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Research Article

A-numerical radius : New inequalities and characterization of equalities

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Abstract

We develop new lower bounds for the A-numerical radius of semi-Hilbertian space operators, and applying these bounds we obtain upper bounds for the A-numerical radius of the commutators of operators. The bounds obtained here improve on the existing ones. Further, we provide characterizations for the equality of the existing A-numerical radius inequalities of semi-Hilbertian space operators.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the $\mathbb{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ and the corresponding norm $\|\cdot\|$. Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, henceforth we reserve the symbol A for positive operator on $\mathcal{H}$. Clearly, A induces a positive semidefinite sesquilinear form $\langle .,. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in \mathcal{H}$. This sesquilinear form induces a seminorm $\|\cdot\|_A : \mathcal{H} \to \mathbb{R}^+$, defined by $\|x\|_A = \sqrt{\langle x, x\rangle_A}$ for all $x \in \mathcal{H}$. Clearly, $\|\cdot\|_A$ is a norm if and only if A is injective. Let $R(A)$ denote the range of A and $\overline{R(A)}$ denote the norm closure of $R(A)$. Let $\mathcal{B}^A(\mathcal{H})$ denote the set of all operators $T \in \mathcal{B}(\mathcal{H})$ for which there exists $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \overline{R(A)}$, and we define

$$\|T\|_A = \sup_{\substack{x \in \overline{R(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \overline{R(A)} \\ \|x\|_A = 1}} \|Tx\|_A < +\infty.$$

For $T \in \mathcal{B}(\mathcal{H})$, if there exists an operator $S \in \mathcal{B}(\mathcal{H})$ satisfying $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in \mathcal{H}$, then S is said to be an A-adjoint of T (see [1]) and in this case $AS = T^*A$, where T^* denotes the Hilbert-adjoint of T. Let $\mathcal{B}_A(\mathcal{H})$ denote the collection of all operators in $\mathcal{B}(\mathcal{H})$, which admit A-adjoint. For $T \in \mathcal{B}_A(\mathcal{H})$, the operator equation $AX = T^*A$ has a unique solution, denoted by $T^\sharp$ (or $T^{\sharp_A}$ as in [7]), satisfying $R(T^\sharp) \subseteq \overline{R(A)}$, where $R(T^\sharp)$ denotes the range of $T^\sharp$. For $T \in \mathcal{B}_A(\mathcal{H})$, the A-numerical range of T, denoted by $W_A(T)$, is defined as $W_A(T) = \{\langle Tx, x\rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}$ and the A-numerical radius of T,

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denoted by $w_A(T)$, is defined as $w_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}$ (see [20]). It is well-known that $w_A(\cdot)$ and $\|\cdot\|_A$ are equivalent seminorms on $\mathcal{B}_A(\mathcal{H})$ satisfying the following inequality (see [2, Prop. 2.5])

$$\frac{1}{2}\|T\|_A \le w_A(T) \le \|T\|_A. \tag{1.1}$$

The inequalities in (1.1) are sharp (see [13]). In particular, $w_A(T) = \frac{1}{2}\|T\|_A$ if $AT^2 = 0$ and $w_A(T) = \|T\|_A$ if $AT = T^*A$. An improvement of (1.1) is given in [14, 15, 21], which is

$$\frac{1}{4}\left\|T^\sharp T + TT^\sharp\right\|_A \le w_A^2(T) \le \frac{1}{2}\left\|T^\sharp T + TT^\sharp\right\|_A. \tag{1.2}$$

More refinements in this direction are also given in [3–7, 10, 11, 17–19].

In this paper, we obtain new refinements of the first inequalities in (1.1) and (1.2). By applying these new refinements, we obtain upper bounds for the A-numerical radius of the commutators of operators. Also, we obtain characterizations for the equality of first inequalities in (1.1) and (1.2). The results obtained here generalize the existing results in [8, 9].

