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Research Article

Cubic rank transmuted inverse rayleigh distribution: Properties and applications

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ABSTRACT

In this paper, we propose a new lifetime distribution called Cubic Rank Transmuted Inverse Rayleigh as an alternative to the inverse Rayleigh distribution. Some distributional properties of the suggested distribution such as moments, incomplete moments, Bonferroni and Lorenz curves, moment generating function, quantile function, median, mean residual life function are examined. We consider five methods such as maximum likelihood, the least squares, weighted least squares, Anderson Darling method, and Crámer–von-Mises method to estimate the parameters of the proposed distribution. Furthermore, a comprehensive Monte Carlo simulation study is performed to compare the performances of the examined estimators according to mean square errors and biases. Finally, a real data application is given to illustrate the usefulness of the proposed distribution.

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INTRODUCTION

In the last decades, many continuous univariate distributions were suggested to model lifetime data in some areas such as engineering, medicine, agriculture, chemistry, and biology. However, current distributions always cannot be enough to model data in these areas. Therefore new extensions are introduced to generate more flexible distributions by many authors. Shaw and Buckley [1, 2] suggested the quadratic rank transmutation map (QRTM) method to generate new distributions. The cumulative distribution function (cdf) and probability density function (pdf) of the distribution constructed based on the QRTM method are

$$F(x)=(1+\lambda)G(x)-\lambda[G(x)]^2 \tag{1}$$

and

$$f(x)=g(x)[1+\lambda-2\lambda G(x)], \tag{2}$$

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respectively, where $\lambda \in [-1,1]$, and $G(x)$ and $g(x)$ denote the cdf and pdf of any distribution respectively. The suggested distributions by using the QRTM are called as transmuted distributions.

Let X be a random variable having inverse Rayleigh (IR) distribution. The cdf and pdf of IR distribution are

$$G(x;\alpha)=\exp\left[-\left(\frac{\alpha}{x^2}\right)\right] \tag{3}$$

and

$$g(x;\alpha)=\frac{2\alpha}{x^3}\exp\left(\frac{-\alpha}{x^2}\right), \tag{4}$$

respectively, where $\alpha > 0$ and $x > 0$. Ahmad et al. [3] suggested transmuted inverse Rayleigh (TIR) distribution by substituting (3)-(4) into (1)-(2). The cdf and pdf of this distribution are given as follows:

$$F_{TIR}(x;\alpha,\lambda)=\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]\left[1+\lambda-\lambda\exp\left(-\left(\frac{\alpha}{x^2}\right)\right)\right] \tag{5}$$

and

$$f_{TIR}(x;\alpha,\lambda)=\frac{2\alpha}{x^3}\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]\begin{bmatrix}1+\lambda-2\lambda\exp\\ \left(-\left(\frac{\alpha}{x^2}\right)\right)\end{bmatrix}, \tag{6}$$

respectively, where $-1 \le \lambda \le 1$, $\alpha > 0$ and $x > 0$.

There are many studies on transmuted distributions in literature, and these can be listed as follows: Granzotto and Louzada [4] introduced the transmuted log-logistic distribution. Alizadeh et al. [5] proposed a generalized transmuted family of distributions. Merovci et al. [6] suggested another generalization of transmuted distributions. Nofal et al. [7] proposed Kumaraswamy transmuted exponentiated additive Weibull distribution and its some properties. Merovci et al. [8] proposed the exponentiated transmuted-G family of distributions. Bhatti et al. [9] introduced a new distribution called the transmuted geometric-quadratic hazard rate distribution. Alizadeh et al. [10] suggested the complementary generalized transmuted Poisson-G family of distributions. Tanış et al. [11] introduced the transmuted complementary exponential power distribution. Saraçoğlu and Tanış [12] proposed a new special case of the family of transmuted distributions called transmuted exponential power distribution.

On the other hand, Granzotto et al. [13] introduced a new method called cubic rank transmutation map (CRTM) to generate new distributions as an alternative to QRTM. The cdf and pdf of a distribution constructed by using the CRTM are given by

$$F(x)=\lambda_1 G(x)+(\lambda_2-\lambda_1)G^2(x)+(1-\lambda_2)G^3(x) \tag{7}$$

and

$$f(x)=g(x)\left[\lambda_1+2(\lambda_2-\lambda_1)G(x)+3(1-\lambda_2)G^2(x)\right], \tag{8}$$

respectively, where $\lambda_1 \in [0,1]$, $\lambda_2 \in [-1,1]$ and $G(x)$ and $g(x)$ denote the cdf and pdf of any distribution respectively. Granzotto et al. [13] suggested two new distributions called cubic rank transmuted Weibull and cubic rank transmuted log-logistic distributions via the CRTM. Aslam et al. [14] provided a new family of transmuted distributions called cubic transmuted-G family of distributions. Then, Saraçoğlu and Tanış [15] proposed cubic rank transmuted Kumaraswamy distribution which is special case of the family of cubic rank transmuted distributions. Bhatti et al. [16] introduced the cubic rank transmuted modified Burr III distribution. Bhatti et al. [17] suggested cubic rank transmuted modified Burr III Pareto distribution. Hameldarbandi and Yılmaz [18] discussed the methodology of cubic rank transmuted distributions.

The purpose of this paper is to suggest a new useful lifetime distribution as an alternative to inverse Rayleigh, and its competitor ones, and to describe some characteristic properties. This study is organized as follows: In Section 2, we introduced a new distribution called as Cubic Rank Transmuted Inverse Rayleigh Distribution (CRTIR) , and its distributional properties. Then, five estimation methods are considered to estimate the parameters of the proposed distribution in Section 3. Section 4 presents an extensive Monte Carlo simulation study to compare the performances of examined estimators in terms of biases and MSEs. In Section 5, we perform a real data application to illustrate the applicability of suggested distribution in real life. Finally, the conclusions are given in Section 6.

