_2)-\Phi_2y(t_1)| &\leq \eta_2 \ |\mathcal{B}_2|\,|t_2-t_1|\left[\ \|\lambda\|_\infty+\ \delta \ r \ \frac{\xi^\gamma}{\Gamma(\gamma+1)}\right] \\
&\quad+|\lambda(t_2)-\lambda(t_1)| \ |y(t_2)-y(t_1)|,
\end{aligned}
\tag{12}
$$

when $t_1\to t_2$, the right hand sides of (11) and (12) tends to zero independently of y. Therefore, $\Phi_j, \ j=1,2$, are equicontinuous operators.
Thanks to Arzelà–Ascoli and Eberlein-Smulian theorems, $\Phi_j, \ j=1,2$, is relatively

weakly compact.

Next, we show that the operator $\Psi_j, j=1,2$, are nonlinear contractions. In view of (H_1) and (H_2), for each $t\in[0,T]$, we obtain

$$\|\Psi_{j,1}y_2-\Psi_{j,1}y_1\|_\infty \leq k_j\ \mathfrak{W}(\ \|y_2-y_1\|_{\mathfrak{E}}\ ),$$

and

$$\|\Psi_{j,2}y_2-\Psi_{j,2}y_1\|_\infty \leq k_{j+2}\ \mathfrak{W}(\ \|y_2-y_1\|_{\mathfrak{E}}\ ),$$

from which we get

$$\|\Psi_j y_2-\Psi_j y_1\|_{\mathfrak{E}} \leq \kappa_j\ \mathfrak{W}(\ \|y_2-y_1\|_{\mathfrak{E}}\ ).$$

In addition, we have to prove condition (iii) of Theorem 1 in two steps.
Phase 1: We verify that $\Psi_j(\mathfrak{E}), j=1,2$ are bounded.
Let $\Psi_j(\mathfrak{E}):=\{\Psi_j(y), y\in\Omega_j\}, j=1,2$, for all $t\in[0,T]$. Thanks to (H_2), we obtain

$$|\Psi_{j,1}y(t)|\leq k_j\ \mathfrak{W}(\eta_j)\ \text{ and }\ |\Psi_{j,2}y(t)|\leq k_{j+2}\ \mathfrak{W}(\eta_j),$$

which simplifies into

$$\|\Psi_j y\|_{\mathfrak{E}} \leq\ \kappa_j\ \mathfrak{W}(\eta_j). \tag{13}$$

Therefore, $\Psi_j(\mathfrak{E}), j=1,2$ are bounded.
Phase 2: Let $z\in\Omega_j, j=1,2$, such that $y=\Psi_j z+\Phi_j y$, so we can write:

$$|y(t)|\leq|\Psi_{j,1}z(t)|\ +\ |\Phi_{j,1}y(t)|\ \text{ and }\ |D^\beta y(t)|\leq|\Psi_{j,2}z(t)|\ +\ |\Phi_{j,2}y(t)|,$$

Thanks to (10) and (13), we obtain

$$\|y\|_{\mathfrak{E}}\leq\ \kappa_j\ \mathfrak{W}(\eta_j)+\eta_j\ \rho_j.$$

Consequently, we have

$$\|y\|_{\mathfrak{E}}\leq\eta_j\Rightarrow y\in\Omega_j.$$

So through the implementation of theorem1, we can state that $\mathcal{H}_j$ has at least one fixed point. Hence problem (1)-(2) has one solution in Ω_j, for $j=1,2$. □

4. An Example

Consider the following example:

$$\begin{cases} D^\alpha\left(D^\beta-\lambda(t)\right)y(t)=f\left(t,y(t),D^\beta y(t)\right), \quad t\in[0,T], \quad 0<\alpha,\beta\leq 1, \\ y(0)=0, \quad D^\beta y(0)=\displaystyle\sum_{i=0}^{r}\delta_i J^\gamma y(\xi_i), \quad 0<\beta\leq 1, \quad \gamma>0 \quad r\in\mathbb{N}^*, \quad \xi_i\in[0,T]. \end{cases}$$