2. Background

It is well-known that $\mathcal{B}^A(\mathcal{H})$ is, in general, not a sub-algebra of $\mathcal{B}(\mathcal{H})$. Note that $\|T\|_A = 0$ if and only if $ATA = 0$. By Douglas theorem [12], it follows that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : R(T^*A) \subseteq R(A)\}.$$

Note that $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A. We note that $\mathcal{B}_A(\mathcal{H}) \left(\subseteq \mathcal{B}^A(\mathcal{H})\right)$ is a sub-algebra of $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}_A(\mathcal{H})$, we have, $AT^\sharp = T^*A$ and $N(T^\sharp) = N(T^*A)$, where $N(T)$ denotes the kernel of T. If $T \in \mathcal{B}_A(\mathcal{H})$, then $T^\sharp \in \mathcal{B}_A(\mathcal{H})$ and $(T^\sharp)^\sharp = P_{\overline{R(A)}} T P_{\overline{R(A)}}$, where $P_{\overline{R(A)}}$ is the orthogonal projection onto $\overline{R(A)}$. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-self-adjoint if AT is self-adjoint, i.e., $AT = T^*A$. For further study on the A-adjoint operator, we refer to [1]. Note that, for $T, S \in \mathcal{B}_A(\mathcal{H})$, $(TS)^\sharp = S^\sharp T^\sharp$, $\|TS\|_A \le \|T\|_A \|S\|_A$ and $\|Tx\|_A \le \|T\|_A \|x\|_A$ for all $x \in \mathcal{H}$. Clearly, for $T \in \mathcal{B}_A(\mathcal{H})$, $\|TT^\sharp\|_A = \|T^\sharp T\|_A = \|T^\sharp\|_A^2 = \|T\|_A^2$. It was shown in [21] that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$w_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{\mathrm{i}\theta} T + (e^{\mathrm{i}\theta} T)^\sharp}{2} \right\|_A.$$

For $T \in \mathcal{B}_A(\mathcal{H})$, we have $\|T\|_A = \sup\{\|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1\} = \sup\{|\langle Tx, y\rangle_A| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}$.

3. Lower bounds for A-numerical radius

We begin with the observation that any $T \in \mathcal{B}_A(\mathcal{H})$ can be expressed as $T = \Re_A(T) + \mathrm{i}\Im_A(T)$, where $\Re_A(T) = \frac{T+T^\sharp}{2}$ and $\Im_A(T) = \frac{T-T^\sharp}{2\mathrm{i}}$. It is easy to verify that $\Re_A(T)$ and $\Im_A(T)$ both are A-self-adjoint, i.e., $A\Re_A(T) = (\Re_A(T))^*A$ and $A\Im_A(T) = (\Im_A(T))^*A$. Therefore, $w_A(\Re_A(T)) = \|\Re_A(T)\|_A$ and $w_A(\Im_A(T)) = \|\Im_A(T)\|_A$. Now, we are in a position to prove our first improvement.

Theorem 3.1. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$w_A(T) \ge \frac{\|T\|_A}{2} + \frac{\left|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A\right|}{2}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then from $T = \Re_A(T) + \mathrm{i}\Im_A(T)$, we have

$$|\langle Tx, x\rangle_A|^2 = |\langle \Re_A(T)x, x\rangle_A|^2 + |\langle \Im_A(T)x, x\rangle_A|^2.$$

This implies that

$$|\langle Tx, x\rangle_A| \geq |\langle \Re_A(T)x, x\rangle_A| \text{ and } |\langle Tx, x\rangle_A| \geq |\langle \Im_A(T)x, x\rangle_A|.$$

Considering supremum over $\|x\|_A = 1$, we get

$$w_A(T) \geq \|\Re_A(T)\|_A \text{ and } w_A(T) \geq \|\Im_A(T)\|_A.$$

Hence,

$$\begin{aligned} w_A(T) &\geq \max\{\|\Re_A(T)\|_A, \|\Im_A(T)\|_A\} \\ &= \frac{\|\Re_A(T)\|_A + \|\Im_A(T)\|_A}{2} + \frac{|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A|}{2} \\ &\geq \frac{\|\Re_A(T) + \mathrm{i}\Im_A(T)\|_A}{2} + \frac{|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A|}{2} \\ &= \frac{\|T\|_A}{2} + \frac{|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A|}{2}. \end{aligned}$$

Thus, we complete the proof. □

Remark 3.2. Clearly, the inequality in Theorem 3.1 is sharper than the first inequality in (1.1), i.e., $w_A(T) \geq \frac{\|T\|_A}{2}$.

In the next theorem we provide a characterization for the equality of lower bound of A-numerical radius mentioned in (1.1).