CUBIC RANK TRANSMUTED INVERSE RAYLEIGH (CRTIR) DISTRIBUTION

In this section, we propose a new lifetime distribution which is a special case based on IR distribution of the family of cubic rank transmuted distributions. The suggested distribution is called cubic rank transmuted inverse Rayleigh (CRTIR) distribution. By substituting (3)-(4) into (7)-(8) the cdf and pdf of CRTIR are obtained by

$$\begin{aligned}F_{CRTIR}(x;\alpha,\lambda_1,\lambda_2)={}&\lambda_1\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]+(\lambda_2-\lambda_1)\\ &\exp\left[-\left(\frac{2\alpha}{x^2}\right)\right]+(1-\lambda_2)\exp\left[-\left(\frac{3\alpha}{x^2}\right)\right]\end{aligned} \tag{9}$$

and

$$f_{CRTIR}\left(x;\alpha,\lambda_1,\lambda_2\right)=\frac{2\alpha}{x^3}\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]$$
$$\left[\lambda_1+2\left(\lambda_2-\lambda_1\right)\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]+3\left(1-\lambda_2\right)\exp\left[-\left(\frac{2\alpha}{x^2}\right)\right]\right], \tag{10}$$

respectively, where $\alpha > 0$, $\lambda_1 \in [0,1]$, $\lambda_2 \in [-1,1]$ and $x > 0$. In this paper, the CRTIR distribution is briefly denoted by CRTIR $(\alpha,\lambda_1,\lambda_2)$. The CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution reduces IR distribution for $\lambda_1 = \lambda_2 = 1$. The hazard function (hf) of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is given by

$$h_{CRTIR}\left(x;\alpha,\lambda_1,\lambda_2\right)=\frac{\left[\begin{array}{l}\lambda_1+2\left(\lambda_2-\lambda_1\right)\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]\\+3\left(1-\lambda_2\right)\exp\left[-\left(\frac{2\alpha}{x^2}\right)\right]\end{array}\right]}{1-\left[\begin{array}{l}\lambda_1\exp\left[-\left(\frac{\alpha}{x^2}\right)\right]+\left(\lambda_2-\lambda_1\right)\\\exp\left[-\left(\frac{2\alpha}{x^2}\right)\right]+\left(1-\lambda_2\right)\\\exp\left[-\left(\frac{3\alpha}{x^2}\right)\right]\end{array}\right]} \times\frac{\frac{2\theta}{x^3}\exp\left[-\left(\frac{\theta}{x^2}\right)\right]}{1-\left[\begin{array}{l}\lambda_1\exp\left[-\left(\frac{\theta}{x^2}\right)\right]+\left(\lambda_2-\lambda_1\right)\\\exp\left[-\left(\frac{2\theta}{x^2}\right)\right]+\left(1-\lambda_2\right)\\\exp\left[-\left(\frac{3\theta}{x^2}\right)\right]\end{array}\right]} \tag{11}$$

Figures 1-2 illustrate some of the possible shapes of the pdf and hf for CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution, respectively.

According to Figure 2, it is seen that the hf of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is upside bathtub shaped for selected values of parameters.

QUANTILE FUNCTION AND MEDIAN

The p^{th} quantile $Q(p)$ of the CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is positive real solution of the Eq. (12);

$$Q(p)=-\frac{\sqrt{-\alpha\log(h)}}{\log(h)} \tag{12}$$

where

$$h=\left(a+\sqrt{a^2+b^3}\right)^{\frac{1}{3}}+\left(a-\sqrt{a^2+b^3}\right)^{\frac{1}{3}}-\frac{\left(\lambda_2-\lambda_1\right)}{3-3\lambda_2}$$

$$a=\frac{9\left(1-\lambda_2\right)\left(\lambda_2-\lambda_1\right)\lambda_1+27\left(1-\lambda_2\right)^2p-2\left(\lambda_2-\lambda_1\right)^3}{54\left(1-\lambda_2\right)^3}$$

$$b=\frac{3\left(1-\lambda_2\right)\lambda_1-\left(\lambda_2-\lambda_1\right)^2}{9\left(1-\lambda_2\right)^2}$$

and, $p \in [0,1]$. Thus, the median can be obtained by taking $p=\frac{1}{2}$ in Eq. (12).

RANDOM NUMBERS GENERATION

In order to generate data from CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution, an acceptance-rejection (AR) sampling method is given in the following algorithm. In this algorithm, the Weibull distribution is chosen as a proposal distribution. The AR algorithm is given as follows:

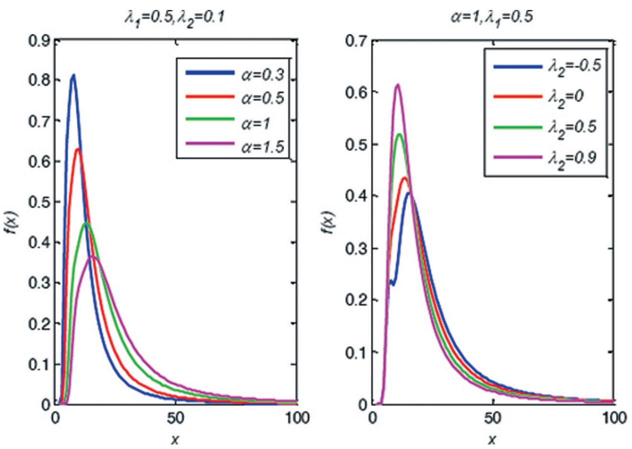

Figure 1. The pdfs of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution for some selected parameters.