We choose $\alpha=0.995,\ \beta=0.995,\ \gamma=1.33,\ \delta=0.75,\ \xi=0.75,\ r=5,$ and $T=1$. Define the continuous function by

$$f(t,x_1,x_2)=\frac{e^{\cos(\pi t)}}{(2-t)^4}\left(\sqrt{|x_1+x_2|}\right), \quad h(t)=\frac{e^{\cos(\pi t)}}{(2-t)^4}, \quad \mathfrak{W}(r)=\sqrt{r}, \quad \lambda(t)=0.1t.$$

From the above data, for $\eta_1 = 4.5$ and $\eta_2 = 1.6$, we have

$$\kappa_1 = 0.5581,\ \ \kappa_2 = 0.4138,\ \ \rho_1 = 0.7293,\ \ \rho_2 = 0.6708.$$

Obviously,

$$\frac{\kappa_1}{(1-\rho_1)} = 2.0622 \leq \frac{\eta_1}{\mathfrak{W}(\eta_1)} = \sqrt{4.5} \sim 2.1213$$
$$\frac{\kappa_2}{(1-\rho_2)} = 1.2575 \leq \frac{\eta_2}{\mathfrak{W}(\eta_2)} = \sqrt{1.6} \sim 1.2649.$$

By Theorem 1, our problem has at least one solution on $[0, 1]$.

5. Numerical Method of Approximation

We recall the following result, which is needed in the next section.

Theorem 4 ([25]). *The three-step Adams-Bashforth scheme for the Caputo Fabrizio fractional derivative is given by*

$$\begin{aligned} y(t_{n+1}) =& y(t_n) + \left(\frac{1-\alpha}{M(\alpha)} + \frac{23\alpha h}{12M(\alpha)}\right) f\left(t_n, y(t_n)\right) \\ &- \left(\frac{1-\alpha}{M(\alpha)} + \frac{16\alpha h}{12M(\alpha)}\right) f\left(t_{n-1}, y(t_{n-1})\right) + \frac{5\alpha h}{12M(\alpha)} f\left(t_{n-2}, y_{n-2}\right). \end{aligned} \tag{14}$$

In what follows, we prove an analogue theorem in the case of Atangana-Baleanu and then in the case of Caputo.

Theorem 5. *The three-step fractional Adams-Bashforth scheme for Atangana-Baleanu derivative in Caputo sense, for $n \in \mathbb{N}$, is given by*

$$\begin{aligned} y(t_{n+1}) =& y(t_n) + \mathfrak{A}\left(f\left(t_n, y_n\right) - f\left(t_{n-1}, y(t_{n-1})\right)\right) \\ &+ f\left(t_n, y\left(t_n\right)\right) \left(\begin{array}{l} \dfrac{h^\alpha \mathfrak{B}(n+1)^\alpha}{2}\left[\dfrac{6}{\alpha} - \dfrac{5(n+1)}{(\alpha+1)} + \dfrac{(n+1)^2}{\alpha+2}\right] \\ - \dfrac{h^\alpha \mathfrak{B} n^\alpha}{2}\left[\dfrac{2}{\alpha} - \dfrac{3n}{\alpha+1} + \dfrac{n^2}{\alpha+2}\right] \end{array}\right) \\ &+ f\left(t_{n-2}, y\left(t_{n-2}\right)\right) \left(\begin{array}{l} \dfrac{h^\alpha \mathfrak{B}(n+1)^\alpha}{2}\left[\dfrac{2}{\alpha} - \dfrac{3(n+1)}{a+1} + \dfrac{(n+1)^2}{\alpha+2}\right] \\ + \dfrac{h^\alpha \mathfrak{B} n^\alpha}{2}\left[\dfrac{n}{\alpha+1} - \dfrac{n^2}{\alpha+2}\right] \end{array}\right) \\ &- 2f\left(t_{n-1}, y\left(t_{n-1}\right)\right) \left(\begin{array}{l} \dfrac{h^\alpha \mathfrak{B}(n+1)^\alpha}{2}\left[\dfrac{3}{\alpha} - \dfrac{4(n+1)}{a+1} + \dfrac{(n+1)^2}{\alpha+2}\right] \\ + \dfrac{h^\alpha \mathfrak{B} n^\alpha}{2}\left[\dfrac{2n}{\alpha+1} - \dfrac{n^2}{\alpha+2}\right] \end{array}\right), \end{aligned} \tag{15}$$

where

$$\mathfrak{A} := \frac{1-\alpha}{B(\alpha)}, \quad \mathfrak{B} := \frac{\alpha}{B(\alpha)\Gamma(\alpha)}.$$