Theorem 3.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$.*
(i) If $w_A(T) = \frac{\|T\|_A}{2}$, then

$$\|\Re_A(T)\|_A = \|\Im_A(T)\|_A = \frac{\|T\|_A}{2}.$$

However, the converse is not necessarily true.
(ii) $w_A(T) = \frac{\|T\|_A}{2}$ if and only if $\|\Re_A(e^{\mathrm{i}\theta}T)\|_A = \|\Im_A(e^{\mathrm{i}\theta}T)\|_A = \frac{\|T\|_A}{2}$ for all $\theta \in \mathbb{R}$.

Proof. (i) It follows from Theorem 3.1 that if $w_A(T) = \frac{\|T\|_A}{2}$, then

$$\|\Re_A(T)\|_A = \|\Im_A(T)\|_A.$$

Also, we get

$$\begin{aligned} \|\Re_A(T)\|_A \leq w_A(T) = \frac{\|T\|_A}{2} = \frac{\|\Re_A(T) + \mathrm{i}\Im_A(T)\|_A}{2} &\leq \frac{\|\Re_A(T)\|_A + \|\Im_A(T)\|_A}{2} \\ &= \|\Re_A(T)\|_A. \end{aligned}$$

This implies that $\|\Re_A(T)\|_A = \frac{\|T\|_A}{2}$, and so $\|\Im_A(T)\|_A = \frac{\|T\|_A}{2}$.

(ii) The "if" part follows from $w_A(T) = \sup_{\theta\in\mathbb{R}} \|\Re_A(e^{\mathrm{i}\theta}T)\|_A$, and so we only need to prove the "only if" part. Let $w_A(T) = \frac{\|T\|_A}{2}$. Clearly $e^{\mathrm{i}\theta}T \in \mathcal{B}_A(\mathcal{H})$ for all $\theta \in \mathbb{R}$. Now, $w_A(e^{\mathrm{i}\theta}T) = w_A(T)$ and $\|e^{\mathrm{i}\theta}T\|_A = \|T\|_A$ for all $\theta \in \mathbb{R}$. Therefore, it follows from (i) that $\|\Re_A(e^{\mathrm{i}\theta}T)\|_A = \|\Im_A(e^{\mathrm{i}\theta}T)\|_A = \frac{\|T\|_A}{2}$ for all $\theta \in \mathbb{R}$. □

Our next improvement of the first inequality in (1.2) reads as follows.

Theorem 3.4. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$w_A(T) \geq \sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A + \frac{1}{2}\left|\ \|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\ \right|}.$$

Proof. We have $w_A(T) \geq \|\Re_A(T)\|_A$ and $w_A(T) \geq \|\Im_A(T)\|_A$ and so

$$\begin{aligned} w_A^2(T) &\geq \max\left\{\|\Re_A(T)\|_A^2, \|\Im_A(T)\|_A^2\right\} \\ &= \frac{\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2}{2} + \frac{\left|\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2} \\ &\geq \frac{\|(\Re_A(T))^2\|_A + \|(\Im_A(T))^2\|_A}{2} + \frac{\left|\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2} \\ &\geq \frac{\|(\Re_A(T))^2 + (\Im_A(T))^2\|_A}{2} + \frac{\left|\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2} \\ &= \frac{1}{4}\left\|T^\sharp T + TT^\sharp\right\|_A + \frac{1}{2}\left|\ \|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\ \right|. \end{aligned}$$

This completes the proof.

□

Remark 3.5. Clearly, the inequality in Theorem 3.4 is sharper than the first inequality in (1.2), i.e., $w_A^2(T) \geq \frac{1}{4}\left\|T^\sharp T + TT^\sharp\right\|_A$.

In the next theorem we prove an equivalent condition for $w_A(T) = \sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A}$.

Theorem 3.6. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then, $w_A(T) = \sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A}$ if and only if $\|\Re_A(e^{i\theta}T)\|_A^2 = \|\Im(e^{i\theta}T)\|_A^2 = \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A$ for all $\theta \in \mathbb{R}$.*

Proof. The "if" part is immediate from $w_A^2(T) = \sup_{\theta\in\mathbb{R}} \|\Re_A(e^{i\theta}T)\|_A^2$, so we only prove the "only if" part. Let $w_A^2(T) = \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A$. Now, $\left(\Re_A(e^{i\theta}T)\right)^2 + \left(\Im_A(e^{i\theta}T)\right)^2 = \frac{T^\sharp T + TT^\sharp}{2}$ for all $\theta \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A &= \frac{1}{2}\left\|\left(\Re_A(e^{i\theta}T)\right)^2 + \left(\Im_A(e^{i\theta}T)\right)^2\right\|_A \\ &\leq \frac{1}{2}\left(\left\|\Re_A(e^{i\theta}T)\right\|_A^2 + \left\|\Im_A(e^{i\theta}T)\right\|_A^2\right) \\ &\leq w_A^2(T) = \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A. \end{aligned}$$

