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Ricci Solitons of Three-Dimensional Lorentzian Bianchi-Cartan-Vranceanu Spaces

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Abstract. In this paper, we obtain explicit formulae for homogenous Ricci solitons on three-dimensional Lorentzian Bianchi-Cartan-Vranceanu spaces. We also give a result about Ricci solitons on a three dimensional Minkowski space.

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1. Introduction

A Ricci soliton metric on a manifold M is defined by the condition

$$L_X g + \rho = \gamma g, \tag{1.1}$$

where X is a smooth vector field on M, $L_X g$ is Lie derivative in the direction of X and γ is a real constant. A Ricci soliton is called *shrinking* if $\gamma > 0$, *steady* if $\gamma = 0$ and *expanding* if $\gamma < 0$. Ricci soliton metrics are a generalization of Einstein metrics.

Ricci solitons and their generalizations have been extensively studied in many works from many points of view, so we may refer [4–6, 11] for more information about geometry of Ricci solitons.

Many researchers have been particularly interested in Ricci solitons on three-dimensional homogenous spaces, such as the Lie group $SL(2,\mathbb{R})$, Heisenberg group Nil_3, Berger spheres S^3_{Berger}, $S^2\times\mathbb{R}$, $H^2\times\mathbb{R}$ and the Lorentzian-Heisenberg group (see [1, 3, 7, 10, 12]).

Bianchi-Cartan-Vranceanu spaces are three-dimensional homogenous spaces with four dimensional isometry group. Ricci solitons on Bianchi-Cartan-Vranceanu spaces were studied by Batat *et al.* in [2].

Lorentzian Bianchi-Cartan-Vranceanu spaces (briefly LBCV-spaces) are considered by several authors in very recent papers, especially when investigating some special curves such as slant, Legendre and biharmonic etc. on it (see [8, 9, 13]).

As we mentioned above, although the subject of Ricci solitons is well-studied on homogenous manifolds, we give a classification of Ricci solitons by obtaining explicit formulae on LBCV-spaces in this paper. In fact, we will prove the following theorem:

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Theorem 1.1. *Let LBCV-spaces with the metric in (2.1) are given. Then, the following statements are true:*
(i) LBCV-spaces do not admit homogenous Ricci solitons when $\lambda \neq 0$ and $\mu > 0$.
(ii) LBCV-spaces admit shrinking homogenous Ricci solitons when $\lambda \neq 0$ and $\mu = 0$.
(iii) LBCV-spaces admit expanding homogenous Ricci solitons when $\lambda \neq 0$ and $\mu < 0$.
(iv) LBCV-spaces admit shrinking homogenous Ricci solitons when $\lambda = 0$ and $\mu > 0$.
(v) LBCV-spaces admit expanding homogenous Ricci solitons when $\lambda = 0$ and $\mu < 0$.

2. Lorentzian Bianchi-Cartan-Vranceanu Spaces (LBCV-Spaces)

In this section, we will recall some fundamental properties of LBCV-spaces (see [8, 13]).

Let $\lambda, \mu \in \mathbb{R}$. An open subset of $\mathbb{R}^3$ is given by

$$D = \{(x, y, z) \in \mathbb{R}^3 : 1 + \mu(x^2 + y^2) > 0\}.$$

The Lorentzian metric is equipped as following:

$$g_{\lambda,\mu} = \frac{dx^2 + dy^2}{(1 + \mu(x^2 + y^2))^2} - \left(dz + \frac{\lambda}{2}\frac{ydx - xdy}{1 + \mu(x^2 + y^2)}\right)^2. \tag{2.1}$$

The pair $(D, g_{\lambda,\mu})$ is called Lorentzian Bianchi-Cartan-Vranceanu spaces and it is denoted by $M_{\lambda,\mu}$.

An orthonormal frame field is given by

$$E_1 = \delta\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z},\ E_2 = \delta\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z},\ E_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

where we write $\delta = 1 + \mu(x^2 + y^2)$.

