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Research Article

Malcev Yang-Baxter equation, weighted $\mathcal{O}$-operators on Malcev algebras and post-Malcev algebras

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Abstract

The purpose of this paper is to study the $\mathcal{O}$-operators on Malcev algebras and discuss the solutions of Malcev Yang-Baxter equation by $\mathcal{O}$-operators. Furthermore we introduce the notion of weighted $\mathcal{O}$-operators on Malcev algebras, which can be characterized by graphs of the semi-direct product Malcev algebra. Then we introduce a new algebraic structure called post-Malcev algebras. Therefore, post-Malcev algebras can be viewed as the underlying algebraic structures of weighted $\mathcal{O}$-operators on Malcev algebras. A post-Malcev algebra also gives rise to a new Malcev algebra. Post-Malcev algebras are analogues for Malcev algebras of post-Lie algebras and fit into a bigger framework with a close relationship with post-alternative algebras.

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1. Introduction

Malcev algebras play an important role in the geometry of smooth loops. Just as the tangent algebra of a Lie group is a Lie algebra, the tangent algebra of a locally analytic Moufang loop is a Malcev algebra [18, 21, 27, 28]. A Malcev algebra is a non-associative algebra A with an anti-symmetric multiplication $[\cdot,\cdot]$ that satisfies the Sagle's identity

$$[[x,z],[y,t]] = [[[x,y],z],t] + [[[y,z],t],x] + [[[z,t],x],y] + [[[t,x],y],z], \forall x,y,z,t \in A.$$

Pre-Malcev algebras have been studied extensively since [26] which are the generalization of pre-Lie algebras, in the sense that any pre-Lie algebra is a pre-Malcev algebra but the converse is not true. Studying pre-Malcev algebras independly is significant not only to its own further development, but also to develop the areas closely connected with such

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algebras. A pre-Malcev algebra is a vector space A endowed with a bilinear product $\triangleright$ satisfying the following identity for $x, y, z, t \in A$,

$$[y,z]\triangleright(x\triangleright t)+[[x,y],z]\triangleright t+y\triangleright([x,z]\triangleright t)-x\triangleright(y\triangleright(z\triangleright t))+z\triangleright(x\triangleright(y\triangleright t))=0, \quad (1.1)$$

where $[x,y] = x \triangleright y - y \triangleright x$. The existence of subadjacent Malcev algebras and compatible pre-Malcev algebras was given in [26, Proposition 5]. For a given pre-Malcev algebra $(A, \triangleright)$, there is a Malcev algebra A^C defined by the commutator $[x,y] = x \triangleright y - y \triangleright x$, and the left multiplication operator in A induces a representation of Malcev algebra A^C.

Rota-Baxter operators were introduced by G. Baxter [7] in 1960 in the study of fluctuation theory in Probability. These operators were then further investigated, by G.-C. Rota [30], Atkinson [1], Cartier [9] and others. In the 1980s, the notion of Rota-Baxter operator of weight 0 was introduced in terms of the classical Yang-Baxter equation for Lie algebras (see [4,5,13–15,17,23] for more details). Later on, B. A. Kupershmidt [19] introduced the notion of $\mathcal{O}$-operator as generalized Rota-Baxter operators to understand classical Yang-Baxter equations and related integrable systems. In fact, a skew-symmetric solution of the CYBE (see [2]) is exactly a special $\mathcal{O}$-operator (associated to the coadjoint representation). Our first goal is to study the connections between $\mathcal{O}$-operators and symmetric solutions of the analogue of CYBE on Malcev algebras motivated by the point of Kupershmidt and Bai.

The notion of post-algebras goes back to Rosenbloom in [29] (1942). An equivalent formulation of the class of post-algebras was given by Rousseau in [31] (1969, 1970) which became a starting point for deep research. Post-Lie algebras have been introduced by Vallette in 2007 [33] in connection with the homology of partition posets and the study of Koszul operads. However, J. L. Loday studied pre-Lie algebras and post-Lie algebras within the context of algebraic operad triples, see for more details [24, 25]. In the last decade, many works [8, 10, 34] intrested in post-Lie algebra structures, motivated by the importance of pre-Lie algebras in geometry and in connection with generalized Lie algebra derivations.

Recently, Post-Lie algebras which are non-associative algebras played an important role in different areas of pure and applied mathematics. They consist of a vector space A equipped with a Lie bracket $[\cdot,\cdot]$ and a binary operation $\triangleright$ satisfying the following axioms

$$x \triangleright [y,z] = [x \triangleright y, z] + [y, x \triangleright z], \quad (1.2)$$

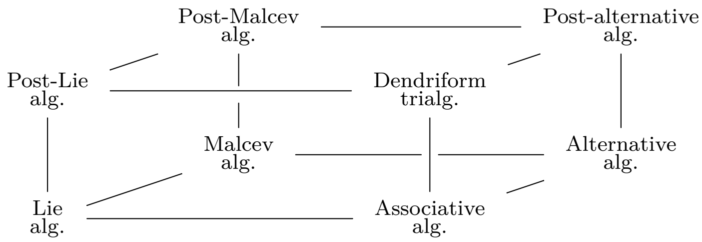
$$[x,y] \triangleright z = as_{\triangleright}(x,y,z) - as_{\triangleright}(y,x,z). \quad (1.3)$$

If the bracket $[\cdot,\cdot]$ is zero, we have exactly a pre-Lie structure. It is worth to note that, in spite the post-Lie product does not yield a Lie bracket by antisymmetrization, the bilinear product $\{\cdot,\cdot\} : A \otimes A \to A$, defined for all $x, y \in A$ by

$$\{x,y\} = x \triangleright y - y \triangleright x + [x,y]. \quad (1.4)$$

defines on A another Lie algebra structure. The varieties of pre- and post-Lie algebras play a crucial role in the definition of any pre and post-algebra through black Manin operads product, see details in [3, 12]. Whereas pre-Lie algebras are intimately associated with euclidean geometry, post-Lie algebras occur naturally in the differential geometry of homogeneous spaces, and are also closely related to Cartan's method of moving frames. Ebrahimi-Fard, Lundervold and Munthe-Kaas [10] studied universal enveloping algebras of post-Lie algebras and the free post-Lie algebra.

In this paper we use weighted $\mathcal{O}$-operators to split operations, although a generalization exists in the alternative setting in terms of bimodules. Diagram (1.5) summarizes the results of the present work.



(1.5)

In Section 2, we study the relationship between $\mathcal{O}$-operators and Malcev Yang-Baxter equation. We construct in Section 3 alternative algebras structure associated to any post-alternative algebra. The multiplication is given by

$$x \star y = x \prec y + y \succ x + x \cdot y.$$

In addition, in Section 4 we investigate the notion of a weighted $\mathcal{O}$-operator to construct a post-alternative algebra structure on the A-bimodule $\mathbb{K}$-algebra of an alternative algebra $(A, \cdot)$. Section 4 is devoted to introduce the notion of post-Malcev algebra and we show that weighted $\mathcal{O}$-operators can be used to construct post-Malcev algebras. We also reveal a relation between post-Malcev algebras and post-alternative algebras by the commutative diagram (1.5).

Throughout this paper, all algebras are finite-dimensional and over a field $\mathbb{K}$ of characteristic 0.

2. $\mathcal{O}$-operators and Malcev Yang-Baxter equation

In this section, we recall the classical result that a skew-symmetric solution of CYBE in a Malcev algebra gives an $\mathcal{O}$-operator through a duality between tensor product and linear maps. Not every $\mathcal{O}$-operator on a Malcev algebra comes from a solution of CYBE in this way. However, any $\mathcal{O}$-operator can be recovered from a solution of CYBE in a larger Malcev algebra.

We first recall the concept of a representation (see [20]) and construct the dual representation.

Definition 2.1 ([20])**.** A **representation** (or **a module**) of a Malcev algebra $(A, [\cdot, \cdot])$ on a vector space V is a linear map $\rho : A \longrightarrow End(V)$ such that, for all $x, y, z \in A$,

$$\rho([[x, y], z]) = \rho(x)\rho(y)\rho(z) - \rho(z)\rho(x)\rho(y) + \rho(y)\rho([z, x]) - \rho([y, z])\rho(x). \tag{2.1}$$

We denote this representation by (V, ρ).

For all $x, y \in A$, define the map $ad : A \longrightarrow End(A)$ by $ad_x(y) = [x, y]$. Then ad is a representation of the Malcev algebra $(A, [\cdot, \cdot])$ on A, which is called the adjoint representation.

Let $(A, [\cdot, \cdot])$ be a Malcev algebra and (V, ρ) is a representation on A. Consider the dual space V^* of V and $End(V^*)$. Define the linear map $\rho^* : A \longrightarrow End(V^*)$ by

$$\langle \rho^*(x)a^*, b\rangle = -\langle a^*, \rho(x)b\rangle, \quad \forall\, x \in A, b \in V, a^* \in V^*, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between V^* and V.

