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COMPARISON OF SOME DYNAMICAL SYSTEMS ON THE QUOTIENT SPACE OF THE SIERPINSKI TETRAHEDRON

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Abstract. In this paper, it is aimed to construct two different dynamical systems on the Sierpinski tetrahedron. To this end, we consider the dynamical systems on a quotient space of $\{0, 1, 2, 3\}^{\mathbb{N}}$ by using the code representations of the points on the Sierpinski tetrahedron. Finally, we compare the periodic points to investigate topological conjugacy of these dynamical systems and we conclude that they are not topologically equivalent.

1. Introduction

In the literature, there are many works to analyze the structures on the fractals [1–17]. Defining different dynamical systems on the fractals is one of these studies [3, 4, 8, 17]. With the method given in [4], dynamical systems are naturally constructed on the self-similar sets using their iterated function systems. Moreover, there are different ways to define the dynamical systems on these sets considering their structures. With the help of the folding, expanding, translation and rotation mappings, many dynamical systems can also be obtained on the fractals as given in [17]. On the other hand, expressing the dynamical systems using the code representations of the points can provide many advantages. The utility of this situation can be seen while showing whether these systems are chaotic or not [3, 17]. For this purpose, we also need to use the intrinsic metrics which are defined by means of the code representations on the related fractals. For instance, the intrinsic metric on the Sierpinski tetrahedron (ST) (see Theorem 1) is required to prove that the dynamical system, defined on the code set of ST, is chaotic [3], and it is also used to show some geometrical properties such as number of the geodesics in [9].

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In this paper, we first focus on the quotient space of the Sierpinski tetrahedron $\{0,1,2,3\}^{\mathbb{N}}/_{\sim}$. On this space, we define two dynamical systems $\{ST;G\}$ and $\{ST;T\}$ in Proposition 3 and Proposition 5 respectively. Then we compare their fixed points and deduce that they are not topologically equivalent in Remark 2. On the other hand, in Proposition 4 and Remark 1, we show that $\{ST;G\}$ is topologically equivalent to $\{ST;F\}$ which is given in [3] (see Proposition 1). Hence, we also conclude that $\{ST;G\}$ is chaotic in the sense of Devaney by the help of the topological conjugacy H.

We now recall some basic notions in the following section:

2. Preliminaries

As a fractal, the Sierpinski tetrahedron with vertices are $P_0 = (0,0,0)$, $P_1 = (1,0,0)$, $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $P_3 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$ is the attractor of the iterated function system (IFS) $\{\mathbb{R}^3; f_0, f_1, f_2, f_3\}$ where

$$\begin{aligned}
f_0(x,y,z) &= \left(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\right), \\
f_1(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y, \frac{1}{2}z\right), \\
f_2(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}, \frac{1}{2}z\right), \\
f_3(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{12}, \frac{1}{2}z + \frac{\sqrt{6}}{6}\right).
\end{aligned}$$

Let $ST_i = f_i(ST)$ for $i = 0,1,2,3$. It is obvious that $ST_i \cap ST_j \neq \emptyset$ for $i \neq j$ where $i,j = 0,1,2,3$ and $\bigcup_{i=0}^{3} ST_i = ST$. Suppose that σ is a word of length $k-1$ on the set $\{0,1,2,3\}$ such as $\sigma = a_1 a_2 a_3 \dots a_{k-1}$ where $a_i \in \{0,1,2,3\}$. Similarly, we get $ST_\sigma = f_{a_{k-1}} \circ f_{a_{k-2}} \circ \cdots \circ f_{a_1} \circ f_{a_0}(ST)$. In the Figure 1, one can see that the sub-tetrahedron ST_{313} of ST for $\sigma = 313$. Since $ST_{a_1}, ST_{a_1 a_2}, ST_{a_1 a_2 a_3}, \dots$ is a sequence of the nested sets such that

$$ST_{a_1} \supset ST_{a_1 a_2} \supset ST_{a_1 a_2 a_3} \supset \ldots \supset ST_{a_1 a_2 \ldots a_n} \supset \ldots,$$

$\bigcap_{k=1}^{\infty} ST_\sigma$ indicates a singleton, A, from the Cantor intersection theorem. The code representations of A is the sequence $a_1 a_2 a_3 \dots$ where $a_i \in \{0,1,2,3\}$.