Therefore, the Lie brackets are obtained as

$$[E_1, E_2] = -2\mu y E_1 + 2\mu x E_2 + \lambda E_3,\ [E_1, E_3] = [E_2, E_3] = 0.$$

Let ∇ and R denote the Levi-Civita connection and the curvature tensor of $M_{\lambda,\mu}$, respectively. We have

$$\begin{aligned}
\nabla_{E_1}E_1 &= 2\mu y E_2,\ \nabla_{E_1}E_2 = -2\mu y E_1 + \frac{\lambda}{2}E_3,\ \nabla_{E_1}E_3 = \frac{\lambda}{2}E_2,\\
\nabla_{E_2}E_1 &= -2\mu x E_2 - \frac{\lambda}{2}E_3,\ \nabla_{E_2}E_2 = 2\mu x E_1,\ \nabla_{E_2}E_3 = -\frac{\lambda}{2}E_1,\\
\nabla_{E_3}E_1 &= \frac{\lambda}{2}E_2,\ \nabla_{E_3}E_2 = -\frac{\lambda}{2}E_1,\ \nabla_{E_3}E_3 = 0.
\end{aligned}$$

The components of the curvature tensor R^l_{ijk} are given by [14]

$$\begin{aligned}
R^1_{121} &= 0,\ R^1_{313} = \frac{\lambda^2}{4},\ R^1_{323} = 0,\ R^1_{221} = -4\mu - \frac{3}{4}\lambda^2,\ R^1_{331} = -\frac{\lambda^2}{4},\\
R^1_{112} &= 0,\ R^1_{223} = 0,\ R^1_{212} = 4\mu + \frac{3}{4}\lambda^2,\ R^1_{332} = 0,\ R^1_{113} = 0,\\
R^2_{121} &= 4\mu + \frac{3}{4}\lambda^2,\ R^2_{313} = 0,\ R^2_{323} = \frac{\lambda^2}{4},\ R^2_{221} = 0,\ R^2_{331} = 0,\\
R^2_{112} &= -4\mu - \frac{3}{4}\lambda^2,\ R^2_{223} = 0,\ R^2_{212} = 0,\ R^2_{332} = -\frac{\lambda^2}{4},\ R^2_{113} = 0,\\
R^3_{121} &= 0,\ R^3_{313} = 0,\ R^3_{323} = 0,\ R^3_{221} = 0,\ R^3_{331} = 0,\\
R^3_{112} &= 0,\ R^3_{223} = -\frac{\lambda^2}{4},\ R^3_{212} = 0,\ R^3_{332} = 0,\ R^3_{113} = -\frac{\lambda^2}{4}.
\end{aligned}$$

Therefore, for the Ricci tensor $\rho(X, Y) = tr\{Z \to R(X, Z)Y\}$ with respect to orthonormal basis (2.2), we obtain

$$\rho_{11} = \rho_{22} = 4\mu + \lambda^2,\ \rho_{33} = \frac{\lambda^2}{2}, \tag{2.3}$$

where we set $\rho_{ij} = \rho(E_i, E_j)$.

3. Ricci Solitons on Lorentzian Bianchi-Cartan-Vranceanu Spaces

In this section, we deal with the Ricci solitons on LBCV-space $M_{\lambda,\mu} = (D, g_{\lambda,\mu})$. Let $X = X_1E_1 + X_2E_2 + X_3E_3$ be an arbitrary vector field on $M_{\lambda,\mu}$, where X_1, X_2, X_3 are smooth functions of the variables x, y, z. Then, the Lie derivative of the metric (2.1) satisfies the following relations:

$$\begin{aligned} L_Xg_{\lambda,\mu}(E_1,E_1) &= 2(E_1(X_1) - 2\mu yX_2), \\ L_Xg_{\lambda,\mu}(E_1,E_2) &= 2\mu xX_2 + 2\mu yX_1 + E_1(X_2) + E_2(X_1), \\ L_Xg_{\lambda,\mu}(E_1,E_3) &= E_3(X_1) - E_1(X_3) - \lambda X_2, \\ L_Xg_{\lambda,\mu}(E_2,E_2) &= 2(E_2(X_2) - 2\mu xX_1), \\ L_Xg_{\lambda,\mu}(E_2,E_3) &= \lambda X_1 - E_2(X_3) + E_3(X_2), \\ L_Xg_{\lambda,\mu}(E_3,E_3) &= -2E_3(X_3). \end{aligned} \tag{3.1}$$