Proposition 2.1. *With the above notations, (V^*, ρ^*) is a representation of A which is called the dual representation of (V, ρ).*

Proof. By (2.1), we have, for $x, y, z \in A$,

$$\rho([[y, x], z]) = -\rho([[x, y], z]) = \rho(z)\rho(y)\rho(x) - \rho(y)\rho(x)\rho(z) - \rho(x)\rho([z, y]) + \rho([x, z])\rho(y).$$

So, for any $x,y,z\in A$, $a^*\in V^*$, $b\in V$, we have

$$\Big\langle \rho^*([[x,y],z])a^*,b\Big\rangle=-\Big\langle a^*,\rho([[x,y],z])b\Big\rangle=-\Big\langle a^*,-\rho([[y,x],z])b\Big\rangle$$
$$=-\Big\langle a^*,(\rho(y)\rho(x)\rho(z)-\rho(z)\rho(y)\rho(x)+\rho(x)\rho([z,y])-\rho([x,z])\rho(y))b\Big\rangle$$
$$=-\Big\langle\big(-\rho^*(z)\rho^*(x)\rho^*(y)+\rho^*(x)\rho^*(y)\rho^*(z)+\rho^*([z,y])\rho^*(x)-\rho^*(y)\rho^*([x,z])\big)a^*,b\Big\rangle.$$

Hence, since $\langle\cdot,\cdot\rangle$ is nondegenerate, we obtain

$$\rho^*([[x,y],z])=\rho^*(x)\rho^*(y)\rho^*(z)-\rho^*(z)\rho^*(x)\rho^*(y)+\rho^*(y)\rho^*([z,x])-\rho^*([y,z])\rho^*(x).\quad\square$$

Definition 2.2. Let $(A,[\cdot,\cdot])$ be a Malcev algebra and $r=\sum_i x_i\otimes y_i\in A\otimes A$. r is called a solution of Malcev Yang-Baxter equation in A if r satisfies

$$[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0,\tag{2.3}$$

where

$$r_{12}=\sum_i x_i\otimes y_i\otimes 1,\quad r_{13}=\sum_i x_i\otimes 1\otimes y_i,\quad r_{23}=\sum_i 1\otimes x_i\otimes y_i,\tag{2.4}$$

and

$$[r_{12},r_{13}]=\sum_{i,j}[x_i,x_j]\otimes y_i\otimes y_j,\quad [r_{13},r_{23}]=\sum_{i,j}x_i\otimes x_j\otimes[y_i,y_j],$$
$$[r_{12},r_{23}]=\sum_{i,j}x_i\otimes[y_i,x_j]\otimes y_j.$$

Let V be a vector space. The **twisting operator** $\sigma: V^{\otimes 2}\to V^{\otimes 2}$ is defined for all $a,b\in V$ by

$$\sigma(a\otimes b)=b\otimes a.$$

We call $r=\sum_i a_i\otimes b_i\in V^{\otimes 2}$ **skew-symmetric** (resp. **symmetric**) if $r=-\sigma(r)$ (resp. $r=\sigma(r)$). Furthermore, r can be regarded as a linear map from V^* to V in the following way

$$\langle a^*,r(b^*)\rangle=\langle a^*\otimes b^*,r\rangle,\quad \forall a^*,b^*\in V^*\tag{2.5}$$

Equation (2.3) gives the tensor form of Malcev Yang-Baxter equation. What we will do next is to replace the tensor form by a linear operator satisfying some conditions.

Theorem 2.1. *Let $(A,[\cdot,\cdot])$ be a Malcev algebra and $r\in A\otimes A$. Then r is a skew-symmetric solution of Malcev Yang-Baxter equation in A if and only if r satisfies for all $x^*,y^*\in A^*$,*

$$[r(x^*),r(y^*)]=r(ad^*r(x^*)(y^*)-ad^*r(y^*)(x^*)).\tag{2.6}$$

Proof. Let $\{e_i,...,e_n\}$ be a basis of A and $\{e_i^*,...,e_n^*\}$ be its dual basis. Suppose that $[e_i,e_j]=\sum_p c_{ij}^p e_p$ and $r=\sum_{i,j}a_{ij}e_i\otimes e_j$. Hence $a_{ij}=-a_{ji}$. Now, we have

$$[r_{12},r_{13}]=\Big[\sum_{i,j}a_{ij}e_i\otimes e_j\otimes 1,\sum_{k,l}a_{kl}e_k\otimes 1\otimes e_l\Big]=\sum_{i,j,k,l,p}a_{ij}a_{kl}c_{ik}^p e_p\otimes e_j\otimes e_l,$$
$$[r_{13},r_{23}]=\Big[\sum_{i,j}a_{ij}e_i\otimes 1\otimes e_j,\sum_{k,l}a_{kl}1\otimes e_k\otimes e_l\Big]=\sum_{i,j,k,l,p}a_{ij}a_{kl}c_{jl}^p e_i\otimes e_k\otimes e_p,$$
$$[r_{12},r_{23}]=\Big[\sum_{i,j}a_{ij}e_i\otimes e_j\otimes 1,\sum_{k,l}a_{kl}1\otimes e_k\otimes e_l\Big]=\sum_{i,j,k,l,p}a_{ij}a_{kl}c_{jk}^p e_i\otimes e_p\otimes e_l.$$

Then r is a solution of the Malcev Yang-Baxter equation in $(A,[\cdot,\cdot])$ if and only if (for any j,p,l)

$$\sum_{i,k}\Big(a_{ij}a_{kl}c_{ik}^p+a_{kp}a_{ij}c_{ki}^l+a_{pi}a_{kl}c_{ik}^j\Big)e_p\otimes e_j\otimes e_l=0.$$

On the other hand, by (2.5), we get $r(e_j^*) = \sum_i a_{ij} e_i$. Then, if we set $x^* = e_j^*$ and $y^* = e_l^*$, by (2.6),

$$\sum_{i,k} \Big(a_{ij} a_{kl} c_{ik}^p + a_{kp} a_{ij} c_{ki}^l + a_{pi} a_{kl} c_{ik}^j \Big) e_p = 0.$$

Therefore, it is easy to see that r is a solution of Malcev Yang-Baxter equation in A if and only if r satisfies (2.6). □

Definition 2.3. Let $(A, [\cdot, \cdot])$ be a Malcev algebra. A symmetric bilinear form B on A is called *invariant* if, for all $x, y, z \in A$,

$$B([x, y], z) = B(x, [y, z]). \tag{2.7}$$

Definition 2.4. Let $(A, [\cdot, \cdot])$ be a Malcev algebra. A Rota-Baxter operator of weight 0 on A is a linear map $\mathcal{R} : A \to A$ satisfying for all $x, y \in A$,

$$[\mathcal{R}(x), \mathcal{R}(y)] = \mathcal{R}([\mathcal{R}(x), y] + [x, \mathcal{R}(y)]).$$

Corollary 2.1. *Let $(A, [\cdot, \cdot])$ be a Malcev algebra and $r \in A \otimes A$. Assume r is skew-symmetric and there exists a nondegenerate symmetric invariant bilinear form B on A. Define a linear map $\varphi : A \to A^*$ by $\langle \varphi(x), y \rangle = B(x, y)$ for any $x, y \in A$. Then r is a solution of the Malcev Yang-Baxter equation in A if and only if $\mathcal{R} = r\varphi : A \to A$ is a Rota-Baxter operator.*

Proof. For any $x, y, z \in A$, we have

$$\langle \varphi(ad(x)y), z \rangle = B([x, y], z) = B(z, [x, y]) = -B(y, [x, z]) = \langle ad^*(x)\varphi(y), z \rangle.$$

Hence $\varphi(ad(x)y) = ad^*(x)\varphi(y)$ for any $x, y \in A$. Let $x^* = \varphi(x)$, $y^* = \varphi(y)$, then by Theorem 2.1, r is a solution of the Malcev Yang-Baxter equation in A if and only if

$$[r\varphi(x), r\varphi(y)] = [r(x^*), r(y^*)] = r(ad^* r(x^*)(y^*) - ad^* r(y^*)(x^*)) = r\varphi([r\varphi(x), y] + [x, r\varphi(y)]).$$

Therefore the conclusion holds. □

Now, we introduce the notion of $\mathcal{O}$-operator of a Malcev algebra.

Definition 2.5. Let $(A, [\cdot, \cdot])$ be a Malcev algebra and let (V, ρ) be a representation of A. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to ρ if for all $a, b \in V$,

$$[T(a), T(b)] = T(\rho(T(a))b - \rho(T(b))a). \tag{2.8}$$

Example 2.1. Let $(A, [\cdot, \cdot])$ be a Malcev algebra. Then a Rota-Baxter operator (of weight zero) is an $\mathcal{O}$-operator of A associated to the adjoint representation (A, ad) and a skew-symmetric solution of Malcev Yang-Baxter equation in A is an $\mathcal{O}$-operator of A associated to the representation (A^*, ad^*).