On the other hand, the intersection of the sequences $ST_\sigma, ST_{\sigma\alpha}, ST_{\sigma\alpha\beta}, ST_{\sigma\alpha\beta\beta}, \dots$ and $ST_\sigma, ST_{\sigma\beta}, ST_{\sigma\beta\alpha}, ST_{\sigma\beta\alpha\alpha}, \dots$ satisfying

$$ST_\sigma \supset ST_{\sigma\alpha} \supset ST_{\sigma\alpha\beta} \supset ST_{\sigma\alpha\beta\beta} \supset \ldots$$

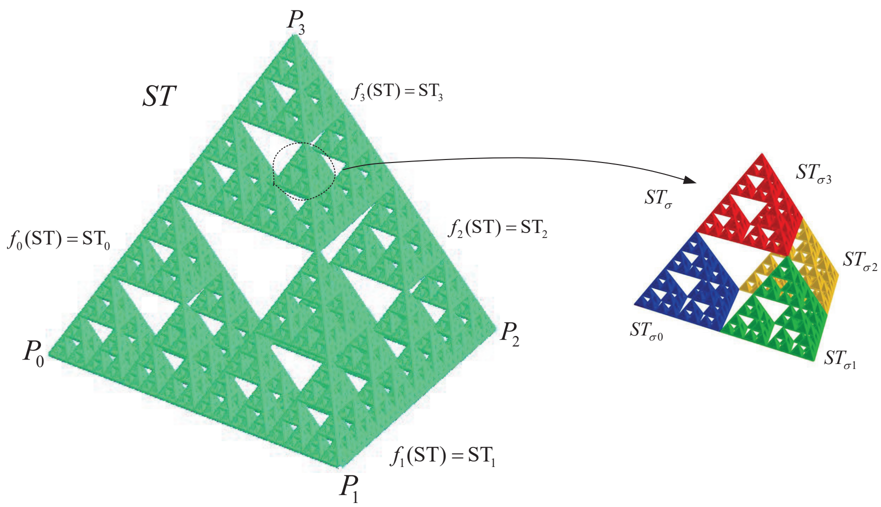

FIGURE 1. The Sierpinski tetrahedron and a small piece ST_σ of ST

and

$$ST_\sigma \supset ST_{\sigma\beta} \supset ST_{\sigma\beta\alpha} \supset ST_{\sigma\beta\alpha\alpha} \supset \dots$$

represents the same point on ST and the code representations of these points are $\sigma\alpha\beta\beta\beta\dots$ and $\sigma\beta\alpha\alpha\alpha\dots$. Therefore, ST can be defined as the quotient space $\{0,1,2,3\}^{\mathbb{N}}/_\sim$ where

$c' \sim c'' \Leftrightarrow c' = c''$ or there are $c_i, \alpha, \beta \in \{0,1,2,3\}$ such that

$c' = c_1c_2\dots c_n\alpha\beta\beta\beta\dots, c'' = c_1c_2\dots c_n\beta\alpha\alpha\alpha\dots$ for an integer n.

The dynamical system, defined in [3] on this quotient space, is given with the following proposition:

Proposition 1. *Let the code representations of points X and Y of the Sierpinski tetrahedron be $x_1x_2x_3\dots$ and $y_1y_2y_3\dots$ respectively. The function $F : ST \to ST$, $F(X) = Y$ such that*

$$y_i \equiv x_{i+1} + x_1 \pmod 4 \tag{1}$$

where $x_i, y_i \in \{0,1,2,3\}$ and $i = 1,2,3,\dots$ is a dynamical system on the code sets of the Sierpinski tetrahedon.

We also give two chaotic dynamical systems on the quotient space of the Sierpinski tetrahedron and we investigate these dynamical systems in terms of topological conjugacy.

Definition 1. *Let $\{X_1; f_1\}$ and $\{X_2; f_2\}$ be two dynamical systems. If there is a homeomorphism $\theta : X_1 \to X_2$ such that $f_2 = \theta \circ f_1 \circ \theta^{-1}$ (or that means $\forall x \in$*

X_1, $\theta(f_1(x)) = f_2(\theta(x)))$, these dynamical systems are equivalent or topologically conjugate. θ is called a topological conjugacy (see [4]).