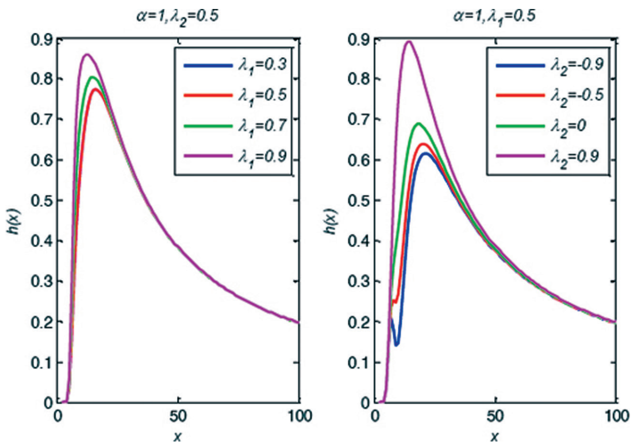

Figure 2. The hfs of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution for some selected parameters.

Algorithm 1

A1. Generate data on random variable Y from Weibull distribution with pdf g given as follows:

$$g(y;\theta,\beta)=\frac{\theta}{\beta}\left(\frac{y}{\beta}\right)^{\theta-1}\exp\left\{\left(\frac{y}{\beta}\right)^{\theta}\right\},$$

A2. Generate U from standard uniform distribution (independent of Y).

$$U<\frac{f_{CRTIR}(Y;\alpha,\lambda_1,\lambda_2)}{k\times g(Y;\theta,\beta)}$$

A3.

Then set $X=Y$("*accept*"); otherwise go back to A1("*reject*"), where pdf of $f_{CRTIR}(\bullet)$ is given in (10) and

$$k=\max_{z\in\mathbb{R}+}\frac{f_{CRTIR}(z;\alpha,\lambda_1,\lambda_2)}{g(z;\theta,\beta)}.$$

The output of this algorithm suggests a random data on X from CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. It is noticed that the Algorithm 1 with the $(\theta,\beta)=(0.2,1)$ is used for all simulations in this study.

MOMENTS

In this subsection, we have obtained r^{th} moment of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. Let X be a random variable having CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. The r^{th} moment of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is given by Theorem 1.

Theorem 1

$$E(X^r)=\alpha^{\frac{r}{2}}\Gamma\left(1-\frac{r}{2}\right)\left[\lambda_1+2^{\frac{r}{2}}(\lambda_2-\lambda_1)+3^{\frac{r}{2}}(1-\lambda_2)\right] \quad (13)$$

Proof

$$\begin{aligned}E(X^r)&=\int_0^\infty x^r f(x;\alpha,\lambda_1,\lambda_2)\\&=\lambda_1\int_0^\infty 2\theta x^{r-3}\exp\left[\left(-\frac{\alpha}{x^2}\right)\right]dx\\&\quad+2(\lambda_2-\lambda_1)\int_0^\infty 2\alpha x^{r-3}\exp\left[\left(-\frac{2\alpha}{x^2}\right)\right]dx\\&\quad+3(1-\lambda_2)\int_0^\infty 2\alpha x^{r-3}\exp\left[\left(-\frac{3\alpha}{x^2}\right)\right]dx\\&=\lambda_1\alpha^{\frac{r}{2}}\Gamma\left(1-\frac{r}{2}\right)+(2\alpha)^{\frac{r}{2}}(\lambda_2-\lambda_1)\Gamma\left(1-\frac{r}{2}\right)\\&\quad+(3\alpha)^{\frac{r}{2}}(1-\lambda_2)\Gamma\left(1-\frac{r}{2}\right)\\&=\alpha^{\frac{r}{2}}\Gamma\left(1-\frac{r}{2}\right)\left[\lambda_1+2^{\frac{r}{2}}(\lambda_2-\lambda_1)+3^{\frac{r}{2}}(1-\lambda_2)\right]\end{aligned}$$

Thus, expected value of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is obtained as follows:

$$E(X)=\sqrt{\alpha}\Gamma\left(\frac{1}{2}\right)\left[\lambda_1+\sqrt{2}(\lambda_2-\lambda_1)+\sqrt{3}(1-\lambda_2)\right] \quad (14)$$

MOMENT GENERATING FUNCTION

The moment generating function of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution, $M_X(t)$, is given by

$$\begin{aligned}M(t)&=\int_0^\infty e^{tx}f(x)dx\\&=\sum_{r=0}^\infty\frac{t^r}{r!}\int_0^\infty x^r f(x)dx\\&=\sum_{r=0}^\infty\frac{t^r}{r!}\lambda_1\alpha^{\frac{r}{2}}\Gamma\left(1-\frac{r}{2}\right)+\sum_{r=0}^\infty\frac{t^r}{r!}(2\alpha)^{\frac{r}{2}}(\lambda_2-\lambda_1)\\&\quad\Gamma\left(1-\frac{r}{2}\right)+\sum_{r=0}^\infty\frac{t^r}{r!}(3\alpha)^{\frac{r}{2}}(1-\lambda_2)\Gamma\left(1-\frac{r}{2}\right)\end{aligned} \quad (15)$$

INCOMPLETE MOMENTS

In this subsection, it is derived r^{th} incomplete moment, $m_r(y)$ of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. The Bonferroni and Lorenz curves are obtained using the first incomplete moment.