Therefore, if we use (2.1), (2.3) and (3.1) in (1.1) and have in mind (2.2), with a standard calculation, we see that a LBCV space is a Ricci soliton if and only if the following system is satisfied:

$$\begin{cases} 2\mu yX_2 - \delta\partial_x X_1 + \frac{\lambda}{2}y\partial_z X_1 = \frac{\rho_{11}-\gamma}{2}, \\ 2\mu xX_2 + 2\mu yX_1 + \delta\partial_x X_2 - \frac{\lambda}{2}y\partial_z X_2 + \delta\partial_y X_1 + \frac{\lambda}{2}x\partial_z X_1 = 0, \\ -\lambda X_2 - \delta\partial_x X_3 + \frac{\lambda}{2}y\partial_z X_3 + \partial_z X_1 = 0, \\ 2\mu xX_1 - \delta\partial_y X_2 - \frac{\lambda}{2}x\partial_z X_2 = \frac{\rho_{11}-\gamma}{2}, \\ \lambda X_1 - \delta\partial_y X_3 - \frac{\lambda}{2}x\partial_z X_3 + \partial_z X_2 = 0, \\ \partial_z X_3 = \frac{\gamma+\rho_{33}}{2}, \end{cases} \tag{3.2}$$

where we set $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_z = \frac{\partial}{\partial z}$.

Equation $(3.2)_6$ implies that

$$X_3 = (\frac{\gamma+\rho_{33}}{2})z + A(x,y), \ A \in C^\infty(M), \tag{3.3}$$

for an arbitrary smooth function $A = A(x,y)$.

Case 1: $\lambda \neq 0$

From $(3.2)_5$ and using (3.3), we get

$$X_1 = \frac{1}{\lambda}\left(\delta\partial_y A - \partial_z X_2 + \lambda(\frac{\gamma+\rho_{33}}{4})x\right). \tag{3.4}$$

Substituting (3.3) and (3.4) in $(3.2)_3$, we occur

$$\lambda^2 X_2 + \partial_z^2 X_2 = \lambda\left(\lambda(\frac{\gamma+\rho_{33}}{4})y - \delta\partial_x A\right).$$

Solution of the above equation gives us

$$X_2 = -\frac{\delta}{\lambda}\partial_x A + (\frac{\gamma+\rho_{33}}{4})y + C_1(x,y)\cos(\lambda z) + C_2(x,y)\sin(\lambda z), \tag{3.5}$$

where C_1 and C_2 are arbitrary smooth functions of the variables x and y.

It follows that

$$X_1 = \frac{\delta}{\lambda}\partial_y A + (\frac{\gamma+\rho_{33}}{4})x + C_1(x,y)\sin(\lambda z) - C_2(x,y)\cos(\lambda z). \tag{3.6}$$

By substituting (3.5) and (3.6) in $(3.2)_1$, we see that

$$\begin{cases} \partial_x C_1 = \left(2\mu + \frac{\lambda^2}{2}\right)\frac{yC_2}{\delta}, \\ \partial_x C_2 = -\left(2\mu + \frac{\lambda^2}{2}\right)\frac{yC_1}{\delta}, \\ (1 + \mu(x^2 - y^2))(\frac{\gamma+\rho_{33}}{4}) + \frac{\delta}{\lambda}\left(2\mu(x\partial_y A + y\partial_x A) + \delta\partial_x\partial_y A\right) = \frac{\gamma-\rho_{11}}{2}. \end{cases} \tag{3.7}$$

Again, by substituting (3.5) and (3.6) in $(3.2)_4$, we obtain

$$\begin{cases} \partial_y C_1 = -\left(2\mu + \frac{\lambda^2}{2}\right)\frac{xC_2}{\delta}, \\ \partial_y C_2 = \left(2\mu + \frac{\lambda^2}{2}\right)\frac{xC_1}{\delta}, \\ (1 - \mu(x^2 - y^2))(\frac{\gamma+\rho_{33}}{4}) - \frac{\delta}{\lambda}\left(2\mu(x\partial_y A + y\partial_x A) + \delta\partial_x\partial_y A\right) = \frac{\gamma-\rho_{11}}{2}. \end{cases} \tag{3.8}$$