Proposition 2. *If the dynamical systems $\{X_1; f_1\}$ and $\{X_2; f_2\}$ have the different number of $n-$periodic points for at least $n \in \mathbb{N}$, then they are not topologically conjugate (see [10]).*

Definition 2. *A dynamical system $\{X; f\}$ is chaotic in the sense of Devaney if it is sensitivite dependence on the initial condition, topologically transitive and it has density of periodic points (see [6]).*

We need a useful metric in order to investigate the dynamical systems are chaotic or not. The intrinsic metric on the quotient space of the Sierpinski tetrahedron is formulated with the following theorem:

Theorem 1. *If $a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots$ and $b_1b_2\ldots b_{k-1}b_kb_{k+1}\ldots$ are two representations of the points A and B respectively on the Sierpinski tetrahedron such that $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$, then the formula*

$$d(A,B) = \min\left\{\sum_{i=k+1}^{\infty}\frac{\alpha_i+\beta_i}{2^i},\ \frac{1}{2^k}+\sum_{i=k+1}^{\infty}\frac{\gamma_i+\delta_i}{2^i},\ \frac{1}{2^k}+\sum_{i=k+1}^{\infty}\frac{\phi_i+\varphi_i}{2^i}\right\} \tag{2}$$

such that

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i = c_k \\ 1, & a_i \neq c_k \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases},$$

$$\phi_i = \begin{cases} 0, & a_i = d_k \\ 1, & a_i \neq d_k \end{cases}, \qquad \varphi_i = \begin{cases} 0, & b_i = d_k \\ 1, & b_i \neq d_k \end{cases}$$

where $a_k \neq c_k \neq b_k$ and $a_k \neq d_k \neq b_k$ and $c_k \neq d_k$ $(a_i, b_i, c_k, d_k \in \{0,1,2,3\},\ i = 1,2,3,\ldots)$ gives the distance $d(A,B)$ between the points A and B.

This metric gives the distance of the shortest path between any points on ST.

3. A Chaotic Dynamical System on the Sierpinski Tetrahedron $\{ST; G\}$

In this section, we construct a dynamical system which is different from (1) on ST and we investigate some periodic points of this dynamical system.

Proposition 3. *Let the code representations of $X, Y \in ST$ be $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively where $i = 1,2,3,\ldots$ and $x_i, y_i \in \{0,1,2,3\}$. Suppose that the function $G : ST \to ST$ is defined according to four different situations of x_1 :*

$$G(0x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 2 \\ 2, & x_{i+1} = 3 \\ 3, & x_{i+1} = 0 \end{cases} \quad (i \geq 1)$$

$$G(1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \\ 3, & x_{i+1} = 3 \end{cases} \quad (i \geq 1)$$

$$G(2x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 3 \\ 1, & x_{i+1} = 0 \\ 2, & x_{i+1} = 1 \\ 3, & x_{i+1} = 2 \end{cases} \quad (i \geq 1)$$

$$G(3x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 2 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \end{cases} \quad (i \geq 1).$$

In this case, $\{ST; G\}$ *states a dynamical system.*

Proof. We know from the hypothesis, there are four different rules in regard to the cases of x_1. If X has a unique code representation, then it is obvious that $G(X)$ also has a unique code representation. For $\alpha, \beta \in \{0, 1, 2, 3\}$ and $\alpha \neq \beta$, let $x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots$ and $x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots$ be two different code representations of X then we have

$$G(x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots) = y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots$$

$$G(x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots) = z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots$$

where $y_i, z_i \in \{0, 1, 2, 3\}$. Therefore, we must show that $y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots$ and $z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots$ are different code representations of $G(X)$.

If $x_1 = 0$, then we get

$$y_i \equiv z_i \equiv x_{i+1} + 3 \pmod{4}$$

for $i = 1, 2, 3, \ldots, n-1$ because of the definition of G. As well, for $i = 1, 2, 3, \ldots$

$$y_n \equiv \alpha + 3 \pmod{4},$$

$$y_{n+i} \equiv \beta + 3 \pmod{4},$$

$$z_n \equiv \beta + 3 \pmod{4},$$

$$z_{n+i} \equiv \alpha + 3 \pmod{4}$$

are obtained. Let us define $s_i \equiv x_{i+1} + 3 \pmod{4}$ and $\alpha + 3 \equiv \gamma \pmod{4}$, $\beta + 3 \equiv \delta \pmod{4}$ for $i = 1, 2, 3, \ldots, n-1$. Thus, we get $\gamma \neq \delta$

$$y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots = s_1s_2s_3\ldots s_{n-1}\gamma\delta\delta\delta\ldots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots.$$