Let X be a random variable having CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. The r^{th} incomplete moment of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is

$$\begin{aligned}m_r(y)&=\int_0^y x^r f(x;\alpha,\lambda_1,\lambda_2)dx\\&=\alpha^{\frac{r}{2}}\lambda_1\Gamma\left(1-\frac{r}{2},\frac{\alpha}{y^2}\right)+(2\alpha)^{\frac{r}{2}}(\lambda_2-\lambda_1)\\&\quad\Gamma\left(1-\frac{r}{2},\frac{2\alpha}{y^2}\right)\\&\quad+(3\alpha)^{\frac{r}{2}}(1-\lambda_2)\Gamma\left(1-\frac{r}{2},\frac{3\alpha}{y^2}\right)\end{aligned} \quad (16)$$

where $\Gamma(a,t)$ is incomplete gamma function is defined as follows:

$$\Gamma(a,t)=\int_t^\infty x^{a-1}e^{-x}dx$$

BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves are suggested by Bonferroni [19]. There are many application fields which these curves used. For example, economics, reliability, demography, insurance and medicine, demography, insurance, etc. The Bonferroni and Lorenz curves for CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution are defined by

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx$$
$$=\frac{\lambda_1\Gamma\left(\frac{1}{2},\frac{\alpha}{q^2}\right)+\sqrt{2}(\lambda_2-\lambda_1)\Gamma\left(\frac{1}{2},\frac{2\alpha}{q^2}\right)+\sqrt{3}(1-\lambda_2)\Gamma\left(\frac{1}{2},\frac{3\alpha}{q^2}\right)}{p\Gamma\left(\frac{1}{2}\right)\left[\lambda_1+\sqrt{2}(\lambda_2-\lambda_1)+\sqrt{3}(1-\lambda_2)\right]} \tag{17}$$

and

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx$$
$$=\frac{\lambda_1\Gamma\left(\frac{1}{2},\frac{\alpha}{q^2}\right)+\sqrt{2}(\lambda_2-\lambda_1)\Gamma\left(\frac{1}{2},\frac{2\alpha}{q^2}\right)+\sqrt{3}(1-\lambda_2)\Gamma\left(\frac{1}{2},\frac{3\alpha}{q^2}\right)}{\Gamma\left(\frac{1}{2}\right)\left[\lambda_1+\sqrt{2}(\lambda_2-\lambda_1)+\sqrt{3}(1-\lambda_2)\right]}, \tag{18}$$

respectively, where μ is the first moment given in (14), and $q=Q(p)$ can be calculated using (12) with probability p. Figure 3 illustrates the Bonferroni and Lorenz curves for selected parameters.

MEAN RESIDUAL LIFE FUNCTION

The mean residual life (MRL) or the expected remaining of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is given as follows:

$$\mu(t)=E(T-t\,|\,T>t)$$
$$=\frac{1}{1-F(t)}\int_t^\infty(1-F(x))dx$$
$$=\frac{(\lambda_1-\lambda_2)\sqrt{2\alpha\pi}\,erf\left(\sqrt{2\alpha}/t\right)-(1-\lambda_2)\left[\sqrt{3\alpha\pi}\,erf\left(\sqrt{3\alpha}/t\right)+t\exp\left[-\left(\frac{3\alpha}{t^2}\right)\right]\right]}{1-\lambda_1\exp\left[-\left(\frac{\alpha}{t^2}\right)\right]-(\lambda_2-\lambda_1)\exp\left[-\left(\frac{2\alpha}{t^2}\right)\right]-(1-\lambda_2)\exp\left[-\left(\frac{3\alpha}{t^2}\right)\right]}$$
$$+\frac{t\left[1+(\lambda_1-\lambda_2)\exp\left[-\left(\frac{2\alpha}{t^2}\right)\right]-\lambda_1\exp\left[-\left(\frac{\alpha}{t^2}\right)\right]\right]-\lambda_1\sqrt{\alpha\pi}\,erf\left(\sqrt{\alpha}/t\right)}{1-\lambda_1\exp\left[-\left(\frac{\alpha}{t^2}\right)\right]-(\lambda_2-\lambda_1)\exp\left[-\left(\frac{2\alpha}{t^2}\right)\right]-(1-\lambda_2)\exp\left[-\left(\frac{3\alpha}{t^2}\right)\right]} \tag{19}$$

where error function, $erf(t)$, is defined as follows:

$$erf(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{x^2}dx$$

POINT ESTIMATION

In this section, we consider five estimation methods including maximum likelihood, least squares, weighted least squares, Anderson-Darling method, and Cramer-von Mises method to estimate the parameters θ,λ_1 and λ_2 of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution.

MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2,\ldots, X_n$ be random variables with independently distributed CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. Then, the log-likelihood function is given by

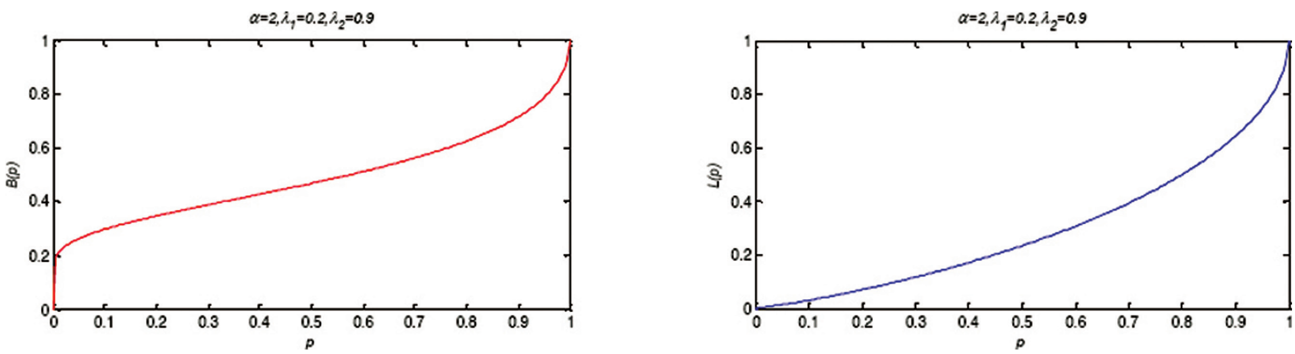

Figure 3. Bonferroni curve (left) Lorenz curve (right) of the CRTIR distribution for selected parameters.