Let $(A, [\cdot, \cdot])$ be a Malcev algebra. Let $\rho^* : A \to gl(V^*)$ be the dual representation of the representation $\rho : A \to gl(V)$ of the Malcev algebra A. A linear map $T : V \to A$ can be identified as an element in $A \otimes V^* \subset (A \ltimes_{\rho^*} V^*) \otimes (A \ltimes_{\rho^*} V)$ as follows. Let $\{e_1, \cdots, e_n\}$ be a basis of A. Let $\{v_1, \cdots, v_m\}$ be a basis of V and $\{v_1^*, \cdots, v_m^*\}$ be its dual basis, that is $v_i^*(v_j) = \delta_{ij}$. Set $T(v_i) = \sum_{j=1}^{n} a_{ij} e_j, i = 1, \cdots, m$. Since as vector spaces, $\mathrm{Hom}(V, A) \cong A \otimes V^*$, we have

$$\begin{aligned} T = \sum_{i=1}^{m} T(v_i) \otimes v_i^* &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} e_j \otimes v_i^* \\ &\in A \otimes V^* \subset (A \ltimes_{\rho^*} V^*) \otimes (A \ltimes_{\rho^*} V^*). \end{aligned} \tag{2.9}$$

Theorem 2.2. *Let $(A, [\cdot, \cdot])$ be a Malcev algebra. Then T is an $\mathcal{O}$-operator of A associated to (V, ρ) if and only if $r = T - \sigma(T)$ is a skew-symmetric solution of the Malcev Yang-Baxter equation in $A \ltimes_{\rho^*} V^*$.*

Proof. From (2.9), we have $r = T - \sigma(T) = \sum_i T(v_i) \otimes v_i^* - v_i^* \otimes T(v_i)$. Thus,

$$\begin{aligned}
[r_{12}, r_{13}] = \sum_{i,k=1}^{m} \{&[T(v_i), T(v_k)] \otimes v_i^* \otimes v_k^* - \rho^*(T(v_i))v_k^* \otimes v_i^* \otimes T(v_k) \\
&+\rho^*(T(v_k))v_i^* \otimes T(v_i) \otimes v_k^*\}, \\
[r_{12}, r_{23}] = \sum_{i,k=1}^{m} \{&-v_i^* \otimes [T(v_i), T(v_k)] \otimes v_k^* - T(v_i) \otimes \rho^*(T(v_k))v_i^* \otimes v_k^* \\
&+v_i^* \otimes \rho^*(T(v_i))v_k^* \otimes T(v_k)\}, \\
[r_{13}, r_{23}] = \sum_{i,k=1}^{m} \{&v_i^* \otimes v_k^* \otimes [T(v_i), T(v_k)] + T(v_i) \otimes v_k^* \otimes \rho^*(T(v_k))v_i^* \\
&-v_i^* \otimes T(v_k) \otimes \rho^*(T(v_i))v_k^*\}.
\end{aligned}$$

By the definition of dual representation, we know $\rho^*(T(v_k))v_i^* = -\sum_{j=1}^{m} v_i^*(\rho(T(v_k))v_j)v_j^*$. Thus,

$$\begin{aligned}
&-\sum_{i,k=1}^{m} T(v_i) \otimes \rho^*(T(v_k))v_i^* \otimes v_k^* = -\sum_{i,k=1}^{m} T(v_i) \otimes [\sum_{j=1}^{m} -v_i^*(\rho(T(v_k))v_j)v_j^*] \otimes v_k^* \\
&= \sum_{i,k=1}^{m} \sum_{j=1}^{m} v_j^*(\rho(T(v_k))v_i)T(v_j) \otimes v_i^* \otimes v_k^* = \sum_{i,k=1}^{m} T(\sum_{j=1}^{m}(v_j^*(\rho(T(v_k))v_i)v_j) \otimes v_i^* \otimes v_k^* \\
&= \sum_{i,k=1}^{m} T(\rho(T(v_k))v_i) \otimes v_i^* \otimes v_k^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\
&= \sum_{i,k=1}^{m} \{([T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k)) \otimes v_i^* \otimes v_k^* \\
&-v_i^* \otimes ([T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k)) \otimes v_k^* \\
&+v_i^* \otimes v_k^* \otimes ([T(v_i), T(v_k)] + T(\rho(T(v_k))v_i) - T(\rho(T(v_i))v_k))\}.
\end{aligned}$$

So r is a classical r-matrix in $A \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator. □

In fact, Theorem 2.2 gives a relation between $\mathcal{O}$-operator and Malcev Yang-Baxter equation. Then, we get a direct conclusion from Theorems 2.1 and 2.2.

Corollary 2.2. *Let $(A, [\cdot, \cdot])$ be a Malcev algebra. Let $\rho : A \to gl(V)$ be a representation of A. Set $\widehat{A} = A \ltimes_{\rho^*} V^*$. Let $T : V \to A$ be a linear map. Then the following three conditions are equivalent:*

(i) *T is an $\mathcal{O}$-operator of A associated to ρ;*
(ii) *$T - \sigma(T)$ is a skew-symmetric solution of the Malcev Yang-Baxter equation in $\widehat{A}$;*
(iii) *$T - \sigma(T)$ is an $\mathcal{O}$-operator of the Malcev algebra $\widehat{A}$ associated to ad^*.*

3. Alternative and post-alternative algebras

In this section, we recall some basic definitions about alternative and pre-alternative algebras studied in [6, 22].

3.1. Some basic results on alternative algebras

Definition 3.1. An **alternative algebra** $(A, \cdot)$ is a vector space A equipped with a bilinear operation $(x, y) \to x \cdot y$ satisfying, for all $x, y, z \in A$,

$$as_A(x, x, y) = as_A(y, x, x) = 0, \tag{3.1}$$

where $as_A(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the **associator**.

Remark 3.1. If the characteristic of the field is not 2, then an alternative algebra $(A, \cdot)$ also satisfies the stronger axioms, for all $x, y, z \in A$,

$$as_A(x, y, z) + as_A(y, x, z) = 0, \tag{3.2}$$

$$as_A(z, x, y) + as_A(z, y, x) = 0. \tag{3.3}$$

Now, recall that an algebra $(A, \cdot)$ is called admissible Malcev algebra if $(A, [\cdot, \cdot])$ is a Malcev algebra, where $[x, y] = x \cdot y - y \cdot x$.

Example 3.1. Any alternative algebra is Malcev admissible. That is if $(A, \cdot)$ be an alternative algebra, then $(A, [\cdot, \cdot])$ is a Malcev algebra, where $[x, y] = x \cdot y - y \cdot x$, for all $x, y \in A$.

Definition 3.2 ([32])**.** Let $(A, \cdot)$ be an alternative algebra and V be a vector space. Let $\mathfrak{l}, \mathfrak{r} : A \to End(V)$ be two linear maps. Then, $(V, \mathfrak{l}, \mathfrak{r})$ is called a representation or a bimodule of A if, for any $x, y \in A$,

$$\mathfrak{r}(x)\mathfrak{r}(y) + \mathfrak{r}(y)\mathfrak{r}(x) - \mathfrak{r}(x \cdot y) - \mathfrak{r}(y \cdot x) = 0, \tag{3.4}$$

$$\mathfrak{l}(x \cdot y) + \mathfrak{l}(y \cdot x) - \mathfrak{l}(x)\mathfrak{l}(y) - \mathfrak{l}(y)\mathfrak{l}(x) = 0, \tag{3.5}$$

$$\mathfrak{l}(x \cdot y) + \mathfrak{r}(y)\mathfrak{l}(x) - \mathfrak{l}(x)\mathfrak{l}(y) - \mathfrak{l}(x)\mathfrak{r}(y) = 0, \tag{3.6}$$

$$\mathfrak{r}(y)\mathfrak{l}(x) + \mathfrak{r}(y)\mathfrak{r}(x) - \mathfrak{l}(x)\mathfrak{r}(y) - \mathfrak{r}(x \cdot y) = 0. \tag{3.7}$$

Definition 3.3. **A pre-alternative algebra** is a triple $(A, \prec, \succ)$, where A is a vector space, $\prec, \succ: A \otimes A \to A$ are bilinear maps satisfying for all $x, y, z \in A$ and $x \cdot y = x \prec y + x \succ y$,

$$(x \succ y) \prec z - x \succ (y \prec z) + (y \prec x) \prec z - y \prec (x \cdot z) = 0, \tag{3.8}$$

$$(x \succ y) \prec z - x \succ (y \prec z) + (z \cdot x) \succ y - z \succ (x \succ y) = 0, \tag{3.9}$$

$$(x \cdot y) \succ z - x \succ (y \succ z) + (y \cdot x) \succ z - y \succ (x \succ z) = 0, \tag{3.10}$$

$$(z \prec x) \prec y - z \prec (x \star y) + (z \prec y) \prec x - z \prec (y \cdot x) = 0. \tag{3.11}$$

Proposition 3.1. *Let $(A, \prec, \succ)$ be a pre-alternative algebra. Then the product $x \cdot y = x \prec y + x \succ y$ defines an alternative algebra A. Furthermore, $(A, L_\succ, R_\prec)$, where $L_\succ(x)y = x \succ y$ and $R_\prec(x)y = y \prec x$, gives a representation of the associated alternative algebra $(A, \cdot)$ on A.*

Proposition 3.2. *Let $(A, \prec, \succ)$ be a pre-alternative algebra. Then the product $x \rhd y = x \succ y - y \prec x$ defines a pre-Malcev structure in A.*