For the case $x_1 = 1$, we obtain $y_i = z_i = x_{i+1}$ for $i = 1, 2, 3, \dots, n-1$. What's more, for $i = 1, 2, 3, \dots$

$$\begin{aligned} y_n &= \alpha, \\ y_{n+i} &= \beta, \\ z_n &= \beta, \\ z_{n+i} &= \alpha \end{aligned}$$

are computed. So, we obtain the following results

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = x_2x_3x_4\dots x_{n-1}\alpha\beta\beta\beta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = x_2x_3x_4\dots x_{n-1}\beta\alpha\alpha\alpha\dots.$$

If $x_1 = 2$, then

$$y_i \equiv z_i \equiv x_{i+1} + 1 \pmod{4}$$

where $i = 1, 2, 3, \dots, n-1$. Moreover, for $i = 1, 2, 3, \dots$, we have

$$\begin{aligned} y_n &\equiv \alpha + 1 \pmod{4}, \\ y_{n+i} &\equiv \beta + 1 \pmod{4}, \\ z_n &\equiv \beta + 1 \pmod{4}, \\ z_{n+i} &\equiv \alpha + 1 \pmod{4}. \end{aligned}$$

Hence, we observe that

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\gamma\delta\delta\delta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots$$

for $i = 1, 2, 3, \dots, n-1$ where $s_i \equiv x_{i+1} + 1 \pmod{4}$ and $\alpha + 1 \equiv \gamma \pmod{4}$, $\beta + 1 \equiv \delta \pmod{4}$.

If $x_1 = 3$, then for $i = 1, 2, 3, \dots, n-1$, we get

$$y_i \equiv z_i \equiv x_{i+1} + 2 \pmod{4}.$$

In addition, for $i = 1, 2, 3, \dots$,

$$\begin{aligned} y_n &\equiv \alpha + 2 \pmod{4}, \\ y_{n+i} &\equiv \beta + 2 \pmod{4}, \\ z_n &\equiv \beta + 2 \pmod{4}, \\ z_{n+i} &\equiv \alpha + 2 \pmod{4} \end{aligned}$$

are satisfied. Here, for $i = 1, 2, 3, \dots, n-1$, $s_i \equiv x_{i+1} + 2 \pmod{4}$ and $\alpha + 2 \equiv \gamma \pmod{4}$ and $\beta + 2 \equiv \delta \pmod{4}$. Since, $\gamma \neq \delta$

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\gamma\delta\delta\delta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots$$

are the different code representations of the point $G(X)$. This shows that G is well-defined on the quotient space of ST. Thus, the proof is completed. □

Proposition 4. *Suppose that the code representations of the points $X,\ X' \in ST$ are $x_1x_2x_3\dots$ and $x_1'x_2'x_3'\dots$ respectively where $x_i,\ x_i' \in \{0,1,2,3\}$ for all $i \in \mathbb{N}$.*

There is a function $H : ST \to ST$ such that

$$H(X) = X',\ x_i' = \begin{cases} 0, & x_i = 3 \\ 1, & x_i = 0 \\ 2, & x_i = 1 \\ 3, & x_i = 2 \end{cases} \tag{3}$$

which satisfies $H(F(X)) = G(H(X))$ is a homoemorphism, where F is defined in Proposition 1.

Proof. It is obvious that H is surjective function and $d(H(X),H(Y)) = d(X,Y)$ for all $X,Y \in ST$. So, we conclude that H is a homeomorphism. One can obtain that $H(F(X)) = G(H(X))$ for all $X \in ST$ with easy computations. □

Remark 1. *Since the function $H : ST \to ST$ defined in (3) is a homeomorphism for $\forall X \in ST$, the dynamical systems $\{ST; F\}$ and $\{ST; G\}$ are topologically conjugate. Therefore, $\{ST; G\}$ is also chaotic since $\{ST; F\}$ is chaotic and $\{ST, d\}$ is compact.*

According to Remark 1, the dynamical systems $\{ST; F\}$ and $\{ST; G\}$ are topologically conjugate. In consequence, the number of periodic points of these systems are equal.