The last equations in (3.7) and (3.8) show that

$$\gamma = 2\rho_{11} + \rho_{33}$$
$$\gamma = 8\mu + \frac{3\lambda^2}{2}.$$

Therefore, (3.7) and (3.8) turn to be

$$\lambda\mu\left(2\mu + \frac{\lambda^2}{2}\right)(x^2 - y^2) + \delta\left(2\mu(x\partial_y A + y\partial_x A) + \delta\partial_x\partial_y A\right) = 0. \tag{3.9}$$

Taking derivative with respect to y in the first equation of (3.7) and with respect to x in the first equation of (3.8), and having in mind $\partial_x C_2$ and $\partial_y C_2$, we see that $C_2 = 0$ (when $\lambda^2 \neq -4\mu$) or $C_2 \in \mathbb{R}$ (when $\lambda^2 = -4\mu$). Similarly, C_1 is zero or constant.

Let the inequality $\lambda^2 \neq -4\mu$ holds. Equation $(3.2)_2$ leads to

$$2\lambda\mu\left(4\mu + \lambda^2\right)xy + \delta[\left(4\mu(y\partial_y A - x\partial_x A) + \delta(\partial_y^2 A - \partial_x^2 A\right)] = 0. \tag{3.10}$$

So, the vector field $X = X_1E_1 + X_2E_2 + X_3E_3$ fulfils (3.2) if and only if

$$X_1 = \tfrac{\delta}{\lambda}\partial_y A + \left(\tfrac{4\mu+\lambda^2}{2}\right)x,$$
$$X_2 = -\tfrac{\delta}{\lambda}\partial_x A + \left(\tfrac{4\mu+\lambda^2}{2}\right)y,$$
$$X_3 = (4\mu + \lambda^2)z + A.$$

Here, the function A satisfies (3.9) and (3.10).

Now, suppose that $\lambda^2 = -4\mu$. In this case, Equations (3.9) and (3.10) remain valid, but the vector field X reduces to

$$\begin{aligned} X_1 &= \tfrac{\delta}{\lambda}\partial_y A + C_1\sin(\lambda z) - C_2\cos(\lambda z),\\ X_2 &= -\tfrac{\delta}{\lambda}\partial_x A + C_1\cos(\lambda z) + C_2\sin(\lambda z),\\ X_3 &= A, \end{aligned} \tag{3.11}$$

$C_1, C_2 \in \mathbb{R}$ and $\gamma = 2\mu$.

(a) If $\mu = 0$, Equations (3.9) and (3.10) turn in to be

$$\partial_x\partial_y A = 0 \text{ and } \partial_y^2 A = \partial_x^2 A.$$

So, we have

$$A = a_1(x^2 + y^2) + a_2x + a_3y + a_4,\ a_1, ..., a_4 \in \mathbb{R}.$$

As a result, when $\mu = 0$, the vector field $X = X_1E_1 + X_2E_2 + X_3E_3$ satisfy the soliton equation (1.1) if and only if

$$X_1 = \tfrac{1}{\lambda}(2a_1y + a_3) - \tfrac{\lambda^2}{4}x,$$
$$X_2 = -\tfrac{1}{\lambda}(2a_1x + a_2) - \tfrac{\lambda^2}{4}y,$$
$$X_3 = -\tfrac{\lambda^2}{2}z + a_1(x^2 + y^2) + a_2x + a_3y + a_4,$$

where $a_1, ..., a_4 \in \mathbb{R}$ and $\gamma = \frac{3\lambda^2}{2} > 0$. Thus, we proved Theorem 1.1 (ii).