$$\ell(\boldsymbol{\theta}\,|\,\mathbf{x}) = n\log(2\alpha) + (2\alpha-3)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\begin{bmatrix}\lambda_1 + 2(\lambda_2-\lambda_1)\exp\left(-\dfrac{\alpha}{x_i^2}\right)\\ +3(1-\lambda_2)\exp\left(-\dfrac{2\alpha}{x_i^2}\right)\end{bmatrix} \quad (20)$$

where, $\boldsymbol{\theta} = (\alpha,\lambda_1,\lambda_2)$ is a parameter vector, and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$. The MLEs, $\hat{\alpha}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of α, λ_1 and λ_2 parameters can be obtained by simultaneous solutions of the likelihood equaitons. These non-linear equations are obtained by differentiating the $\ell(\alpha,\lambda_1,\lambda_2|\mathbf{x})$ with according to α, λ_1 and λ_2 parameters, and equating to zero. The likelihood equations are

$$\frac{\ell(\boldsymbol{\theta}\,|\,\mathbf{x})}{\partial\alpha} = \frac{n}{\alpha} + 2\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\frac{2(\lambda_2-\lambda_1)\exp\left(\dfrac{-\alpha}{x_i^2}\right) + 6(1-\lambda_2)\exp\left(\dfrac{-2\alpha}{x_i^2}\right)}{x_i^2\begin{pmatrix}\lambda_1 + 2(\lambda_2-\lambda_1)\exp\left(\dfrac{-\alpha}{x_i^2}\right)\\ +3(1-\lambda_2)\exp\left(\dfrac{-2\alpha}{x_i^2}\right)\end{pmatrix}} = 0, \quad (21)$$

$$\frac{\ell(\boldsymbol{\theta}\,|\,\mathbf{x})}{\partial\lambda_1} = \sum_{i=1}^{n}\frac{1-2\exp\left(\dfrac{-\alpha}{x_i^2}\right)}{\lambda_1 + 2(\lambda_2-\lambda_1)\exp\left(\dfrac{-\alpha}{x_i^2}\right) + 3(1-\lambda_2)\exp\left(\dfrac{-2\alpha}{x_i^2}\right)} = 0, \quad (22)$$

and

$$\frac{\ell(\boldsymbol{\theta}\,|\,\mathbf{x})}{\partial\lambda_2} = \sum_{i=1}^{n}\frac{2\exp\left(\dfrac{-\theta}{x_i^2}\right) - 3\exp\left(\dfrac{-2\theta}{x_i^2}\right)}{\lambda_1 + 2(\lambda_2-\lambda_1)\exp\left(\dfrac{-\theta}{x_i^2}\right) + 3(1-\lambda_2)\exp\left(\dfrac{-2\theta}{x_i^2}\right)} = 0. \quad (23)$$

LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATION

Least square estimation (LSE) and weighted least square estimation (WLSE) methods were introduced by Swain et al. [20] for the estimation problem of Beta distribution parameters. The LSEs and WLSEs can be derived by minimizing the following two functions which are used to obtain LSE and WLSE, respectively,

$$Q_{LSE}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\left(F_{CRTIR}\left(X_{(i)}\right) - \frac{i}{n+1}\right)^2 \quad (24)$$

and

$$Q_{WLSE}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(F_{CRTIR}\left(X_{(i)}\right) - \frac{i}{n+1}\right)^2, \quad (25)$$

where $F_{CRTIR}(.)$ is defined in (9).

ANDERSON-DARLING ESTIMATION

The Anderson-Darling estimators (ADEs) of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution can be obtained minimizing following function given in Equation (26)

$$Q_{ADE}(\boldsymbol{\theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\begin{pmatrix}\log\left[F_{CRTIR}\left(X_{(i)}\right)\right]\\ +\log\left[1-F_{CRTIR}\left(X_{(i)}\right)\right]\end{pmatrix}^2. \quad (26)$$

CRÁMER-VON-MISES ESTIMATION

The Crámer–von-Mises estimators (CvMEs) of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution can be obtained minimizing function given in Equation (27)

$$Q_{CvME}(\boldsymbol{\theta}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_{CRTIR}\left(X_{(i)}\right) - \frac{2i-1}{2n}\right)^2. \quad (27)$$

The examined five estimators can be obtained by optim function in R with BFGS algorithm.

SIMULATION STUDY

In this section, a Monte Carlo simulation study is carried out to assess the MLEs of unknown parameters for CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution according to MSEs and biases. In simulation study, we consider inverse transform techniques for generating random variables from CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. In this regard, a total of 5000 random samples are generated by using (12). The parameter settings are as follows: Case 1: α=0.5, λ_1=0.2, λ_2=0.9, Case 2: α=0.9, λ_1=0.1, λ_2=0.6, Case 3: α=1, λ_1=0.3, λ_2=0.8 and Case 4: α=2, λ_1=0.25, λ_2=−0.5. The biases and mean square errors (MSEs) of the MLEs are given in Tables 1-4.