3.2. A-bimodule alternative algebras, weighted $\mathcal{O}$-operators and post-alternative algebras

Definition 3.4. Let $(A, \cdot)$ be an alternative algebra. Let $(V, \cdot_V)$ be an alternative algebra and $\mathfrak{l}, \mathfrak{r} : A \to End(V)$ be two linear maps. We say that $(V, \cdot_V, \mathfrak{l}, \mathfrak{r})$ is an **A-bimodule alternative algebra** if $(V, \mathfrak{l}, \mathfrak{r})$ is a representation of $(A, \cdot)$ such that the following compatibility conditions hold (for all $x \in A,\ a, b \in V$)

$$\mathfrak{r}(x)(a \cdot_V b) - a \cdot_V (\mathfrak{r}(x)b) + \mathfrak{r}(x)(b \cdot_V a) - b \cdot_V (\mathfrak{r}(x)a) = 0, \tag{3.12}$$

$$(\mathfrak{l}(x)a) \cdot_V b - \mathfrak{l}(x)(a \cdot_V b) + (\mathfrak{l}(x)b) \cdot_V a - \mathfrak{l}(x)(b \cdot_V a) = 0, \tag{3.13}$$

$$(\mathfrak{l}(x)a)\cdot_V b - a\cdot_V(\mathfrak{l}(x)b) + (\mathfrak{r}(x)a)\cdot_V b - \mathfrak{l}(x)(a\cdot_V b) = 0, \tag{3.14}$$
$$(\mathfrak{r}(x)a)\cdot_V b - a\cdot_V(\mathfrak{l}(x)b) + \mathfrak{r}(x)(a\cdot_V b) - a\cdot_V(\mathfrak{r}(x)b) = 0. \tag{3.15}$$

Proposition 3.3. *Let $(A,\cdot)$ and $(V,\cdot_V)$ be two alternative algebras and $\mathfrak{l},\mathfrak{r}: A\to End(V)$ be two linear maps. Then $(V,\cdot_V,\mathfrak{l},\mathfrak{r})$ is an A-bimodule alternative algebra if and only if the direct sum $A\oplus V$ of vector spaces is an alternative algebra (the semi-direct sum) with the product on $A\oplus V$ defined for all $x,y\in A,\ a,b\in V$ by*

$$(x+a)*(y+b) = x\cdot y + \mathfrak{l}(x)b + \mathfrak{r}(y)a + a\cdot_V b. \tag{3.16}$$

We denote this algebra by $A\ltimes_{\mathfrak{l},\mathfrak{r}} V$ or simply $A\ltimes V$. Further, if $(A,\cdot)$ is an alternative algebra, then it is easy to see that $(A,\cdot,L_.,R_.)$ is an A-bimodule alternative algebra, where $L_.$ and $R_.$ are the left and right multiplication operators corresponding to the multiplication $\cdot$.

Proof. For any $x,y,z\in A, a,b,c\in V$

$$\begin{aligned}
&as_{A\oplus V}(x+a,y+b,z+c) + as_{A\oplus V}(y+b,x+a,z+c)\\
=&((x+a)*(y+b))*(z+c) - (x+a)*((y+b)*(z+c)) + ((y+b)*(x+a))*(z+c)\\
&-(y+b)*((x+a)*(z+c))\\
=&(x\cdot y + \mathfrak{l}(x)b + \mathfrak{r}(y)a + a\cdot_V b)*(z+c) - (x+a)*(y\cdot z + \mathfrak{l}(y)c + \mathfrak{r}(z)b + b\cdot_V c)\\
&+(y\cdot x + \mathfrak{l}(y)a + \mathfrak{r}(x)b + b\cdot_V a)*(z+c) - (y+b)*(x\cdot z + \mathfrak{l}(x)c + \mathfrak{r}(z)a + a\cdot_V c)\\
=&(x\cdot y)\cdot z + \mathfrak{l}(x\cdot y)c + \mathfrak{r}(z)\big(\mathfrak{l}(x)b + \mathfrak{r}(y)a + a\cdot_V b\big) + \big(\mathfrak{l}(x)b + \mathfrak{r}(y)a + a\cdot_V b\big)\cdot_V c\\
&-x\cdot(y\cdot z) - \mathfrak{l}(x)\big(\mathfrak{l}(y)c + \mathfrak{r}(z)b + b\cdot_V c\big) - \mathfrak{r}(y\cdot z)a - a\cdot_V\big(\mathfrak{l}(y)c + \mathfrak{r}(z)b + b\cdot_V c\big)\\
&+(y\cdot x)\cdot z + \mathfrak{l}(y\cdot x)c + \mathfrak{r}(z)\big(\mathfrak{l}(y)a + \mathfrak{r}(x)b + b\cdot_V a\big) + \big(\mathfrak{l}(y)a + \mathfrak{r}(x)b + b\cdot_V a\big)\cdot_V c\\
&-y\cdot(x\cdot z) - \mathfrak{l}(y)\big(\mathfrak{l}(x)c + \mathfrak{r}(z)a + a\cdot_V c\big) - \mathfrak{r}(x\cdot z)b - b\cdot_V\big(\mathfrak{l}(x)c + \mathfrak{r}(z)a + a\cdot_V c\big).
\end{aligned}$$

Hence, $as_{A\oplus V}(x+a,y+b,z+c) + as_{A\oplus V}(y+b,x+a,z+c) = 0$ if and only if (3.2), (3.12) and (3.14) hold.
Analogously, $as_{A\oplus V}(z+c,x+a,y+b) + as_{A\oplus V}(z+c,y+b,x+a) = 0$ if and only if (3.3), (3.13) and (3.15) hold. □

Definition 3.5 ([3]). A **post-alternative algebra** $(A,\prec,\succ,\cdot)$ is a vector space A equipped with bilinear operations $\prec,\succ,\cdot: A\otimes A\to A$ obeying the following equations for $\star = \prec + \succ + \cdot$ and all $x,y,z\in A$,

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) + (y\cdot x)\cdot z - y\cdot(x\cdot z) = 0, \tag{3.17}$$
$$(z\cdot x)\cdot y - z\cdot(x\cdot y) + (z\cdot y)\cdot x - z\cdot(y\cdot x) = 0, \tag{3.18}$$
$$(x\cdot y)\prec z - x\cdot(y\prec z) + (y\cdot x)\prec z - y\cdot(x\prec z) = 0, \tag{3.19}$$
$$(x\succ y)\cdot z - x\succ(y\cdot z) + (x\succ z)\cdot y - x\succ(z\cdot y) = 0, \tag{3.20}$$
$$(y\succ x)\cdot z - x\cdot(y\succ z) + (x\prec y)\cdot z - y\succ(x\cdot z) = 0, \tag{3.21}$$
$$(z\prec x)\cdot y - z\cdot(x\succ y) + (z\cdot y)\prec x - z\cdot(y\prec x) = 0, \tag{3.22}$$
$$(x\succ y)\prec z - x\succ(y\prec z) + (y\prec x)\prec z - y\prec(x\star z) = 0, \tag{3.23}$$
$$(x\succ y)\prec z - x\succ(y\prec z) + (z\star x)\succ y - z\succ(x\succ y) = 0, \tag{3.24}$$
$$(x\star y)\succ z - x\succ(y\succ z) + (y\star x)\succ z - y\succ(x\succ z) = 0, \tag{3.25}$$
$$(z\prec x)\prec y - z\prec(x\star y) + (z\prec y)\prec x - z\prec(y\star x) = 0. \tag{3.26}$$

Remark 3.2. Let $(A,\prec,\succ,\cdot)$ be a post-alternative algebra. If the operation $\cdot$ is trivial, then it is a pre-alternative algebra.

Let $(A,\prec,\succ,\cdot)$ be a post-alternative algebra, it is obvious that $(A,\cdot)$ is an alternative algebra. On the other hand, it is straightforward to get the following conclusion:

Theorem 3.1. *If $(A,\prec,\succ,\cdot)$ is a post-alternative algebra, then with a new bilinear operation $\star : A\times A\to A$ on A defined for all $x,y\in A$ by*

$$x\star y = x\prec y + x\succ y + x\cdot y, \tag{3.27}$$

$(A,\star)$ becomes an alternative algebra. It is called the associated alternative algebra of $(A,\prec,\succ,\cdot)$.

Proof. In fact, for any $x,y,z\in A$, we have

$$\begin{aligned}
&as_A(x,y,z)+as_A(y,x,z) = (x\star y)\star z - x\star(y\star z)+(y\star x)\star z - y\star(x\star z)\\
&= (x\star y)\prec z + (x\star y)\succ z + (x\star y)\cdot z - x\prec (y\star z) - x\succ (y\star z) - x\cdot (y\star z)\\
&\quad + (y\star x)\prec z + (y\star x)\succ z + (y\star x)\cdot z - y\prec (x\star z) - y\succ (x\star z) - y\cdot (x\star z)\\
&= (x\prec y)\prec z + (x\succ y)\prec z + (x\cdot y)\prec z + (x\star y)\succ z + (x\prec y)\cdot z + (x\succ y)\cdot z\\
&\quad + (x\cdot y)\cdot z - x\prec (y\star z) - x\succ (y\prec z) - x\succ (y\succ z) - x\succ (y\cdot z) - x\cdot (y\prec z)\\
&\quad - x\cdot (y\succ z) - x\cdot (y\cdot z) + (y\prec x)\prec z + (y\succ x)\prec z + (y\cdot x)\prec z + (y\star x)\succ z\\
&\quad + (y\prec x)\cdot z + (y\succ x)\cdot z + (y\cdot x)\cdot z - y\prec (x\star z) - y\succ (x\prec z) - y\succ (x\succ z)\\
&\quad - y\succ (x\cdot z) - y\cdot (x\prec z) - y\cdot (x\succ z) - y\cdot (x\cdot z) = 0,
\end{aligned}$$

and then replacing (x,y,z) in this computation by (z,x,y) yields $as_A(z,x,y)+as_A(z,y,x)=0$, which completes the proof according to Definition 3.1 and Remark 3.1. □

The following terminology is motivated by the notion of λ-weighted $\mathcal{O}$-operator as a generalization of (the operator form of) the classical Yang-Baxter equation in [2, 19].