(b) Now, suppose that $\mu \neq 0$. Set $f = \delta A$ and $\Delta = \lambda\mu\left(2\mu + \frac{\lambda^2}{2}\right)$. Then, Equations (3.9) and (3.10) imply

$$\partial_x\partial_y f = \frac{\Delta(y^2 - x^2)}{1 + \mu(x^2 + y^2)}, \tag{3.12}$$

$$\partial_x^2 f - \partial_y^2 f = \frac{4\Delta xy}{1 + \mu(x^2 + y^2)}. \tag{3.13}$$

If we integrate (3.12) with respect to y, we get

$$\partial_x f = \Delta\left[\frac{y}{\mu} - \frac{(1 + 2\mu x^2)}{|\mu|^{3/2}\sqrt{1 + \mu x^2}}\arctan\left(\frac{\sqrt{|\mu|}\,y}{\sqrt{1 + \mu x^2}}\right)\right] + \alpha(x), \tag{3.14}$$

and if we integrate (3.12) with respect to x, we obtain

$$\partial_y f = \Delta\left[-\frac{x}{\mu} + \frac{(1 + 2\mu y^2)}{|\mu|^{3/2}\sqrt{1 + \mu y^2}}\arctan\left(\frac{\sqrt{|\mu|}\,x}{\sqrt{1 + \mu y^2}}\right)\right] + \beta(y), \tag{3.15}$$

where α and β are smooth functions. Remark that if $\mu < 0$, we have *arctanh* instead of *arctan*. Differentiating (3.14) by x and (3.15) by y, replacing into (3.13), we deduce that there is a solution if and only if $\Delta = 0$, that is, if $\mu = -\frac{\lambda^2}{4} < 0$. This shows that when $\mu > 0$ the solution does not exist which proves the statement Theorem 1.1 (i). Moreover, we occur that

$$f = a_1(x^2 + y^2) + a_2 x + a_3 y + a_4,$$
$$\text{and } A(x, y) = \frac{a_1(x^2 + y^2) + a_2 x + a_3 y + a_4}{1 + \mu(x^2 + y^2)}.$$

Thus, if $\mu > 0$, Equation (1.1) has no solution and if $\mu < 0$ it is satisfied only for $\mu = -\frac{\lambda^2}{4}$. Then, from (3.11), we obtain the corresponding solutions as follows:

$$\begin{aligned} X_1 &= \tfrac{-2a_2\mu xy + a_3\left(\mu(x^2-y^2)+1\right) - 2a_4\mu y + 2a_1 y}{\lambda(1+\mu(x^2+y^2))} \\ &\quad + a_5 \sin(\lambda z) - a_6 \cos(\lambda z), \\ X_2 &= \tfrac{2\mu x(a_3 y + a_4) + a_2\left(\mu(x^2-y^2)-1\right) - 2a_1 x}{\lambda(1+\mu(x^2+y^2))} \\ &\quad + a_5 \cos(\lambda z) + a_6 \sin(\lambda z), \\ X_3 &= \tfrac{a_1(x^2+y^2)+a_2x+a_3y+a_4}{1+\mu(x^2+y^2)}, \end{aligned}$$

with $a_1, ..., a_6 \in \mathbb{R}$ and $\gamma = -\frac{\lambda^2}{2} < 0$. This completes the proof of Theorem 1.1 (iii). Remark that in this case associated the solitons are Killing vector fields also.

Case 2: $\lambda = 0,\ \mu \neq 0$

In this case the system (3.2) reduces to

$$\begin{aligned} 2\mu y X_2 - \delta\partial_x X_1 &= \tfrac{4\mu-\gamma}{2}, \\ 2\mu x X_2 + 2\mu y X_1 + \delta\partial_x X_2 + \delta\partial_y X_1 &= 0, \\ -\delta\partial_x X_3 + \partial_z X_1 &= 0, \\ 2\mu x X_1 - \delta\partial_y X_2 &= \tfrac{4\mu-\gamma}{2}, \\ -\delta\partial_y X_3 + \partial_z X_2 &= 0, \\ \partial_z X_3 &= \tfrac{\gamma}{2}. \end{aligned} \tag{3.16}$$

From the equations $(3.16)_3$, $(3.16)_5$ and $(3.16)_6$, we obtain

$$\begin{aligned} X_1 &= \delta(\partial_x A)z + F(x, y), \\ X_2 &= \delta(\partial_y A)z + E(x, y), \\ X_3 &= \tfrac{\gamma}{2}z + A(x, y), \end{aligned} \tag{3.17}$$

where A, E and F are smooth functions of x and y. Putting these expressions of X_1 and X_2 in $(3.16)_1$ gives us

$$-\delta[2\mu(x\partial_x A - y\partial_y A) + \delta\partial_x^2 A]z + 2\mu y E - \delta\partial x F = \frac{4\mu - \gamma}{2}.$$