The MSE and bias are computed by

$$MSE(\hat{\boldsymbol{\theta}}) = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\boldsymbol{\theta}}_{(i)} - \boldsymbol{\theta}\right)^2 \quad (28)$$

and

$$Bias\left(\hat{\boldsymbol{\theta}}\right)=\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{\boldsymbol{\theta}}_{(i)}-\boldsymbol{\theta}\right\}, \quad (29)$$

respectively, where n denotes sample size and $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2)$.

According to Tables 1-4, it is seen that as the sample size increases , the biases and MSEs of all estimators decrease. The MLE is the estimator with the smallest MSE in all cases. The CvME has the smallest absolute biases and MSE for λ_2 in Case 1. As a result, in point estimation for CRTIR ($\alpha,\lambda_1,\lambda_2$) distribution, it can be recommended the MLE for α and λ_1 parameters. The ADE can be recommended as an alternative to MLE for α and λ_1 parameters. Furthermore, CvME can be recommended as an alternative to MLE for λ_2.

REAL DATA APPLICATION

In this section, we provide a real application to illustrate the potential of CRTIR ($\alpha,\lambda_1,\lambda_2$) distribution in modeling lifetime data. In this data modeling, the superiority of the CRTIR ($\alpha,\lambda_1,\lambda_2$) distribution over some competing distributions such as transmuted inverse Rayleigh (TIR) [3], inverse Rayleigh (IR), Kumaraswamy-inverse Rayleigh (Kw-IR) [21], Lindley (L), Frechet (F) has been shown. We consider the -2×log-likelihood value, Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling statistics (A*), Cramer-von-Mises statistics (W*), Kolmogorov-Smirnov test statistics (KS) and its (p-value) as comparison statistics for fitted distributions.

The data set consist of 30 observations of March precipitation (in inches) in Minneapolis/St Paul. These data are obtained by [22], and studied by some authors such as [23-25]. The data set is given as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05. Table 5 shows that the pdfs of fitted distributions. The MLEs and standard errors (in parenthesis) and selection criteria statistics for precipitation data are given in Tables 6-7, respectively. Also, Figure 4 illustrates fitted cdfs and pdfs for precipitation data.

Table 1. Average biases and MSEs of examined estimators for Case 1

		bias			MSE		
Estimator	***n***	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
	50	0.0516	**0.3106**	-0.7902	**0.0353**	**0.1520**	**0.8872**
	100	**0.0240**	**0.2183**	-0.6309	**0.0259**	**0.1087**	0.6167
MLE	200	**0.0202**	**0.1749**	-0.4882	**0.0260**	**0.0972**	0.4099
	500	**0.0079**	**0.1058**	-0.3378	**0.0197**	**0.0703**	0.2351
	1000	**-0.01**	**0.0496**	-0.2685	**0.0140**	**0.0422**	0.1622
	50	0.1486	0.5083	-0.8080	0.1083	0.4695	1.2262
	100	0.1500	0.4539	-0.5831	0.0877	0.3883	0.7277
LSE	200	0.1406	0.3783	-0.3566	0.0654	0.2912	0.3776
	500	0.1247	0.3165	-0.2212	0.0503	0.2265	0.1886
	1000	0.1084	0.2516	-0.1032	0.0359	0.1548	0.0980
	50	0.0564	0.4044	-0.9696	0.0694	0.3143	1.3692
	100	0.0909	0.3515	-0.6130	0.0606	0.2680	0.6682
WLSE	200	0.1015	0.3063	-0.3701	0.0490	0.2148	0.3346
	500	0.0767	0.2201	-0.2299	0.0319	0.1436	**0.1582**
	1000	0.0674	0.1776	-0.1404	0.0250	0.1035	0.0982
	50	**0.0493**	0.3257	-0.8341	0.0499	0.2161	1.0881
	100	0.0641	0.2932	-0.6058	0.0456	0.2014	0.6526
ADE	200	0.0733	0.2552	-0.3936	0.0388	0.1693	0.3553
	500	0.0644	0.2004	-0.2504	0.0293	0.1295	0.1786
	1000	0.0530	0.1512	-0.1539	0.0214	0.0874	0.1065
	50	0.1824	0.4379	**-0.5447**	0.1162	0.3969	0.8881
	100	0.1691	0.4225	**-0.4428**	0.0899	0.3511	**0.5802**
CvME	200	0.1487	0.3637	**-0.2939**	0.0661	0.2771	**0.3317**
	500	0.1257	0.3056	**-0.1939**	0.0492	0.2166	**0.1726**
	1000	0.1105	0.2487	**-0.0867**	0.0356	0.1519	**0.0902**