Hence, $\left\|\Re_A(e^{i\theta}T)\right\|_A^2 + \left\|\Im_A(e^{i\theta}T)\right\|_A^2 = \frac{1}{2}\|T^\sharp T + TT^\sharp\|_A$. Now, $\sup_{\theta\in\mathbb{R}}\left\|\Re_A(e^{i\theta}T)\right\|_A^2 = \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A = \sup_{\theta\in\mathbb{R}}\left\|\Im_A(e^{i\theta}T)\right\|_A^2$. Therefore, $\|\Re_A(e^{i\theta}T)\|_A^2 = \|\Im_A(e^{i\theta}T)\|_A^2 = \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A$ for all $\theta \in \mathbb{R}$. □

Again, we obtain another characterizations for the equalities $w_A(T) = \frac{1}{2}\|T\|_A$ and $w_A(T) = \sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A}$, respectively. First we need to prove the following lemma.

Lemma 3.7. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then, $\|\Re_A(e^{i\theta}T)\|_A = k$ (i.e., a constant) for all $\theta \in \mathbb{R}$ if and only if $W_A(T)$ is a circular disk with center at the origin and radius k.*

Proof. The "if" part is trivial, we only prove the "only if" part. Let $\left\|\Re_A(e^{i\theta}T)\right\|_A = k$ for all $\theta \in \mathbb{R}$. Then, $\sup_{\|x\|_A=1} |\langle\Re_A(e^{i\theta}T)x, x\rangle_A| = k$ for all $\theta \in \mathbb{R}$, i.e., $\sup_{\|x\|_A=1} |Re(e^{i\theta}\langle Tx, x\rangle_A)| = k$ for all $\theta \in \mathbb{R}$. Thus, for each $\theta \in \mathbb{R}$, there exists a sequence $\{x_n^\theta\} \subseteq \mathcal{H}$ with $\|x_n^\theta\|_A = 1$ such that $|Re(e^{i\theta}\langle Tx_n^\theta, x_n^\theta\rangle_A)| \to k$. This implies that the boundary of $W_A(T)$ must be a circle with center at the origin and radius k. Since $W_A(T)$ is a convex subset of $\mathbb{C}$ (see in [2, Th. 2.1]), so $W_A(T)$ is a circular disk with center at the origin and radius k.

□

Theorem 3.8. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then, the following results hold.*
(i) $w_A(T) = \frac{1}{2}\|T\|_A$ if and only if $W_A(T)$ is a circular disk with center at the origin and radius $\frac{1}{2}\|T\|_A$.
(ii) $w_A(T) = \sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A}$ if and only if $W_A(T)$ is a circular disk with center at the origin and radius $\sqrt{\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A}$.

Proof. The proof of (i) and (ii) follow from Theorem 3.3 (ii) and Theorem 3.6, respectively, by using Lemma 3.7. □

Another improvement of the first inequality in (1.1) reads as follows:

Theorem 3.9. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$w_A(T) \geq \frac{\|T\|_A}{2} + \frac{|\ \|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\ |}{2\sqrt{2}}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then, we have

$$\begin{aligned} |\ \langle Tx, x\rangle_A\ | &= \sqrt{|\langle \Re_A(T)x, x\rangle_A|^2 + |\langle \Im_A(T)x, x\rangle_A|^2} \\ &\geq \frac{1}{\sqrt{2}}(|\ \langle \Re_A(T)x, x\rangle_A\ | + |\ \langle \Im_A(T)x, x\rangle\ |) \\ &\geq \frac{1}{\sqrt{2}}\ |\ \langle (\Re_A(T) \pm \Im_A(T))x, x\rangle_A\ |\ . \end{aligned}$$