Since this equation holds for all z, we have

$$2\mu(x\partial_x A - y\partial_y A) + \delta\partial_x^2 A = 0,\ 2\mu y E - \delta\partial x F = \frac{4\mu - \gamma}{2}. \tag{3.18}$$

Again, substituting the expressions of X_1 and X_2 in (3.17) into $(3.16)_4$ and $(3.16)_2$ we obtain, respectively

$$2\mu(y\partial_y A - x\partial_x A) + \delta\partial_y^2 A = 0,\ 2\mu x F - \delta\partial y E = \frac{4\mu - \gamma}{2} \tag{3.19}$$

and

$$2\mu(x\partial_y A + y\partial_x A) + \delta\partial_x\partial_y A = 0,\ 2\mu(xE + yF) + \delta(\partial_x E + \partial_y F) = 0. \tag{3.20}$$

Combining the first equations in (3.18) and (3.19), we get

$$\partial_x^2 A + \partial_y^2 A = 0. \tag{3.21}$$

If we derive the first equation in (3.18) with respect to x and the first equation with respect to y in (3.19), and have in mind (3.21), we occur

$$2\partial_x A + x\partial_x^2 A + y\partial_x\partial_y A = 0. \tag{3.22}$$

Now, if we derive the first equation in (3.18) with respect to y and the first equation with respect to x in (3.19), and by virtue of (3.21), we deduce

$$2\partial_y A - y\partial_x^2 A + x\partial_x\partial_y A = 0. \tag{3.23}$$

Therefore, from (3.22) and (3.23), after using the first equation in (3.18), we obtain that $\partial_x^2 A = \partial_y^2 A = 0$. So, the first equations in (3.18) and (3.19) become $x\partial_x A - y\partial_y A = 0$, which together with the first equation of (3.20) shows that A is a constant function.

Similarly, by considering the second equations of (3.18), (3.19) and (3.20), we have

$$\partial_y(\delta E) - \partial_x(\delta F) = 0,\ \partial_x(\delta E) + \partial_y(\delta F) = 0.$$

The solution of this system is $\delta E = c_1$, $\delta F = c_2$, where $c_1, c_2 \in \mathbb{R}$. Putting this in (3.18), we obtain $E = F = 0$ with $\gamma = 4\mu$. So, by setting $A = a \in \mathbb{R}$, the system (3.17) turns in to be

$$X_1 = X_2 = 0,\ X_3 = 2\mu z + a.$$

This completes the proof of Theorem 1.1.

Case 3: $\lambda = \mu = 0$

In this final case we deal with a Minkowski three-space. If $\lambda = \mu = 0$, the system (3.17) becomes

$$\begin{aligned} \delta\partial_x X_1 &= \tfrac{\gamma}{2}, \\ \partial_x X_2 + \partial_y X_1 &= 0, \\ -\partial_x X_3 + \partial_z X_1 &= 0, \\ \partial_y X_2 &= \tfrac{\gamma}{2}, \\ -\partial_y X_3 + \partial_z X_2 &= 0, \\ \partial_z X_3 &= \tfrac{\gamma}{2}. \end{aligned}$$

By direct computation, we see that, for $X = X_1E_1 + X_2E_2 + X_3E_3$, the corresponding soliton has the following form:

$$\begin{aligned} X_1 &= \tfrac{\gamma}{2}x - a_1 y + a_2 z + a_3, \\ X_2 &= a_1 x + \tfrac{\gamma}{2}y + a_4 z + a_5, \\ X_3 &= a_2 x + a_4 y + \tfrac{\gamma}{2}z + a_6, \end{aligned}$$

for every $\gamma \in \mathbb{R}$ with $a_1, ..., a_6 \in \mathbb{R}$.

4. Conclusion

In this work, we gave a classification for Ricci solitons on Lorentzian Bianchi-Cartan-Vranceanu spaces. We showed that there exist significant differences from the Riemannian case, which is studied in the reference [2], when $\lambda \neq 0$.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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