Definition 3.6. Let $(A,\cdot)$ be an alternative algebra and $(V,\cdot_V,\mathfrak{l},\mathfrak{r})$ be an A-bimodule alternative algebra. A linear map $T:V\to A$ is called a λ-weighted $\mathcal{O}$-operator associated to $(V,\cdot_V,\mathfrak{l},\mathfrak{r})$ if T satisfies, for all $a,b\in V$,

$$T(a)\cdot T(b) = T\big(\mathfrak{l}(T(a))b + \mathfrak{r}(T(b))a + \lambda a\cdot_V b\big). \tag{3.28}$$

When $(V,\cdot_V,\mathfrak{l},\mathfrak{r}) = (A,\cdot,L_\cdot,R_\cdot)$, the condition (3.28) becomes

$$\mathcal{R}(x)\cdot\mathcal{R}(y) = \mathcal{R}(\mathcal{R}(x)\cdot y + x\cdot\mathcal{R}(y) + \lambda x\cdot y). \tag{3.29}$$

The property (3.29) implies that $\mathcal{R}:A\to A$ is a λ-weighted Rota-Baxter operator on the alternative algebra $(A,\cdot)$.

Theorem 3.2. *Let $(A,\cdot)$ be an alternative algebra and $(V,\cdot_V,\mathfrak{l},\mathfrak{r})$ be an A-bimodule alternative algebra. Let $T:V\to A$ be a λ-weighted $\mathcal{O}$-operator associated to $(V,\cdot_V,\mathfrak{l},\mathfrak{r})$. Define three new bilinear operations $\prec,\succ,\circ : V\otimes V\to V$ on V as follows:*

$$a\succ b = \mathfrak{l}(T(a))b,\quad a\prec b = \mathfrak{r}(T(b))a,\quad a\circ b = \lambda a\cdot_V b. \tag{3.30}$$

Then $(V,\prec,\succ,\circ)$ becomes a post-alternative algebra and T is a homomorphism of alternative algebras.

Proof. Since A is an alternative algebra, (3.17) and (3.18) obviously hold. Furthermore, for any $a,b,c\in V$, we have

$$\begin{aligned}
&(a\circ b)\prec c - a\circ(b\prec c) + (b\circ a)\prec c - b\circ(a\prec c)\\
=&(\lambda a\cdot_V b)\prec c - a\circ(\mathfrak{r}(T(c))b) + (\lambda b\cdot_V a)\prec c - b\circ(\mathfrak{r}(T(c))a)\\
=&\lambda\big(\mathfrak{r}(T(c))(a\cdot_V b) - a\cdot_V(\mathfrak{r}(T(c))b) + \mathfrak{r}(T(c))(b\cdot_V a) - b\cdot_V(\mathfrak{r}(T(c))a)\big) = 0.
\end{aligned}$$

So, (3.19) holds. Moreover, (3.20) holds. Indeed,

$$\begin{aligned}
&(a\succ b)\circ c - a\succ(b\circ c) + (a\succ c)\circ b - a\succ(c\circ b)\\
=&(\mathfrak{l}(T(a))b)\circ c - a\succ(\lambda b\cdot_V c) + (\mathfrak{l}(T(a))c)\circ b - a\succ(\lambda c\cdot_V b)\\
=&\lambda\big((\mathfrak{l}(T(a))b)\cdot_V c - \mathfrak{l}(T(a))(b\cdot_V c) + (\mathfrak{l}(T(a))c)\cdot_V b - \mathfrak{l}(T(a))(c\cdot_V b)\big) = 0.
\end{aligned}$$

To prove identity (3.21), we compute as follows

$$\begin{aligned}&(b \succ a) \circ c - a \circ (b \succ c) + (a \prec b) \circ c - b \succ (a \circ c)\\ =&(\mathfrak{l}(T(b))a) \circ c - a \circ (\mathfrak{l}(T(b))c) + (\mathfrak{r}(T(b))a) \circ c - b \succ (\lambda a \cdot_V c)\\ =&\lambda\big(\mathfrak{l}(T(b))a) \cdot_V c - a \cdot_V (\mathfrak{l}(T(b))c) + (\mathfrak{r}(T(b))a) \cdot_V c - \mathfrak{l}(T(b))(a \cdot_V c)\big) = 0.\end{aligned}$$

The other identities can be shown similarly. □

Corollary 3.1. *Let $(A, \cdot)$ be an alternative algebra and $\mathcal{R} : A \to A$ be a λ-weighted Rota-Baxter operator for A. Then $(A, \prec, \succ, \circ)$ is a post-alternative algebra with the operations*

$$x \prec y = x \cdot \mathcal{R}(y), \quad x \succ y = \mathcal{R}(x) \cdot y, \quad x \circ y = \lambda x \cdot y.$$

4. Weighted $\mathcal{O}$-operators and post-Malcev algebras

We start this section by introducing the notion of post-Malcev algebra together with some of its basic properties. We will also briefly discuss the post-Malcev algebra structure underneath the λ-weighted $\mathcal{O}$-operators. We then show that there is a close relationship between post-Malcev algebras and post-alternative algebras in parallel to the relationship between pre-Malcev and pre-alternative algebras.

4.1. A-module Malcev algebras and weighted $\mathcal{O}$-operators

Now, we extend the concept of a module to that of an A-module algebra by replacing the $\mathbb{K}$-module V by a Malcev algebra. Next, we introduce λ-weighted $\mathcal{O}$-operators on Malcev algebras and study some basic properties.

Definition 4.1. Let $(A, [\cdot,\cdot])$ and $(V, [\cdot,\cdot]_V)$ be two Malcev algebras. Let $\rho : A \to End(V)$ be a linear map such that (V, ρ) is a representation of $(A, [\cdot,\cdot])$ and the following compatibility conditions hold for all $x, y, \in A,\ a, b, c \in V$:

$$\rho([x,y])[a,b]_V = \rho(x)[\rho(y)a,b]_V - [\rho(y)\rho(x)a,b]_V - [\rho(x)\rho(y)b,a]_V + \rho(y)[\rho(x)b,a]_V, \tag{4.1}$$

$$[\rho(x)a,\rho(y)b]_V = [\rho([x,y])a,b]_V - \rho(x)[\rho(y)a,b]_V + \rho(y)\rho(x)[a,b]_V + [\rho(y)\rho(x)b,a]_V, \tag{4.2}$$

$$[\rho(x)a,[b,c]_V]_V = [[\rho(x)b,a]_V,c]_V - \rho(x)[[b,a]_V,c]_V - [\rho(x)[a,c]_V,b]_V - [[\rho(x)c,b]_V,a]_V. \tag{4.3}$$

Then $(V, [\cdot,\cdot]_V, \rho)$ is called an **A-module Malcev algebra**.

In the sequel, an A-module Malcev algebra is denoted by $(V; [\cdot,\cdot]_V, \rho)$. It is straightforward to get the following:

Proposition 4.1. *Let $(A, [\cdot,\cdot])$ and $(V, [\cdot,\cdot]_V)$ be two Malcev algebras and $(V; [\cdot,\cdot]_V, \rho)$ be an A-module Malcev algebra. Then $(A \oplus V, [\cdot,\cdot]_\rho)$ carries a new Malcev algebra structure with bracket*

$$[x+a, y+b]_\rho = [x,y] + \rho(x)b - \rho(y)a + [a,b]_V, \quad \forall x,y \in A, \quad a,b \in V. \tag{4.4}$$

This is called the semi-direct product, often denoted by $A \ltimes_\rho V$ or simply $A \ltimes V$.

Proof. For $x, y, z, t \in A$ and $a, b, c, d \in V$,

$$\begin{aligned}[[x+a,z+c]_\rho,[y+b,t+d]_\rho]_\rho =& [[x,z],[y,t]] + \rho([x,z])\rho(y)d - \rho([x,z])\rho(t)b\\ &+ \rho([x,z])[b,d]_V - \rho([y,t])\rho(x)c + \rho([y,t])\rho(z)a - \rho([y,t])[a,c]_V + [\rho(x)c,\rho(y)d]_V\\ &- [\rho(x)c,\rho(t)b]_V + [\rho(x)c,[b,d]_V]_V - [\rho(z)a,\rho(y)d]_V + [\rho(z)a,\rho(t)b]_V - [\rho(z)a,[b,d]_V]_V\\ &+ [[a,c]_V,\rho(y)d]_V - [[a,c]_V,\rho(t)b]_V + [[a,c]_V,[b,d]_V]_V,\end{aligned}$$

$$[[[x+a,y+b]_\rho,z+c]_\rho,t+d]_\rho = [[[x,y],z],t] + \rho([[x,y],z])d - \rho(t)\rho([x,y])c$$

$$
\begin{aligned}
&+\rho(t)\rho(z)\rho(x)b-\rho(t)\rho(z)\rho(y)a+\rho(t)\rho(z)[a,b]_V-\rho(t)[\rho(x)b,c]_V+\rho(t)[\rho(y)a,c]_V\\
&-\rho(t)[[a,b]_V,c]_V+[\rho([x,y])c,d]_V-[\rho(z)\rho(x)b,d]_V+[\rho(z)\rho(y)a,d]_V-[\rho(z)[a,b]_V,d]_V\\
&+[[\rho(x)b,c]_V,d]_V-[[\rho(y)a,c]_V,d]_V+[[[a,b]_V,c]_V,d]_V,
\end{aligned}
$$