Proof. To approach the fractional derivative of Atangana-Baleanu we use [27, 28]. First, we take the following differential equation

$$^{ABC}D_t^{\alpha} y(t) = f(t, y(t)).$$

With respect to the integral representation, we find that

$$y(t) - y(0) = \frac{1-\alpha}{B(\alpha)} f(t, y(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau.$$

At t_{n+1}, we get

$$y(t_{n+1}) - y(0) = \frac{1-\alpha}{B(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} f(t, y(t)) dt,$$

thus

$$y(t_{n+1}) - y(t_n) = \mathfrak{A}\left(f(t_n, y_n) - f(t_{n-1}, y(t_{n-1}))\right) + C_1 - C_2, \tag{16}$$

where,

$$C_1 := \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} f(t, y(t)) dt,$$
$$C_2 := \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{\alpha-1} f(t, y(t)) dt.$$

To approximate the integral parts, we must use the polynomial approximation for $f(t, y(t))$ that passes through $f(t_n, y(t_n)), f(t_{n-1}, y(t_{n-1}))$, and $f(t_{n-2}, y_{n-2})$, which is given by

$$\Pi_2(t) = \sum_{i=0}^{2} f(t_{n-i}, y_{n-i}) L_i(t),$$

where $L_i(t)$ is the Lagrange polynomial for the interpolation points on t_n, t_{n-1} and t_{n-2}, as

$$\begin{aligned}\Pi_2(t) =& \frac{f(t_{n-2}, y(t_{n-2}))}{2h^2}(t-t_n)(t-t_{n-1}) - \frac{f(t_{n-1}, y(t_{n-1}))}{h^2}(t-t_n) \\ & \times (t-t_{n-2}) + \frac{f(t_n, y(t_n))}{2h^2}(t-t_{n-1})(t-t_{n-2}).\end{aligned}$$

Now, using $u = (t_{n+1} - t)/h$ in C_1, we get

$$C_1 = \frac{h^\alpha (n+1)^\alpha}{2} \left(\begin{array}{l} \left[\frac{6}{\alpha} - \frac{5(n+1)}{(\alpha+1)} + \frac{(n+1)^2}{\alpha+2} \right] f(t_n, y(t_n)) \\ -2 \left[\frac{3}{\alpha} - \frac{4(n+1)}{\alpha+1} + \frac{(n+1)^2}{\alpha+2} \right] \times f(t_{n-1}, y(t_{n-1})) \\ + \left[\frac{2}{\alpha} - \frac{3(n+1)}{a+1} + \frac{(n+1)^2}{\alpha+2} \right] f(t_{n-2}, y(t_{n-2})) \end{array} \right). \quad (17)$$

Similarly, taking $u = (t_n - t)/h$ in C_2, we obtain

$$\begin{aligned} C_2 = \frac{h^\alpha (n)^\alpha}{2} \quad & \left(\left[\frac{n^2}{\alpha+2} - \frac{3n}{\alpha-1} + \frac{2}{\alpha} \right] f(t_n, y(t_n)) + 2 \left[\frac{2n}{\alpha+1} - \frac{n^2}{\alpha+2} \right] \right. \\ & \left. \times f(t_{n-1}, y(t_{n-1})) - \left[\frac{n}{\alpha+1} - \frac{n^2}{\alpha+2} \right] f(t_{n-2}, y(t_{n-2})) \right). \end{aligned} \quad (18)$$

Substituting (17) and (18) into (16), we find (15). □

Theorem 6. *The three-step fractional Adams-Bashforth scheme for Caputo derivative, for $n \in \mathbb{N}$, is defined by:*