Taking supremum over $\|x\|_A = 1$, we get

$$w_A(T) \geq \frac{1}{\sqrt{2}}\|\Re_A(T) \pm \Im_A(T)\|_A.$$

Therefore, we have

$$\begin{aligned} w_A(T) &\geq \frac{1}{\sqrt{2}}\max\{\|\Re_A(T) + \Im_A(T)\|_A, \|\Re_A(T) - \Im_A(T)\|_A\} \\ &= \frac{1}{\sqrt{2}}\Big\{\frac{\|\Re_A(T) + \Im_A(T)\|_A + \|\Re_A(T) - \Im_A(T)\|_A}{2} \\ &\quad + \frac{|\ \|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\ |}{2}\Big\} \\ &\geq \frac{1}{\sqrt{2}}\Big\{\frac{\|(\Re_A(T) + \Im_A(T)) - \mathrm{i}(\Re_A(\mathrm{T}) - \Im_A(\mathrm{T}))\|_A}{2} \\ &\quad + \frac{|\ \|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\ |}{2}\Big\} \\ &= \frac{1}{\sqrt{2}}\Big\{\frac{\|(1-\mathrm{i})\mathrm{T}\|_A}{2} + \frac{|\ \|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\ |}{2}\Big\} \\ &= \frac{\|T\|_A}{2} + \frac{|\ \|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\ |}{2\sqrt{2}}, \end{aligned}$$

as desired. □

Remark 3.10. (i) Clearly, the inequality in Theorem 3.9 is sharper than the first inequality in (1.1), i.e., $w_A(T) \geq \frac{\|T\|_A}{2}$.
(ii) The inequalities obtained in Theorem 3.1 and Theorem 3.9 are not comparable, in general. As for example, we consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, Theorem 3.1 gives $w_A(T) \geq 1$, whereas Theorem 3.9 gives $w_A(T) \geq \frac{1}{2}$. Again, if we consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $T = \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}$, then Theorem 3.1 gives $w_A(T) \geq \frac{1}{\sqrt{2}}$, whereas Theorem 3.9 gives $w_A(T) \geq \frac{2}{\sqrt{2}}$.

Another refinement of the first inequality in (1.2) reads as follows:

Theorem 3.11. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$w_A(T) \geq \sqrt{\frac{\|T^\sharp T + TT^\sharp\|_A}{4} + \frac{\left|\, \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \,\right|}{4}}.$$

Proof. Following the proof of Theorem 3.9, we have

$$\begin{aligned}
w_A^2(T) &\geq \frac{1}{2}\max\{\|\Re_A(T) + \Im_A(T)\|_A^2, \|\Re_A(T) - \Im_A(T)\|_A^2\} \\
&= \frac{1}{2}\Big\{\frac{\|\Re_A(T) + \Im_A(T)\|_A^2 + \|\Re_A(T) - \Im_A(T)\|_A^2}{2} \\
&\quad + \frac{\left|\, \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \,\right|}{2}\Big\} \\
&\geq \frac{1}{2}\Big\{\frac{\|(\Re_A(T) + \Im_A(T))^2 + (\Re_A(T) - \Im_A(T))^2\|_A}{2} \\
&\quad + \frac{\left|\, \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \,\right|}{2}\Big\} \\
&= \frac{\|T^\sharp T + TT^\sharp\|_A}{4} + \frac{\left|\, \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \,\right|}{4}.
\end{aligned}$$

This completes the proof. □

Remark 3.12. (i) Clearly, the inequality in Theorem 3.11 is sharper than the first inequality in (1.2), i.e., $w_A^2(T) \geq \frac{1}{4}\left\|T^\sharp T + TT^\sharp\right\|_A$.
(ii) Considering the same examples as in Remark 3.10 (ii), we conclude that the inequalities obtained in Theorem 3.4 and Theorem 3.11 are not comparable, in general.

4. Applications

In this section we obtain new inequalities for the A-numerical radius of the generalized commutators of operators by applying Theorems 3.4 and 3.11. First we prove the following lemma.

Lemma 4.1. *If $T, X, Y \in \mathcal{B}_A(\mathcal{H})$, then*

$$w_A(TX \pm YT) \leq \max\{\|X\|_A, \|Y\|_A\}\sqrt{2\|T^\sharp T + TT^\sharp\|_A}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$ and $\max\{\|X\|_A, \|Y\|_A\} \leq 1$. Then by Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|\langle (TX \pm YT)x, x\rangle_A| &\leq |\langle Xx, T^\sharp x\rangle_A| + |\langle Tx, Y^\sharp x\rangle_A| \\
&\leq \|T^\sharp x\|_A + \|Tx\|_A \ (\text{since } \|Y\|_A = \|Y^\sharp\|_A \leq 1) \\
&\leq \sqrt{2}\left(\|T^\sharp x\|_A^2 + \|Tx\|_A^2\right)^{\frac{1}{2}} \\
&= \sqrt{2}\langle (T^\sharp T + TT^\sharp)x, x\rangle_A^{\frac{1}{2}} \\
&\leq \sqrt{2}\|T^\sharp T + TT^\sharp\|_A^{\frac{1}{2}}.
\end{aligned}$$