$$
\begin{aligned}
[[[y+b,z+c]_\rho,t+d]_\rho,x+a]_\rho&=[[[y,z],t],x]+\rho([[y,z],t])a-\rho(x)\rho([y,z])d\\
&+\rho(x)\rho(t)\rho(y)c-\rho(x)\rho(t)\rho(z)b+\rho(x)\rho(t)[b,c]_V-\rho(x)[\rho(y)c,d]_V+\rho(x)[\rho(z)b,d]_V\\
&-\rho(x)[[b,c]_V,d]_V+[\rho([y,z])d,a]_V-[\rho(t)\rho(y)c,a]_V+[\rho(t)\rho(z)b,a]_V-[\rho(t)[b,c]_V,a]_V\\
&+[[\rho(y)c,d]_V,a]_V-[[\rho(z)b,d]_V,a]_V+[[[b,c]_V,d]_V,a]_V,
\end{aligned}
$$

$$
\begin{aligned}
[[[z+c,t+d]_\rho,x+a]_\rho,y+b]_\rho&=[[[z,t],x],y]+\rho([[z,t],x])b-\rho(y)\rho([z,t])a\\
&+\rho(y)\rho(x)\rho(z)d-\rho(y)\rho(x)\rho(t)c+\rho(y)\rho(x)[c,d]_V-\rho(y)[\rho(z)d,a]_V+\rho(y)[\rho(t)c,a]_V\\
&-\rho(y)[[c,d]_V,a]_V+[\rho([z,t])a,b]_V-[\rho(x)\rho(z)d,b]_V+[\rho(x)\rho(t)c,b]_V-[\rho(x)[c,d]_V,b]_V\\
&+[[\rho(z)d,a]_V,b]_V-[[\rho(t)c,a]_V,b]_V+[[[c,d]_V,a]_V,b]_V,
\end{aligned}
$$

$$
\begin{aligned}
[[[t+d,x+a]_\rho,y+b]_\rho,z+c]_\rho&=[[[t,x],y],z]+\rho([[t,x],y])c-\rho(z)\rho([t,x])b\\
&+\rho(z)\rho(y)\rho(t)a-\rho(z)\rho(y)\rho(x)d+\rho(z)\rho(y)[d,a]_V-\rho(z)[\rho(t)a,b]_V+\rho(z)[\rho(x)d,b]_V\\
&-\rho(z)[[d,a]_V,b]_V+[\rho([t,x])b,c]_V-[\rho(y)\rho(t)a,c]_V+[\rho(y)\rho(x)d,c]_V-[\rho(y)[d,a]_V,c]_V\\
&+[[\rho(t)a,b]_V,c]_V-[[\rho(x)d,b]_V,c]_V+[[[d,a]_V,b]_V,c]_V.
\end{aligned}
$$

Then $A\oplus V$ is a Malcev algebra if and only if (V,ρ) is a representation on A satisfying (4.1)-(4.3). □

Remark 4.1. More generally, if we define a λ-semi-direct product denoted by $A\ltimes^\lambda V$ as follow

$$[x+a,y+b]_\rho^\lambda=[x,y]+\rho(x)b-\rho(y)a+\lambda[a,b]_V,\quad \forall x,y\in A,\quad a,b\in V \tag{4.5}$$

we obtain the same characterization given in the above Proposition.

Example 4.1. It is known that (A,ad) is a representation of A called the adjoint representation. Then $(A,[\cdot,\cdot],ad)$ is an A-module Malcev algebra.

Proposition 4.2. *Let $(A,\cdot)$ be an alternative algebra. Then the triplet $(V;[\cdot,\cdot]_V,\mathfrak{l}-\mathfrak{r})$ defines an A-module Malcev admissible algebra of $(A,[\cdot,\cdot])$.*

Proof. By Proposition 3.3, $A\ltimes_{\mathfrak{l},\mathfrak{r}}V$ is an alternative algebra. For its associated Malcev algebra $(A\oplus V,\overbrace{[\cdot,\cdot]})$, we have

$$
\begin{aligned}
\overbrace{[x+a,y+b]}&=(x+a)*(y+b)-(y+b)*(x+a)\\
&=x\cdot y+\mathfrak{l}(x)b+\mathfrak{r}(y)a+a\cdot_V b-y\cdot x-\mathfrak{l}(y)a-\mathfrak{r}(x)b-b\cdot_V a\\
&=[x,y]+(\mathfrak{l}-\mathfrak{r})(x)b-(\mathfrak{l}-\mathfrak{r})(y)a+[a,b]_V.
\end{aligned}
$$

According to (4.4), we deduce that $(V;[\cdot,\cdot]_V,\mathfrak{l}-\mathfrak{r})$ is an A-module Malcev admissible algebra of $(A,[\cdot,\cdot])$. □

Definition 4.2. Let $(A,[\cdot,\cdot])$ be a Malcev algebra and $(V;[\cdot,\cdot]_V,\rho)$ be an A-module Malcev algebra. A linear map $T:V\to A$ is said to be a **λ-weighted $\mathcal{O}$-operator** associated to $(V;[\cdot,\cdot]_V,\rho)$ if for all $a,b\in V$,

$$[T(a),T(b)]=T\big(\rho(T(a))b-\rho(T(b))a+\lambda[a,b]_V\big). \tag{4.6}$$

Obviously, a λ-weighted $\mathcal{O}$-operator associated to $(A,[\cdot,\cdot],ad)$ is just a λ-weighted Rota-Baxter operator on A. A λ-weighted $\mathcal{O}$-operator can be viewed as the relative version of a Rota-Baxter operator in the sense that the domain and range of an $\mathcal{O}$-operator might be different.

Example 4.2. (i) A Rota-Baxter operator on A is simply a 0-weighted $\mathcal{O}$-operator.
(ii) The identity map $id : A \to A$ is a (-1)-weighted $\mathcal{O}$-operator.
(iii) If $f : A \to A$ is a Malcev algebra homomorphism and $f^2 = f$ (idempotent condition), then f is a (-1)-weighted $\mathcal{O}$-operator.
(iv) If T is a λ-weighted $\mathcal{O}$-operator, then for any $\nu \in \mathbb{K}$, the map νT is a $(\nu\lambda)$-weighted $\mathcal{O}$-operator.
(v) If T is a λ-weighted $\mathcal{O}$-operator, then $-\lambda id - T$ is a λ-weighted $\mathcal{O}$-operator.

In the following, we characterize λ-weighted $\mathcal{O}$-operators in terms of their graph.

Proposition 4.3. *Let $(V; [\cdot,\cdot]_V, \rho)$ be an A-module Malcev algebra. Then a linear map $T : V \to A$ is a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot,\cdot]_V, \rho)$ if and only if the graph*

$$\mathrm{Gr}(T) = \{T(a) + a|\ a \in V\}$$

of the map T is a subalgebra of the λ-semi-direct product $A \ltimes^{\lambda} V$.

Proof. Let $T : V \to A$ be a linear map. For all $a, b \in V$, we have

$$[T(a) + a, T(b) + b]^{\lambda}_{\rho} = [T(a), T(b)] + \rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V,$$

which implies that the graph $\mathrm{Gr}(T) = \{T(a) + a|\ a \in V\}$ is a subalgebra of the Malcev algebra $A \ltimes^{\lambda} V$ if and only if T satisfies

$$[T(a), T(b)] = T\big(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V\big),$$

which means that T is a λ-weighted $\mathcal{O}$-operator. $\square$

As a consequence of the above proposition, we get the following.

Corollary 4.1. *Let $T : V \to A$ be a λ-weighted $\mathcal{O}$-operator. Since $\mathrm{Gr}(T)$ is isomorphism to V as a vector space, we get that V inherits a new Malcev algebra structure with the bracket*

$$[a, b]_T := \rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V, \quad \textit{for } a, b \in V.$$

In other words, $(V, [\cdot,\cdot]_T)$ is a Malcev algebra, denoted by V_T (called the induced Malcev algebra). Moreover, $T : V_T \to A$ is a homomorphism of Malcev algebras.

Let $T, T' : (A, [\cdot,\cdot]) \to (V, [\cdot,\cdot]_V)$ be two λ-weighted $\mathcal{O}$-operators. A **homomorphism** from T to T' consists of Malcev algebra homomorphisms $\phi : A \to A$ and $\psi : V \to V$ such that

$$\phi \circ T = T' \circ \psi, \tag{4.7}$$

$$\psi(\rho(x)a) = \rho(\phi(x))(\psi(a)), \quad \forall x \in A, a \in V. \tag{4.8}$$

In particular, if both ϕ and ψ are invertible, (ϕ, ψ) is called an **isomorphism** from T to T'.