$$\begin{aligned} y(t_{n+1}) = & y(t_n) + f(t_n, y(t_n)) \left(\begin{array}{l} \frac{h^\alpha (n+1)^\alpha}{2\Gamma(\alpha)} \left[\frac{6}{\alpha} - \frac{5(n+1)}{(\alpha+1)} + \frac{(n+1)^2}{\alpha+2} \right] \\ - \frac{h^\alpha n^\alpha}{2\Gamma(\alpha)} \left[\frac{2}{\alpha} - \frac{3n}{\alpha+1} + \frac{n^2}{\alpha+2} \right] \end{array} \right) \\ & + f(t_{n-2}, y(t_{n-2})) \left(\begin{array}{l} \frac{h^\alpha (n+1)^\alpha}{2\Gamma(\alpha)} \left[\frac{2}{\alpha} - \frac{3(n+1)}{a+1} + \frac{(n+1)^2}{\alpha+2} \right] \\ + \frac{h^\alpha n^\alpha}{2\Gamma(\alpha)} \left[\frac{n}{\alpha+1} - \frac{n^2}{\alpha+2} \right] \end{array} \right) \\ & - 2f(t_{n-1}, y(t_{n-1})) \left(\begin{array}{l} \frac{h^\alpha (n+1)^\alpha}{2\Gamma(\alpha)} \left[\frac{3}{\alpha} - \frac{4(n+1)}{a+1} + \frac{(n+1)^2}{\alpha+2} \right] \\ + \frac{h^\alpha n^\alpha}{2\Gamma(\alpha)} \left[\frac{2n}{\alpha+1} - \frac{n^2}{\alpha+2} \right] \end{array} \right), \end{aligned} \quad (19)$$

Proof. For Caputo derivative, we examine the following differential equation

$$^cD_t^\alpha y(t) = f(t, y(t)).$$

The integral representation is given by

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau.$$

In a similar manner as before, we obtain (19) □

We further extend the feasibility of the suggested new scheme to explore issues modeled in many applications. In order to reproduce some existing chaotic problems, we adequately replace the classical time derivative by the fractional derivative of Caputo, Caputo-Fabrizio, and Atangana-Baleanu, then we faithfully perform the

simulation with the three-step Adams Bashforth fractional method as it was constructed above.
We note that (1) can be reduced to the following system:

$$\begin{array}{rclcl} D^{\beta}y(t) & = & z(t)+\lambda(t)\ y(t) & = & f_1(t,y(t)) \\ D^{\alpha}z(t) & = & f(t,y(t),D^{\beta}y(t)) & = & f_2(t,y(t),D^{\beta}y(t)). \end{array}$$

We can therefore stipulate the conditions (2) as follows

$$y(0)=0, \quad z(0)=\sum_{i=0}^{r}\delta_i J^{\gamma}y(\xi_i), \quad \gamma>0. \tag{20}$$

By (14), (15) and (19), the above system is transformed into the following:
Caputo case

$$\begin{aligned} y(t_{n+1}) =& y(t_n)+f_1\left(t_n,y\left(t_n\right)\right)C_{1,\beta}+f_1\left(t_{n-2},y\left(t_{n-2}\right)\right)C_{2,\beta} \\ &-2f_1\left(t_{n-1},y\left(t_{n-1}\right)\right)C_{3,\beta} \\ z(t_{n+1}) =& z(t_n)+f_2\left(t_n,z\left(t_n\right)\right)C_{1,\alpha}+f_2\left(t_{n-2},z\left(t_{n-2}\right)\right)C_{2,\alpha} \\ &-2f_2\left(t_{n-1},z\left(t_{n-1}\right)\right)C_{3,\alpha}. \end{aligned}$$

Caputo Fabrizio case

$$\begin{aligned} y(t_{n+1}) =& y(t_n)+f_1\left(t_n,y\left(t_n\right)\right)F_{1,\beta}+f_1\left(t_{n-2},y\left(t_{n-2}\right)\right)F_{2,\beta} \\ &-2f_1\left(t_{n-1},y\left(t_{n-1}\right)\right)F_{3,\beta} \\ z(t_{n+1}) =& z(t_n)+f_2\left(t_n,z\left(t_n\right)\right)F_{1,\alpha}+f_2\left(t_{n-2},z\left(t_{n-2}\right)\right)F_{2,\alpha} \\ &-2f_2\left(t_{n-1},z\left(t_{n-1}\right)\right)F_{3,\alpha}. \end{aligned}$$