Therefore, taking supremum over $\|x\|_A = 1$, we get

$$w_A(TX \pm YT) \leq \sqrt{2\|T^\sharp T + TT^\sharp\|_A}.$$

If $X, Y \in \mathcal{B}_A(\mathcal{H})$ are arbitrary with $\max\{\|X\|_A, \|Y\|_A\} \neq 0$, then it follows from the above inequality that

$$w_A(TX \pm YT) \leq \max\{\|X\|_A, \|Y\|_A\} \sqrt{2\|T^\sharp T + TT^\sharp\|_A}.$$

Also, if $\max\{\|X\|_A, \|Y\|_A\} = 0$, then the above inequality holds trivially. This completes the proof. □

Theorem 4.2. *If $T, X, Y \in \mathcal{B}_A(\mathcal{H})$, then*

(i) $w_A(TX \pm YT) \leq 2\sqrt{2} \max\{\|X\|_A, \|Y\|_A\} \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2 \right|}{2}}$.

and

(ii) $w_A(TX \pm YT)$

$$\leq 2\sqrt{2} \max\{\|X\|_A, \|Y\|_A\} \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \right|}{4}}.$$

Proof. By applying the inequalities in Theorem 3.4 and Theorem 3.11 in Lemma 4.1, we have (i) and (ii), respectively. □

It should be mentioned here that the inequalities (i) and (ii) in Theorem 4.2 are not comparable, in general. Considering $X = Y = S$ in Theorem 4.2, we get the following corollary.

Corollary 4.3. *If $T, S \in \mathcal{B}_A(\mathcal{H})$, then*

$$(i)\ w_A(TS \pm ST) \leq 2\sqrt{2}\|S\|_A \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2 \right|}{2}}.$$

and

$$\begin{aligned} (ii) \qquad & w_A(TS \pm ST) \\ & \leq 2\sqrt{2}\|S\|_A \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \right|}{4}}. \end{aligned}$$

Now, interchanging T and S in Corollary 4.3 (i), we get

$$w_A(TS \pm ST) \quad \leq \quad 2\sqrt{2} \min\{\alpha_1, \alpha_2\}, \tag{4.1}$$

where

$$\begin{aligned} \alpha_1 &= \|S\|_A \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2 \right|}{2}}, \\ \alpha_2 &= \|T\|_A \sqrt{w_A^2(S) - \frac{\left| \|\Re_A(S)\|_A^2 - \|\Im_A(S)\|_A^2 \right|}{2}}. \end{aligned}$$

Note that the inequality (4.1) is also obtained in [16, Theorem 2.10]. Again, interchanging T and S in Corollary 4.3 (ii), we get

$$w_A(TS \pm ST) \quad \leq \quad 2\sqrt{2} \min\{\beta_1, \beta_2\}, \tag{4.2}$$

where

$$\begin{aligned} \beta_1 &= \|S\|_A \sqrt{w_A^2(T) - \frac{\left| \|\Re_A(T) + \Im_A(T)\|_A^2 - \|\Re_A(T) - \Im_A(T)\|_A^2 \right|}{4}}, \\ \beta_2 &= \|T\|_A \sqrt{w_A^2(S) - \frac{\left| \|\Re_A(S) + \Im_A(S)\|_A^2 - \|\Re_A(S) - \Im_A(S)\|_A^2 \right|}{4}}. \end{aligned}$$

Remark 4.4. In [21, Th. 4.2], Zamani proved that if $T, S \in \mathcal{B}_A(\mathcal{H})$, then

$$w_A(TS \pm ST) \leq 2\sqrt{2}\min\{\|T\|_A w_A(S), \|S\|_A w_A(T)\}.$$

Clearly, both the inequalities in (4.1) and (4.2) are stronger than the inequality in [21, Theorem 4.2].

Statements and Declarations

Data sharing not applicable to this article as no datasets were generated or analysed during the current study. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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