While the periodic points of F are known, the periodic points of G can be found with the help of the homeomorphism H in (3). We have the fixed points and $2-$periodic points of F from [3]. Because of the fixed points of F, which are

$$\bullet\overline{0} = 000\dots,\quad \bullet\overline{1032} = 10321032\dots,\quad \bullet\overline{20} = 202020\dots,\quad \bullet\overline{3012} = 30123012\dots$$

the fixed points of G are obtained as follows

$$\bullet H(\overline{0}) = \overline{1},\quad \bullet H(\overline{1032}) = \overline{2103},\quad \bullet H(\overline{20}) = \overline{31},\quad \bullet H(\overline{3012}) = \overline{0123}.$$

Similarly, the $2-$ periodic points of G are

$$\bullet H(\overline{13023120}) = \overline{20130231},\quad \bullet H(\overline{0220}) = \overline{1331},\quad \bullet H(\overline{01302312}) = \overline{12013023}$$

$$\bullet H(\overline{03102132}) = \overline{10213203},\quad \bullet H(\overline{12}) = \overline{23},\quad \bullet H(\overline{11223300}) = \overline{22330011}$$

$$\bullet H(\overline{2200}) = \overline{3311},\quad \bullet H(\overline{21100332}) = \overline{32211003},\quad \bullet H(\overline{23300112}) = \overline{30011223}$$

$$\bullet H(\overline{31021320}) = \overline{02132031},\quad \bullet H(\overline{32}) = \overline{03},\quad \bullet H(\overline{33221100}) = \overline{00332211}.$$

4. A Dynamical System on the Sierpinski Tetrahedron $\{ST;T\}$

We now define a new dynamical system which is not topologically conjugate with $\{ST;G\}$ and automatically with $\{ST;F\}$.

Proposition 5. *The code representations of* $X,Y\in ST$ *are* $x_1x_2x_3\ldots$ *and* $y_1y_2y_3\ldots$ *respectively. The function* $T:ST\to ST$ *are defined for* $i=1,2,3,\ldots$ *and* $x_i,y_i\in\{0,1,2,3\}$ *as follows*

$$T(0x_2x_3\ldots)=x_2x_3x_4\ldots$$

$$T(1x_2x_3\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=3\\ 1, & x_{i+1}=0\\ 2, & x_{i+1}=2\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1).$$

If $x_1=2$*, there are four situations:*

Case 1:

$$T(222\ldots 20x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=0\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1)$$

Case 2:

$$T(222\ldots 21x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=1\\ 3, & x_{i+1}=0\end{cases}\ (i\geq 1)$$

Case 3:

$$T(22\ldots 23x_s\ldots 0x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,$$

where $x_s\in\{2,3\}$ *for* $s<k$

$$y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=0\\ 2, & x_{i+1}=3\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1)$$

Case 4:

$$T(22\ldots 23x_s\ldots 1x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,$$

where $x_s\in\{2,3\}$ *for* $s<k$

$$y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=1\\ 2, & x_{i+1}=3\\ 3, & x_{i+1}=0\end{cases}\ (i\geq 1).$$

(Note that, due to above rules $T(\overline{2})=\overline{0}$, $T(2\overline{3})=\overline{2}$ and $T(23\overline{2})=2\overline{0}$ are obtained.)
If $x_1=3$, then

$$T(3x_2x_3\dots)=y_1y_2y_3\dots,\qquad y_i=\begin{cases}0, & x_{i+1}=1\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=2\\ 3, & x_{i+1}=0\end{cases}\ (i\geq 1).$$

Then, $\{ST;T\}$ is a dynamical system.

Proof. To state that $\{ST;T\}$ is a dynamical system, the images of the points expressed by two different code representations must indicate the same point. For example, $0\overline{1}$ and $1\overline{0}$ or $23\overline{0}$ and $20\overline{3}$ indicates the same point on ST. Thus, we investigate the images of following points $0\overline{1}, 0\overline{2}, 0\overline{3}, 1\overline{0}, 1\overline{2}, 1\overline{3}, 2\overline{0}, 2\overline{1}, 2\overline{3}, 3\overline{0}, 3\overline{1}, 3\overline{2}, 00\overline{1}, 01\overline{0}, 00\overline{2}, 02\overline{0}, 00\overline{3}, 03\overline{0}, 01\overline{2}, 02\overline{1}, 01\overline{3}, 03\overline{1}, 02\overline{3}, 03\overline{2}$ $11\overline{0}, 10\overline{1}, 10\overline{2}, 12\overline{0}, 10\overline{3}, 13\overline{0}, 12\overline{1}, 11\overline{2}, 11\overline{3}, 13\overline{1}, 12\overline{3}, 13\overline{2}, 20\overline{1}, 21\overline{0}, 20\overline{2}, 22\overline{0}, 20\overline{3}, 23\overline{0}, 21\overline{2}, 22\overline{1}, 21\overline{3}, 23\overline{1}, 23\overline{2}, 22\overline{3}$ and $30\overline{1}, 31\overline{0}, 30\overline{2}, 32\overline{0}, 30\overline{3}, 33\overline{0}, 31\overline{2}, 32\overline{1}, 31\overline{3}, 33\overline{1}, 32\overline{3}, 33\overline{2}$. So, we get the following results,