Atangana-Baleanu case

$$\begin{aligned} y(t_{n+1}) =& y(t_n)+f_1\left(t_n,y\left(t_n\right)\right)A_{1,\beta}+f_1\left(t_{n-2},y\left(t_{n-2}\right)\right)A_{2,\beta} \\ &-2f_1\left(t_{n-1},y\left(t_{n-1}\right)\right)A_{3,\beta} \\ z(t_{n+1}) =& z(t_n)+f_2\left(t_n,z\left(t_n\right)\right)A_{1,\alpha}+f_2\left(t_{n-2},z\left(t_{n-2}\right)\right)A_{2,\alpha} \\ &-2f_2\left(t_{n-1},z\left(t_{n-1}\right)\right)A_{3,\alpha}, \end{aligned}$$

where $A_{i,\alpha},\ A_{i,\beta}, F_{i,\alpha},\ F_{i,\beta},\ C_{i,\alpha}, C_{i,\beta}$, constants obtained from (14), (15), (19) respectively.

6. Numerical Experiments

We use a variety of real-world examples to assess the performance of the new method on our problem, see [10,14,16,29,30,36]. The integration is carried out using the three-step fractional Adams-Bashforth methods for Caputo, Caputo Fabrizio, and Atangana-Baleanu. The classic case is plotted using the three-step Adams-Bashforth method for comparison.

For all the examples, we take $n = 8000$, so $T = n \times h$, $\alpha = 0.999999999$, $\beta = 0.99999999$.

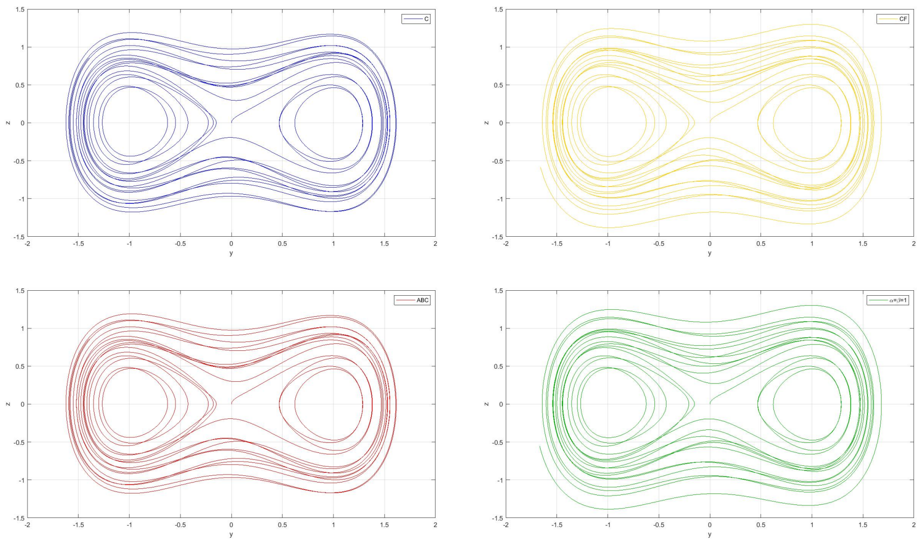

FIGURE 1. 2-D phase portraits for the numerical simulation for (21)

Example 1 (see [14])**.** *We consider the following general nonlinear Helmholtz–Duffing oscillator:*

$$\begin{aligned} &D^{\alpha}\left(D^{\beta}-\delta\right)y(t)=\gamma\cos(\omega t)+y-(1-\sigma)y^{2}-\sigma y^{3}-0.000001D^{\beta}y(t),\\ &t\in[0,T],\ \ 0<\alpha,\beta\leq 1, \end{aligned} \tag{21}$$

the equation (21) *can be reduced to the following system:*

$$\begin{aligned} D^{\beta}y(t)&=z(t)+\delta y(t),\\ D^{\alpha}z(t)&=\gamma\cos(\omega t)+y-(1-\sigma)y^{2}-\sigma y^{3}-0.000001D^{\beta}y(t). \end{aligned}$$

With initial conditions $(0, 0.00025)$*,* $h = 0.01$*,* $\delta = 0.01$*,* $\sigma = 1$*,* $\omega = 0.068$*,* $\gamma = 1$*.*