Table 2. Average biases and MSEs of examined estimators for Case 2

		bias			MSE		
Estimator	n	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
	50	**0.1454**	**0.2957**	-0.6144	**0.1361**	**0.1440**	**0.6133**
	100	**0.1275**	**0.2418**	-0.4679	**0.1427**	**0.1167**	**0.4492**
MLE	200	**0.0857**	**0.1639**	-0.3361	**0.1097**	**0.0776**	**0.2984**
	500	**0.0823**	**0.1167**	-0.1657	**0.0857**	**0.0549**	**0.1665**
	1000	**0.0507**	**0.0736**	-0.1177	**0.0582**	**0.0336**	**0.1196**
	50	0.4128	0.5565	-0.6204	0.6151	0.5567	1.3237
	100	0.3997	0.4655	-0.3463	0.4803	0.4032	0.7762
LSE	200	0.3914	0.4049	-0.1705	0.4251	0.3272	0.4761
	500	0.3208	0.3398	-0.1096	0.3206	0.2482	0.3041
	1000	0.2451	0.2596	-0.0786	0.2372	0.1717	0.2465
	50	0.2736	0.4631	-0.6895	0.4118	0.3742	1.2191
	100	0.2521	0.3868	-0.5030	0.3371	0.2936	0.7468
WLSE	200	0.1986	0.2937	-0.3719	0.2581	0.2072	0.4719
	500	0.1960	0.2353	-0.1683	0.2003	0.1532	0.2307
	1000	0.1422	0.1727	-0.1295	0.1487	0.1069	0.1743
	50	0.1612	0.3854	-0.7840	0.2677	0.2657	1.2348
	100	0.2134	0.3447	-0.5000	0.2754	0.2371	0.7475
ADE	200	0.1901	0.2766	-0.3514	0.2354	0.1831	0.4799
	500	0.1876	0.2256	-0.1643	0.1865	0.1400	0.2433
	1000	0.1365	0.1681	-0.1325	0.1411	0.0998	0.1851
	50	0.4483	0.4750	**-0.4132**	0.6477	0.4709	1.1745
	100	0.4105	0.4226	**-0.2491**	0.4861	0.3621	0.7321
CvME	200	0.4010	0.3839	**-0.1128**	0.4300	0.3085	0.4574
	500	0.3168	0.3265	**-0.0932**	0.3152	0.2370	0.3029
	1000	0.2404	0.2521	**-0.0761**	0.2342	0.1662	0.2498

Table 3. Average biases and MSEs of examined estimators for Case 3

		bias			MSE		
Estimator	n	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
MLE	50	**0.0263**	**0.2149**	-0.7445	**0.0947**	**0.1027**	**0.7991**
	100	**0.0211**	**0.1628**	-0.5655	**0.0944**	**0.0892**	**0.5182**
	200	**0.0243**	**0.1236**	-0.4082	**0.0828**	**0.0769**	**0.3059**
	500	**0.0197**	**0.0774**	-0.2530	**0.0591**	**0.0555**	**0.1450**
	1000	**0.0173**	**0.0532**	-0.1651	**0.0385**	**0.0341**	**0.0799**
LSE	50	0.2347	0.4304	-0.7652	0.3797	0.3909	1.2035
	100	0.2000	0.3427	-0.5449	0.2638	0.2851	0.6953
	200	0.1926	0.2861	-0.3615	0.1984	0.2139	0.3771
	500	0.1382	0.2041	-0.2472	0.1362	0.1408	0.2171
	1000	0.1192	0.1650	-0.1674	0.1002	0.0961	0.1442
WLSE	50	0.0691	0.3153	-0.8591	0.2257	0.2399	1.2084
	100	0.0894	0.2467	-0.5768	0.1713	0.1842	0.6324
	200	0.0932	0.1959	-0.3913	0.1284	0.1402	0.3325
	500	0.0685	0.1319	-0.2509	0.0888	0.0922	0.1672
	1000	0.0516	0.0927	-0.1681	0.0590	0.0566	0.1017
ADE	50	0.0466	0.2476	-0.7755	0.1706	0.1731	1.0175
	100	0.0632	0.2065	-0.5569	0.1403	0.1438	0.5950
	200	0.0774	0.1727	-0.3803	0.1121	0.1175	0.3240
	500	0.0551	0.1181	-0.2557	0.0819	0.0824	0.1744
	1000	0.0377	0.0831	-0.1871	0.0580	0.0526	0.1168
CvME	50	0.2932	0.3615	**-0.5253**	0.4089	0.3323	0.9274
	100	0.2262	0.3062	**-0.4191**	0.2653	0.2554	0.5822
	200	0.2079	0.2715	**-0.3005**	0.2001	0.2039	0.3348
	500	0.1460	0.2000	**-0.2220**	0.1372	0.1386	0.2049
	1000	0.1252	0.1637	**-0.1502**	0.1001	0.0951	0.1362

Table 4. Average biases and MSEs of examined estimators for Case 4

		bias			MSE		
Estimator	n	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\alpha}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
	50	**0.6080**	**0.1169**	0.1841	**1.0308**	**0.0503**	**0.2509**
	100	**0.3567**	**0.0484**	0.1053	**0.5717**	**0.0233**	**0.1930**
MLE	200	**0.1532**	**0.0147**	**0.0209**	**0.1957**	**0.0103**	**0.1178**
	500	**0.0568**	**0.0084**	-0.0222	**0.0417**	**0.0041**	**0.0477**
	1000	**0.0359**	**0.0066**	-0.0221	**0.0167**	**0.0020**	**0.0239**
	50	0.8443	0.4555	-0.3943	3.6576	0.3837	1.3042
	100	0.7374	0.2974	-0.1712	2.9431	0.2043	0.9303
LSE	200	0.6806	0.2175	-0.0401	2.7434	0.1410	0.7412
	500	0.5088	0.1234	0.0690	1.7555	0.0668	0.5020
	1000	0.3138	0.0653	0.0656	0.9385	0.0301	0.3324
	50	0.7596	0.3669	-0.2865	3.4343	0.2813	0.6131
	100	0.6925	0.2272	-0.0683	2.8400	0.1601	0.3981
WLSE	200	0.4636	0.1161	0.0229	1.6693	0.0709	0.3369
	500	0.1930	0.0410	**0.0135**	0.5105	0.0185	0.1566
	1000	0.0869	0.0180	**-0.0058**	0.1637	0.0056	0.0748
	50	0.6447	0.2630	**-0.1511**	2.3614	0.1811	0.5321
	100	0.5072	0.1592	**-0.0490**	1.7457	0.0954	0.3748
ADE	200	0.2576	0.0710	-0.0278	0.7327	0.0312	0.2684
	500	0.1021	0.0253	-0.0218	0.1776	0.0067	0.1190
	1000	0.0508	0.0140	-0.0258	0.0558	0.0025	0.0581
	50	0.7083	0.3814	-0.3965	3.1821	0.3300	1.4664
	100	0.6590	0.2593	-0.1820	2.7157	0.1810	0.9733
CvME	200	0.6185	0.1923	-0.0479	2.5014	0.1239	0.7577
	500	0.4871	0.1150	0.0629	1.6989	0.0636	0.5018
	1000	0.3040	0.0615	0.0623	0.9130	0.0287	0.3322