$$\begin{array}{lll}
T(0\overline{1})=\ \overline{1}, & T(0\overline{2})=\ \overline{2}, & T(0\overline{3})=\ \overline{3},\\
T(1\overline{0})=\ \overline{1}, & T(2\overline{0})=\ \overline{2}, & T(3\overline{0})=\ \overline{3},\\[1ex]
T(1\overline{2})=\ \overline{2}, & T(1\overline{3})=\ \overline{0}, & T(2\overline{3})=\ \overline{2},\\
T(2\overline{1})=\ \overline{2}, & T(3\overline{1})=\ \overline{0}, & T(3\overline{2})=\ \overline{2},
\end{array}$$

$$\begin{array}{lll}
T(00\overline{1})=\ 0\overline{1}, & T(00\overline{2})=\ 0\overline{2}, & T(00\overline{3})=\ 0\overline{3},\\
T(01\overline{0})=\ 1\overline{0}, & T(02\overline{0})=\ 2\overline{0}, & T(03\overline{0})=\ 3\overline{0},\\[1ex]
T(01\overline{2})=\ 1\overline{2}, & T(01\overline{3})=\ 1\overline{3}, & T(02\overline{3})=\ 2\overline{3},\\
T(02\overline{1})=\ 2\overline{1}, & T(03\overline{1})=\ 3\overline{1}, & T(03\overline{2})=\ 3\overline{2},\\[1ex]
T(10\overline{1})=\ 1\overline{3}, & T(10\overline{2})=\ 1\overline{2}, & T(10\overline{3})=\ 1\overline{0},\\
T(11\overline{0})=\ 3\overline{1}, & T(12\overline{0})=\ 2\overline{1}, & T(13\overline{0})=\ 0\overline{1},\\[1ex]
T(11\overline{2})=\ 3\overline{2}, & T(11\overline{3})=\ 3\overline{0}, & T(12\overline{3})=\ 2\overline{0},\\
T(12\overline{1})=\ 2\overline{3}, & T(13\overline{1})=\ 0\overline{3}, & T(13\overline{2})=\ 0\overline{2},\\[1ex]
T(20\overline{1})=\ 2\overline{3}, & T(20\overline{2})=\ 2\overline{0}, & T(20\overline{3})=\ 2\overline{1},\\
T(21\overline{0})=\ 2\overline{3}, & T(22\overline{0})=\ 0\overline{2}, & T(23\overline{0})=\ 2\overline{1},\\[1ex]
T(21\overline{2})=\ 2\overline{0}, & T(21\overline{3})=\ 2\overline{1}, & T(22\overline{3})=\ 0\overline{2},\\
T(22\overline{1})=\ 0\overline{2}, & T(23\overline{1})=\ 2\overline{1}, & T(23\overline{2})=\ 2\overline{0},\\[1ex]
T(30\overline{1})=\ 3\overline{0}, & T(30\overline{2})=\ 3\overline{2}, & T(30\overline{3})=\ 3\overline{1},\\
T(31\overline{0})=\ 0\overline{3}, & T(32\overline{0})=\ 2\overline{3}, & T(33\overline{0})=\ 1\overline{3},\\[1ex]
T(31\overline{2})=\ 0\overline{2}, & T(31\overline{3})=\ 0\overline{1}, & T(32\overline{3})=\ 2\overline{1}\\
T(32\overline{1})=\ 2\overline{0}, & T(33\overline{1})=\ 1\overline{0}, & T(33\overline{2})=\ 1\overline{2}
\end{array}.$$

As seen from above, the image of the different code representations of the same points state the same addresses.