Proposition 4.4. *Let (ϕ, ψ) be a homomorphism of λ-weighted $\mathcal{O}$-operators from T to T'. Then $\psi : V \to V$ is a homomorphism of induced Malcev algebras from $(V, [\cdot,\cdot]_T)$ to $(V, [\cdot,\cdot]_{T'})$.*

Proof. For any $a, b \in V$, we have

$$\begin{aligned}\psi([a, b]_T) &= \psi\big(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V\big)\\ &= \rho(\phi(T(a)))(\psi(b)) - \rho(\phi(T(b)))(\psi(a)) + \lambda[\psi(a), \psi(b)]_V\\ &= \rho(T'(\psi(a)))(\psi(b)) - \rho(T'(\psi(b)))(\psi(a)) + \lambda[\psi(a), \psi(b)]_V = [\psi(a), \psi(b)]_{T'}.\end{aligned}$$

This shows that $\psi : (V, [\cdot,\cdot]_T) \to (V, [\cdot,\cdot]_{T'})$ is a homomorphism of Malcev algebras. $\square$

In the sequel, we characterize λ-weighted $\mathcal{O}$-operators associated to $(V;[\cdot,\cdot]_V,\rho)$ in terms of the Nijenhuis operators. Recall that a Nijenhuis operator on a Malcev algebra $(A,[\cdot,\cdot])$ is a linear map $N:A\to A$ satisfying, for all $x,y\in A$,

$$[N(x),N(y)]=N\big([N(x),y]-[N(y),x]-N([x,y])\big).$$

Proposition 4.5. *Let $(V;[\cdot,\cdot]_V,\rho)$ be an A-module Malcev algebra. Then a linear map $T:V\to A$ is a λ-weighted $\mathcal{O}$-operator associated to $(V;[\cdot,\cdot]_V,\rho)$ if and only if*

$$N_T=\begin{bmatrix}\lambda id & -T\\ 0 & 0\end{bmatrix}:A\oplus V\to A\oplus V$$

is a Nijenhuis operator on the semi-direct product Malcev algebra $A\ltimes V$.

Proof. For all $x,y\in A$, $a,b\in V$, on the one hand, we have

$$\begin{aligned}[N_T(x+a),N_T(y+b)]_\rho&=[\lambda x-T(a),\lambda y-T(b)]_\rho\\&=\lambda^2[x,y]-\lambda[x,T(b)]-\lambda[T(a),y]+[T(a),T(b)].\end{aligned}$$

On the other hand, since $N_T^2=N_T$, we have

$$\begin{aligned}&N_T\big([N_T(x+a),y+b]_\rho-[N_T(y+b),x+a]_\rho-N_T([x+a,y+b]_\rho)\big)\\=&N_T\big([\lambda x-T(a),y+b]_\rho-[\lambda y-T(b),x+a]_\rho-N_T([x,y]+\rho(x)b-\rho(y)a+[a,b]_V)\big)\\=&\lambda^2[x,y]-\lambda[x,T(b)]-\lambda[T(a),y]+T\big(\rho(T(a))b-\rho(T(b))a+\lambda[a,b]_V\big).\end{aligned}$$

Therefore, N_T is a Nijenhuis operator on the semi-direct product Malcev algebra $A\ltimes V$ if and only if (4.6) is satisfied. □

Corollary 4.2. *A linear map $T:V\to A$ is a λ-weighted $\mathcal{O}$-operator associated to $(V;[\cdot,\cdot]_V,\rho)$ if and only if the operator*

$$N_T=\begin{bmatrix}id & -T\\ 0 & 0\end{bmatrix}:A\oplus V\to A\oplus V$$

is a Nijenhuis operator on the λ-semi-direct product Malcev algebra $(A\oplus V,[\cdot,\cdot]_\rho^\lambda)$.

4.2. Definition and constructions of post-Malcev algebras

In this section, we introduce the notion of post-Malcev algebras. We show that post-Malcev algebras arise naturally from a λ-weighted $\mathcal{O}$-operators. Therefore, post-Malcev algebras can be viewed as the underlying algebraic structures of λ-weighted $\mathcal{O}$-operators on Malcev algebras. Finally, we study some properties of post-Malcev algebras.

Definition 4.3. A **post-Malcev algebra** $(A,[\cdot,\cdot],\rhd)$ is a Malcev algebra $(A,[\cdot,\cdot])$ together with a bilinear map $\rhd:A\otimes A\to A$ such that for all $x,y,z\in A$, and $\{x,y\}=x\rhd y-y\rhd x+[x,y]$,

$$\{x,z\}\rhd[y,t]=x\rhd[z\rhd y,t]-[z\rhd(x\rhd y),t]-[x\rhd(z\rhd t),y]+z\rhd[x\rhd t,y], \tag{4.9}$$

$$[x\rhd z,y\rhd t]=[\{x,y\}\rhd z,t]-x\rhd[y\rhd z,t]+y\rhd(x\rhd[z,t])+[y\rhd(x\rhd t),z], \tag{4.10}$$

$$[x\rhd z,[y,t]]=[[x\rhd y,z],t]-x\rhd[[y,z],t]-[x\rhd[z,t],y]-[[x\rhd t,y],z], \tag{4.11}$$

$$\{\{x,y\},z\}\rhd t=x\rhd(y\rhd(z\rhd t))-z\rhd(x\rhd(y\rhd t))-y\rhd(\{x,z\}\rhd t)-\{y,z\}\rhd(x\rhd t). \tag{4.12}$$

Example 4.3.

(1) A pre-Malcev algebra is a post-Malcev algebra with an abelian Malcev algebra $(A,[\cdot,\cdot]=0,\rhd)$. (See [16, 26] for more details.)
(2) Post-Malcev algebras generalize post-Lie algebras.
(3) If $(A,[\cdot,\cdot])$ is a Malcev algebra, then $(A,[\cdot,\cdot],\rhd)$ is a post-Malcev algebra, where $x\rhd y=[y,x]$ for all $x,y\in A$.

Let $(A,[\cdot,\cdot],\rhd)$ and $(A',[\cdot,\cdot]',\rhd')$ be two post-Malcev algebras. A homomorphism of post-Malcev algebras is a linear map $f: A\to A'$ such that $f([x,y])=[f(x),f(y)]'$ and $f(x\rhd y)=f(x)\rhd' f(y)$.

Proposition 4.6. *Let $(A,[\cdot,\cdot],\rhd)$ be a post-Malcev algebra. Then the bracket*

$$\{x,y\}=x\rhd y-y\rhd x+[x,y] \tag{4.13}$$

defines a Malcev algebra structure on A. We denote this algebra by A^C and we call it the sub-adjacent Malcev algebra of A.

Proof. The skew symmetry is obvious. For all $x,y,z,t\in A$, we have

$$\begin{aligned}
&\{\{x,z\},\{y,t\}\}=\{x,z\}\rhd\{y,t\}-\{y,t\}\rhd\{x,z\}+[\{x,z\},\{y,t\}]\\
&=\{x,z\}\rhd(y\rhd t)-\{x,z\}\rhd(t\rhd y)+\{x,z\}\rhd[y,t]-\{y,t\}\rhd(x\rhd z)\\
&\quad+\{y,t\}\rhd(z\rhd x)-\{y,t\}\rhd[x,z]+[x\rhd z,y\rhd t]-[x\rhd z,t\rhd y]\\
&\quad+[x\rhd z,[y,t]]-[z\rhd x,y\rhd t]+[z\rhd x,t\rhd y]-[z\rhd x,[y,t]]\\
&\quad+[[x,z],y\rhd t]-[[x,z],t\rhd y]+[[x,z],[y,t]],
\end{aligned}$$

$$\begin{aligned}
&\{\{\{x,y\},z\},t\}=\{\{x,y\},z\}\rhd t-t\rhd\{\{x,y\},z\}+[\{\{x,y\},z\},t]\\
&=\{\{x,y\},z\}\rhd t-t\rhd(\{x,y\}\rhd z)+t\rhd(z\rhd(x\rhd y))-t\rhd(z\rhd(y\rhd x))\\
&\quad+t\rhd(z\rhd[x,y])-t\rhd[x\rhd y,z]+t\rhd[y\rhd x,z]-t\rhd[[x,y],z]\\
&\quad+[\{x,y\}\rhd z,t]-[z\rhd(x\rhd y),t]+[z\rhd(y\rhd x),t]-[z\rhd[x,y],t]\\
&\quad+[[x\rhd y,z],t]-[[y\rhd x,z],t]+[[[x,y],z],t],
\end{aligned}$$

$$\begin{aligned}
&\{\{\{y,z\},t\},x\}=\{\{y,z\},t\}\rhd x-x\rhd\{\{y,z\},t\}+[\{\{y,z\},t\},x]\\
&=\{\{y,z\},t\}\rhd x-x\rhd(\{y,z\}\rhd t)+x\rhd(t\rhd(y\rhd z))-x\rhd(t\rhd(z\rhd y))\\
&\quad+x\rhd(t\rhd[y,z])-x\rhd[y\rhd z,t]+x\rhd[z\rhd y,t]-x\rhd[[y,z],t]\\
&\quad+[\{y,z\}\rhd t,x]-[t\rhd(y\rhd z),x]+[t\rhd(z\rhd y),x]-[t\rhd[y,z],x]\\
&\quad+[[y\rhd z,t],x]-[[z\rhd y,t],x]+[[[y,z],t],x],
\end{aligned}$$

$$\begin{aligned}
&\{\{\{z,t\},x\},y\}=\{\{z,t\},x\}\rhd y-y\rhd\{\{z,t\},x\}+[\{\{z,t\},x\},y]\\
&=\{\{z,t\},x\}\rhd y-y\rhd(\{z,t\}\rhd x)+y\rhd(x\rhd(z\rhd t))-y\rhd(x\rhd(t\rhd z))\\
&\quad+y\rhd(x\rhd[z,t])-y\rhd[z\rhd t,x]+y\rhd[t\rhd z,x]-y\rhd[[z,t],x]\\
&\quad+[\{z,t\}\rhd x,y]-[x\rhd(z\rhd t),y]+[x\rhd(t\rhd z),y]-[x\rhd[z,t],y]\\
&\quad+[[z\rhd t,x],y]-[[t\rhd z,x],y]+[[[z,t],x],y],
\end{aligned}$$

$$\begin{aligned}
&\{\{\{t,x\},y\},z\}=\{\{t,x\},y\}\rhd z-z\rhd\{\{t,x\},y\}+[\{\{t,x\},y\},z]\\
&=\{\{t,x\},y\}\rhd z-z\rhd(\{t,x\}\rhd y)+z\rhd(y\rhd(t\rhd x))-z\rhd(y\rhd(x\rhd t))\\
&\quad+z\rhd(y\rhd[t,x])-z\rhd[t\rhd x,y]+z\rhd[x\rhd t,y]-z\rhd[[t,x],y]\\
&\quad+[\{t,x\}\rhd y,z]-[y\rhd(t\rhd x),z]+[y\rhd(x\rhd t),z]-[y\rhd[t,x],z]\\
&\quad+[[t\rhd x,y],z]-[[x\rhd t,y],z]+[[[t,x],y],z].
\end{aligned}$$