Table 5. The list of pdfs of fitted distribution for precipitation data

$f_{Kw-IR}(x)=\frac{2\alpha\lambda}{x^3}\exp\left(-\frac{\lambda}{x^2}\right)\left[1-\exp\left(-\frac{\lambda}{x^2}\right)\right]^{\alpha-1},$	$\alpha, \lambda > 0, x > 0$
$f_{Lindley}(x)=\frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x)$	$\theta > 0, x > 0$
$f_{Frechet}(x)=\mu\sigma^{\mu}x^{-(\mu+1)}\exp\left(-\left(\frac{\sigma}{x}\right)^{\mu}\right)$	$\mu, \sigma > 0, x > 0$

Table 6. MLEs (standard errors) for precipitation data

Distribution	MLEs (standard errors)
CRTIR	$\hat{\alpha}=0.4722(0.1121)$, $\hat{\lambda}_1=0.7214(0.3988)$, $\hat{\lambda}_2=-0.8903(0.7762)$
TIR	$\hat{\alpha}=0.6285(0.1582)$, $\hat{\lambda}=-0.6700(0.2661)$
IR	$\hat{\alpha}=0.8587(0.1567)$
Kw-IR	$\hat{\alpha}=0.7314(0.1716)$, $\hat{\lambda}=0.6867(0.1853)$
L	$\hat{\theta}=0.9096(0.1247)$
F	$\hat{\mu}=1.5496(0.2026)$, $\hat{\sigma}=1.0162(0.1272)$

Table 7. Selection criteria statistics for precipitation data

Distribution	-2log	AIC	BIC	A*	W*	K-S	p value
CRTIR	**78.9324**	**84.9324**	**89.1360**	**0.2932**	**0.0341**	**0.0891**	**0.9709**
TIR	84.2022	88.2022	91.0046	1.1316	0.2117	0.1817	0.2748
IR	88.2730	90.2730	91.6742	2.1823	0.4308	0.2396	0.0636
Kw-IR	86.4023	90.4023	93.2047	1.2099	0.2278	0.1984	0.1882
L	86.2874	88.2874	89.6886	1.5910	0.2618	0.1882	0.2382
F	83.8340	87.8340	90.6364	0.7596	0.1201	0.1523	0.4892

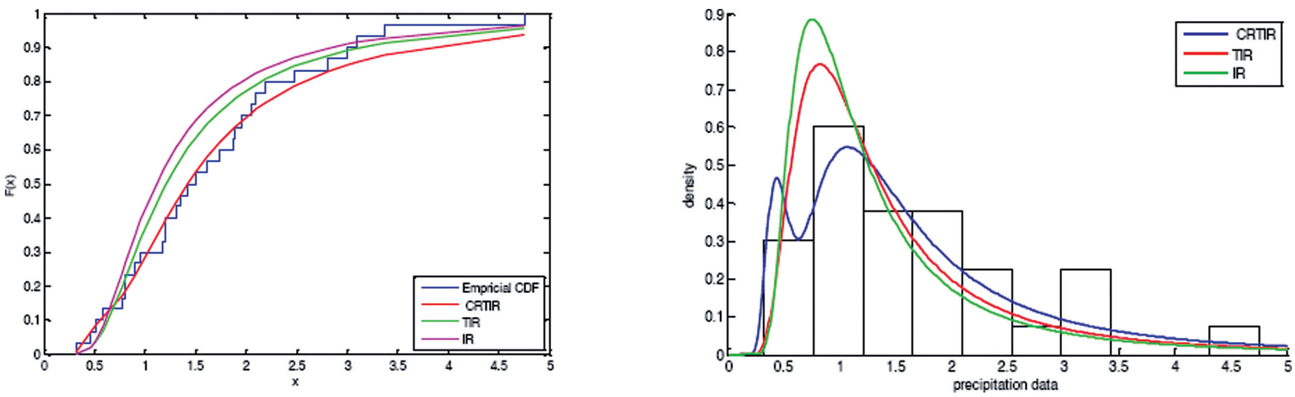

Figure 4. Fitted cdfs (left) and pdfs (right) for precipitation data.

CONCLUSION

In this study, we introduce a new lifetime distribution an alternative to IR distribution called CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution using the extension suggested by Granzotto et al. (2017). The hf of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution is upside bathtub shaped according to Figure 2. We provide five estimators of unknown parameters of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. Also, we perform a Monte Carlo simulation study in order to evaluate these estimators in terms of biases and MSEs at different samples. The results of the simulation study show that there is a decrease in biases and MSEs of all estimators as the size of the sample increases for all cases. We recommend the maximum likelihood method for point estimation of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution. We present a real data application to illustrate the usefulness of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution for modeling data. It is compared the fits of CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution and five statistical distributions. Table 7 illustrates that the best-fitted model is CRTIR $(\alpha,\lambda_1,\lambda_2)$ distribution among all fitted distributions in modeling precipitation data.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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