Example 2 (see [16])**.** *We consider the following problem in light of the Josephson Junction pendulum description and the pendulum system for ultra-subharmonic resonance:*

$$\begin{aligned} D^{\alpha}\left(D^{\beta}-\delta\right)y(t)=&-ay-[1+f_0\cos(\Omega t+\Psi)]\sin y+f_1\cos(\omega t)\sin(y-\gamma)\\ &-5*10^{(-5)}D^{\beta}y(t),\quad t\in[0,T],\quad 0<\alpha,\beta\leq 1. \end{aligned} \tag{22}$$

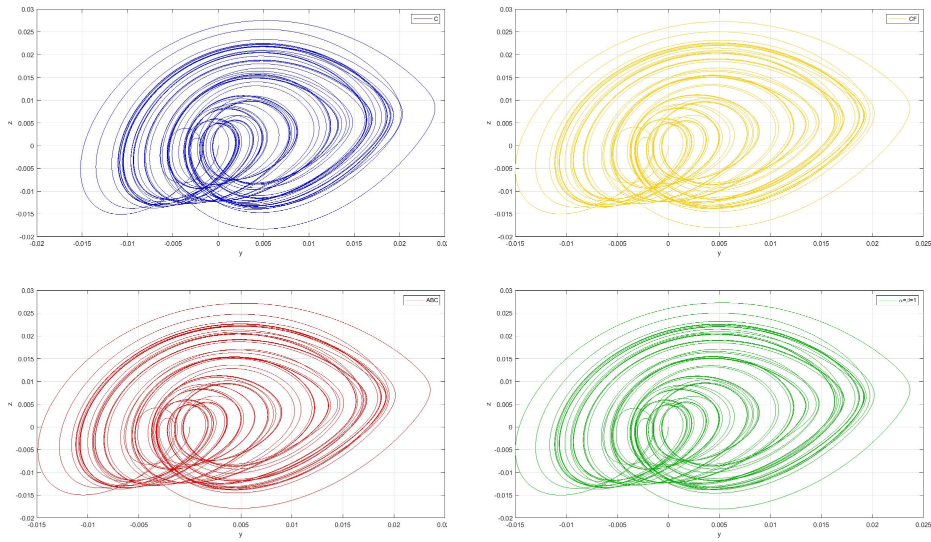

FIGURE 2. 2-D phase portraits for the numerical simulation for (22)

The equation (22) *can be reduced to the following system:*

$$\begin{aligned} D^{\beta}y(t) =& z(t)+\delta y(t), \\ D^{\alpha}z(t) =& -ay-[1+f_0\cos(\Omega t+\Psi)]\sin y+f_1\cos(\omega t)\sin(y-\gamma) \\ & -5*10^{(-5)}D^{\beta}y(t). \end{aligned}$$

The initial conditions are: $(0,0)$, $h=0.01$, $\delta=0.1$, $a=0.1$, $\Omega=0.75$, $\omega=1.5$, $\Psi=\ 7\pi/4$, $f_0=0.2$, $f_1=1.381$, $\gamma=0.01$.

Example 3 (see [30])**.** *We examine the resulting chaos of a simple nonlinear damped and driven pendulum motion:*

$$\begin{aligned} & D^{\alpha}\left(D^{\beta}-q\right)y(t)=a\Omega^2\cos(\Omega_D t)-\Omega^2\sin(y(t))+0.001D^{\beta}y(t), \\ & t\in[0,T], \quad 0<\alpha,\beta\leq 1. \end{aligned} \tag{23}$$

The equation (23) *can be reduced to the following system:*

$$\begin{aligned} D^{\beta}y(t) &= z(t)+qy(t), \\ D^{\alpha}z(t) &= a\Omega^2\cos(\Omega_D t)-\Omega^2\sin(y(t))+0.001D^{\beta}y(t). \end{aligned}$$

The initial conditions: $(0,0.8)$, $h=0.045$. $q=-0.4$, $a=1.4$, $\Omega=1$, $\Omega_D=2/3$.