In general, if $\sigma = x_1x_2x_3 \ldots x_n$ then $\sigma 0\overline{1}$ and $\sigma 1\overline{0}$, $\sigma 1\overline{2}$ and $\sigma 2\overline{1}$, $\sigma 0\overline{2}$ and $\sigma 2\overline{0}$, $\sigma 0\overline{3}$ and $\sigma 3\overline{0}$, $\sigma 1\overline{3}$ and $\sigma 3\overline{1}$, $\sigma 3\overline{2}$ and $\sigma 2\overline{3}$, $\sigma 00\overline{1}$ and $\sigma 01\overline{0}$, $\sigma 00\overline{2}$ and $\sigma 02\overline{0}$, $\sigma 00\overline{3}$ and $\sigma 03\overline{0}$, $\sigma 01\overline{2}$ and $\sigma 02\overline{1}$, $\sigma 01\overline{3}$ and $\sigma 03\overline{1}$, $\sigma 02\overline{3}$ and $\sigma 03\overline{2}$, $\sigma 11\overline{0}$ and $\sigma 10\overline{1}$, $\sigma 10\overline{2}$ and $\sigma 12\overline{0}$, $\sigma 10\overline{3}$ and $\sigma 13\overline{0}$, $\sigma 12\overline{1}$ and $\sigma 11\overline{2}$, $\sigma 11\overline{3}$ and $\sigma 13\overline{1}$, $\sigma 12\overline{3}$ and $\sigma 13\overline{2}$, $\sigma 20\overline{1}$ and $\sigma 21\overline{0}$, $\sigma 20\overline{2}$ and $\sigma 22\overline{0}$, $\sigma 20\overline{3}$ and $\sigma 23\overline{0}$, $\sigma 21\overline{2}$ and $\sigma 22\overline{1}$, $\sigma 21\overline{3}$ and $\sigma 23\overline{1}$, $\sigma 22\overline{3}$ and $\sigma 23\overline{2}$, $\sigma 30\overline{1}$ and $\sigma 31\overline{0}$, $\sigma 30\overline{2}$ and $\sigma 32\overline{0}$, $\sigma 30\overline{3}$ and $\sigma 33\overline{0}$, $\sigma 31\overline{2}$ and $\sigma 32\overline{1}$, $\sigma 31\overline{3}$ and $\sigma 33\overline{1}$, $\sigma 32\overline{3}$ and $\sigma 33\overline{2}$ are different representations of same points and the image of these sequences represents the same addresses independently of σ. This shows that T is well-defined on ST. □

We can compute the $n-$ periodic points of T by using the equation

$$T^n(x_1x_2x_3\ldots) = x_1x_2x_3\ldots.$$

Since $T(\overline{0}) = \overline{0}$, $T(\overline{103}) = \overline{103}$, $T(\overline{301}) = \overline{301}$, $T(\overline{20}) = \overline{20}$ and $T(\overline{2130}) = \overline{2130}$,

$$\bullet\overline{0} = 00\ldots, \quad \bullet\overline{103} = 103103\ldots, \quad \bullet\overline{301} = 301301\ldots,$$

$$\bullet\overline{20} = 2020\ldots, \quad \bullet\overline{2130} = 21302130\ldots$$

are the fixed points of T.

Moreover,

$$\bullet\overline{013} = 013013\ldots, \quad \bullet\overline{031} = 031031\ldots, \quad \bullet\overline{0220} = 02200220\ldots$$

$$\bullet\overline{02211330} = 0221133002211330\ldots, \quad \bullet\overline{1} = 111\ldots, \quad \bullet\overline{130} = 130130\ldots$$

$$\bullet\overline{2010} = 20102010\ldots, \quad \bullet\overline{201030} = 201030201030\ldots$$

$$\bullet\overline{2200} = 22002200\ldots, \quad \bullet\overline{22113300} = 2211330022113300\ldots, \quad \bullet\overline{2320} = 23202320\ldots$$

$$\bullet\overline{232120} = 232120232120\ldots, \quad \bullet\overline{2120} = 21202120\ldots, \quad \bullet\overline{2030} = 20302030\ldots$$

$$\bullet\overline{210} = 210210\ldots, \quad \bullet\overline{230} = 230230\ldots, \quad \bullet\overline{21031230} = 2103123021031230\ldots$$

$$\bullet\overline{23120130} = 2312013023120130\ldots, \quad \bullet\overline{310} = 310310\ldots$$

are $2-$ periodic points of T.

Remark 2. *Since $\{ST; G\}$ and $\{ST; T\}$ have the different number of fixed points, they are not topologically conjugate (see Proposition 2).*

5. Conclusion

This paper gives a way to define different dynamical systems on the Sierpinski tetrahedron. This method can be also used for the other fractals which have the intrinsic metrics defined by using the code representations of the points.

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