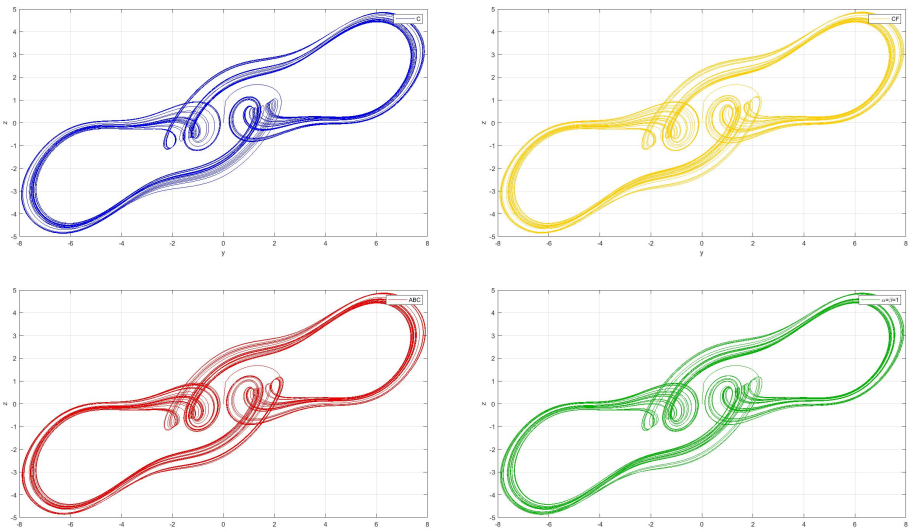

FIGURE 3. 2-D phase portraits for the numerical simulation for (23)

Example 4 (see [10])**.** *We employ numerical techniques to display chaotic attractors on the dynamics of a vertically driven damped planar pendulum:*

$$D^{\alpha}\left(D^{\beta}-\gamma\right)y(t)=(\chi-\psi\cos\tau)y(t)+0.001D^{\beta}y(t),\quad t\in[0,T],\quad 0<\alpha,\beta\leq 1. \tag{24}$$

The equation (24) *can be reduced to:*

$$\begin{aligned}D^{\beta}y(t)&=z(t)+\gamma y(t),\\ D^{\alpha}z(t)&=(\chi-\psi\cos\tau)y(t)+0.001D^{\beta}y(t).\end{aligned}$$

As initial conditions: $(0,0.05)$*, and* $h=0.05$*,* $\gamma=-0.001$*,* $\chi=-0.1$*,* $\psi=0.545$.

Example 5 (see [24])**.** *We examine the Mixed Rayleigh Lienard Oscillator Driven by Parametric Periodic Pimping and External Excitation given by:*

$$\begin{aligned}D^{\alpha}\left(D^{\beta}-\left(\alpha_1+\eta\cos vt\right)\right)y(t)=&\,\omega_0^2\left(F_0+F_1\cos\omega t\right)-\beta_0(D^{\beta}y(t))^2\\ &-\beta_1(D^{\beta}y(t))^3+\omega_0^2y(t)-\gamma y(t)^3.\end{aligned} \tag{25}$$

The equation (25) *can be reduced to the following system:*

$$\begin{aligned}D^{\beta}y(t)&=z(t)+\left(\alpha_1+\eta\cos vt\right)y(t),\\ D^{\alpha}z(t)&=\omega_0^2\left(F_0+F_1\cos\omega t\right)-\beta_0(D^{\beta}y(t))^2-\beta_1(D^{\beta}y(t))^3+\omega_0^2y(t)-\gamma y(t)^3.\end{aligned}$$

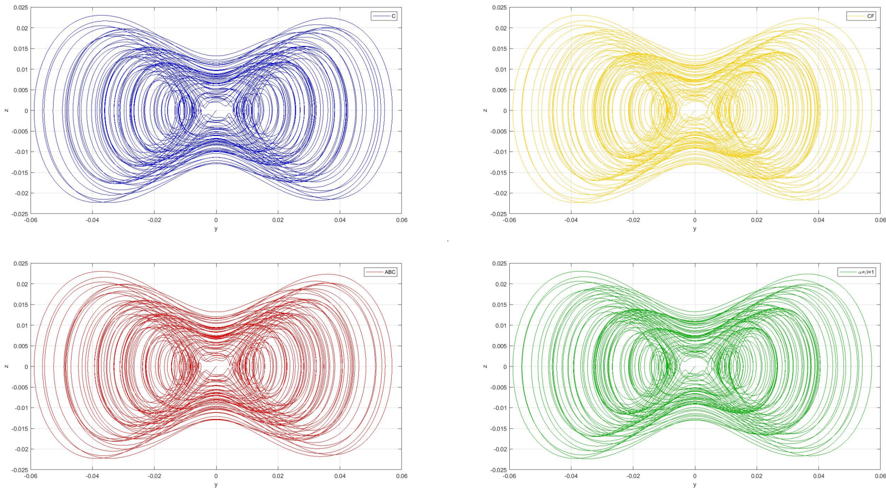

FIGURE 4. 2-D phase portraits for the numerical simulation for (24)

For initial conditions: $(0, -0.5)$, $\omega_0 = F_0 = 0.25$, $\alpha_0 = 0.015$, $\alpha_1 = 0.025$,$\gamma = 1$, $F_1 = 0.5$, $\beta_0 = 0.01$, $\beta_1 = 0.005$, *and* $\omega = v = 0.618$, $v = \frac{\sqrt{5}-1}{2}$, $\eta = 4$.

TABLE 1. Error summary table for each approach

Errors \ Examples	Example 1	Example 2	Example 3	Example 4	Example 5
$\Vert y_{AB3} - y_{ABc}\Vert_2$	1.29947854	0.00055533	0.82869035	0.05410987	0.4834849
$\Vert y_{AB3} - y_{ABcf}\Vert_2$	0.00062860	0.00000003	0.00009742	0.000023851	0.0011151
$\Vert y_{AB3} - y_{ABab}\Vert_2$	0.97326548	0.00047625	0.82859426	0.13071068	0.49796754

- The appearance of chaos under specific parameters demonstrates the convenience and pertinence of the proposed method.
- It is important to underline that some derivatives are more appropriate than others for particular cases but not for others.

7. CONCLUSION

In this study, we have examined the existence of solutions to the above fractional differential Langevin equation with Caputo-Fabrizio and Atangana-Baleanu derivatives. To achieve this, we have used a fixed point theorem based on the sum of two weakly sequentially continuous mappings.

Following that, we have proposed a novel three-step Adam Bashforth approach based on Caputo and Atangan Baleanu fractional derivatives. Numerous nonlinear

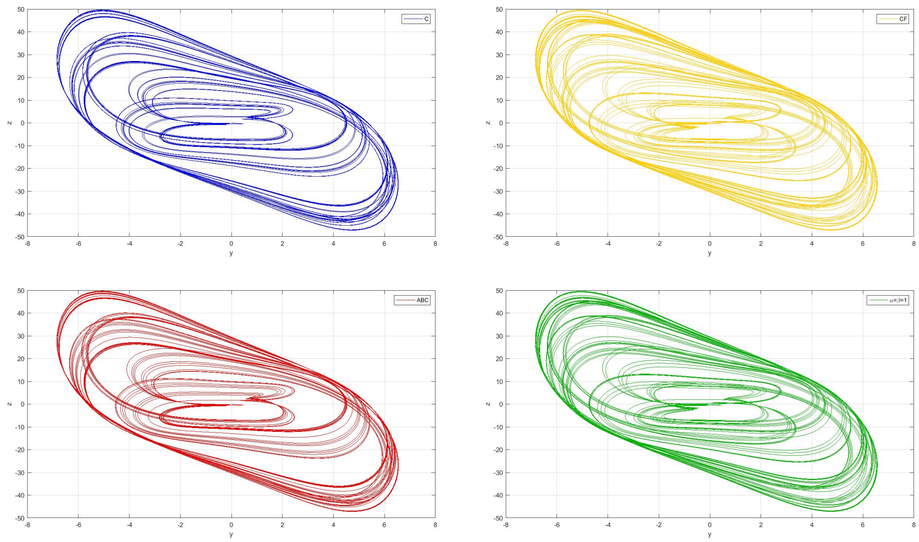

FIGURE 5. 2-D phase portraits for the numerical simulation for (25)

fractional differential equations have been exposed to a range of quantitative experiments. To assess the accuracy of the innovative numerical approach, the classical solution was compared towards the numerical solution for various values. Computational simulation results, for particular instances of α, β, are endowed with chaotic attractors.

Author Contribution Statements Z. DAHMANI proposed the problem and corrected the analytical study of Belhamiti. M.M. BELHAMITI studied the problem in its analytical and numerical studies. M.Z. SARIKAYA organized the paper and corrected some other analytical aspects.

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