By the identity of Malcev algebra and (4.9)-(4.12), we have

$$\{\{x,z\},\{y,t\}\}-\{\{\{x,y\},z\},t\}-\{\{\{y,z\},t\},x\}-\{\{\{z,t\}x\},y\}-\{\{\{t,x\},y\},z\}=0. \quad\square$$

Remark 4.2. Let $(A,[\cdot,\cdot],\rhd)$ be a post-Malcev algebra. If $\rhd$ is commutative, $x\rhd y=y\rhd x$, then the two Malcev brackets $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ coincide.

Corollary 4.3. *If $(A,[\cdot,\cdot],\rhd)$ be a post-Malcev algebra, then $(A,\circ)$ is an admissible Malcev algebra, with the product $\circ$ defined for all $x,y\in A$ by*

$$x\circ y=x\rhd y+\frac{1}{2}[x,y]. \tag{4.14}$$

Proposition 4.7. *Let $(A, [\cdot,\cdot], \rhd)$ be a post-Malcev algebra. Define $L_\rhd : A \to A$ by $L_\rhd(x)y = x \rhd y$ for any $x, y \in A$. Then $(A; [\cdot,\cdot], L_\rhd)$ is an A-module Malcev algebra of $(A^C, \{\cdot,\cdot\})$.*

Proof. By (4.12), $L_\rhd$ is a representation of $(A^C, \{\cdot,\cdot\})$. Indeed, for $x, y, z, t \in A$,

$$\begin{aligned}L_\rhd(\{\{x,y\},z\})t &= \{\{x,y\},z\} \rhd t\\ &= x \rhd (y \rhd (z \rhd t)) - z \rhd (x \rhd (y \rhd t)) - y \rhd (\{x,z\} \rhd t) - \{y,z\} \rhd (x \rhd t)\\ &= L_\rhd(x)L_\rhd(y)L_\rhd(z)t - L_\rhd(z)L_\rhd(x)L_\rhd(y)t - L_\rhd(y)L_\rhd(\{x,z\})t\\ &\quad - L_\rhd(\{y,z\})L_\rhd(x)t.\end{aligned}$$

To prove (4.1), according to (4.9) we compute

$$\begin{aligned}&L_\rhd(\{x,z\})[y,t]\\ &= \{x,z\} \rhd [y,t] = x \rhd [z \rhd y, t] - [z \rhd (x \rhd y), t] - [x \rhd (z \rhd t), y] + z \rhd [x \rhd t, y]\\ &= L_\rhd(x)[L_\rhd(z)y, t] - [L_\rhd(z)L_\rhd(x)y, t] - [L_\rhd(x)L_\rhd(z)t, y] + L_\rhd(z)[L_\rhd(x)t, y].\end{aligned}$$

Similarly, by (4.10) and (4.11), we have

$$\begin{aligned}&[L_\rhd(x)z, L_\rhd(y)t]\\ &= [x \rhd z, y \rhd t] = [\{x,y\} \rhd z, t] - x \rhd [y \rhd z, t] + y \rhd (x \rhd [z,t]) + [y \rhd (x \rhd t), z]\\ &= [L_\rhd(\{x,y\})z, t] - L_\rhd(x)[L_\rhd(y)z, t] + L_\rhd(y)L_\rhd(x)[z,t] + [L_\rhd(y)L_\rhd(x)t, z],\end{aligned}$$

$$\begin{aligned}&[L_\rhd(x)z, [y,t]]\\ &= [x \rhd z, [y,t]] = [[x \rhd y, z], t] - x \rhd [[y,z],t] - [x \rhd [z,t], y] - [[x \rhd t, y], z]\\ &= [[L_\rhd(x)y, z], t] - L_\rhd(x)[[y,z],t] - [L_\rhd(x)[z,t], y] - [[L_\rhd(x)t, y], z].\end{aligned}$$

Therefore $(A; [\cdot,\cdot], L_\rhd)$ is an A-module Malcev algebra of $(A^C, \{\cdot,\cdot\})$. □

Proposition 4.8. *If $(A, [\cdot,\cdot], \rhd)$ is a post-Malcev algebra, then $(A, -[\cdot,\cdot], \blacktriangleright)$ is also a post-Malcev algebra, where for all $x, y \in A$,*

$$x \blacktriangleright y = x \rhd y + [x,y]. \tag{4.15}$$

Moreover, $(A, [\cdot,\cdot], \rhd)$ and $(A, -[\cdot,\cdot], \blacktriangleright)$ have the same sub-adjacent Malcev algebra A^C.

Proof. We check only that $(A, -[\cdot,\cdot], \blacktriangleright)$ verifies the first post-Malcev identity. The other identities can be verified similarly. In fact, for all $x, y, z, t \in A$,

$$\begin{aligned}&- \{x,z\} \blacktriangleright [y,t] + x \blacktriangleright [z \blacktriangleright y, t] - [z \blacktriangleright (x \blacktriangleright y), t] - [x \blacktriangleright (z \blacktriangleright t), y] + z \blacktriangleright [x \blacktriangleright t, y]\\ &= -\{x,z\} \rhd [y,t] - [\{x,z\},[y,t]] + x \rhd [z \rhd y, t] + x \rhd [[z,y],t] + [x, [z \rhd y, t]]\\ &\quad + [x, [[z,y],t]] - [z \rhd (x \rhd y), t] - [z \rhd [x,y], t] - [[z, x \rhd y], t] - [[z,[x,y]], t]\\ &\quad - [x \rhd (z \rhd t), y] - [x \rhd [z,t], y] - [[x, z \rhd t], y] - [[x,[z,t]], y] + z \rhd [x \rhd t, y]\\ &\quad + z \rhd [[x,t],y] + [z, [x \rhd t, y]] + [z, [[x,t],y]] = 0.\end{aligned}$$

□

Theorem 4.1. *If $(A, [\cdot,\cdot], \rhd)$ is a post-Malcev algebra, then $(A \times A, \cdot, \cdot)$ is a Malcev algebra, with the double bracket product $\cdot, \cdot$ on $A \times A$ defined for all $a, b, x, y \in A$ by*

$$(a,x),(b,y) = (a \rhd b - b \rhd a + [a,b], \quad a \rhd y - b \rhd x + [x,y]). \tag{4.16}$$

Proof. Let $x, y, z, t, a, b, c, d \in A$. It is obvious that $(a,x),(b,y) = -(b,y),(a,x)$. On the other hand,

$$\begin{aligned}&(a,x),(c,z),(b,y),(d,t) =\\ &\{\{a,c\},\{b,d\}\},\ (a \rhd c) \rhd (b \rhd t) - (a \rhd c) \rhd (d \rhd y) - (c \rhd a) \rhd (b \rhd t)\\ &+ (c \rhd a) \rhd (d \rhd y) + [a,c] \rhd (b \rhd t) - [a,c] \rhd (d \rhd y) + \{a,c\} \rhd [y,t]\\ &- (b \rhd d) \rhd (a \rhd z) + (b \rhd d) \rhd (c \rhd x) + (d \rhd b) \rhd (a \rhd z) - (d \rhd b) \rhd (c \rhd x)\end{aligned}$$

$$
\begin{aligned}
&- [b,d] \rhd (a \rhd z) + [b,d] \rhd (c \rhd x) - \{b,d\} \rhd [x,z] + [a \rhd z, b \rhd t] \\
&- [a \rhd z, d \rhd y] - [c \rhd x, b \rhd t] + [c \rhd x, d \rhd y] - [a \rhd z, [y,t]] + [c \rhd x, [y,t]] \\
&+ [[x,z], b \rhd t] - [[x,z], d \rhd y] - [[x,z],[y,t]],
\end{aligned}
$$

$$
\begin{aligned}
&(a,x),(b,y),(c,z),(d,t) = \\
&\{\{\{a,b\},c\},d\},\ ((a \rhd b) \rhd c) \rhd t - ((b \rhd a) \rhd c) \rhd t - (c \rhd (a \rhd b)) \rhd t \\
&+ (c \rhd (b \rhd a)) \rhd t + [a \rhd b, c] \rhd t - [b \rhd a, c] \rhd t + \{[a,b],c\} \rhd t \\
&- d \rhd ((a \rhd b) \rhd z) + d \rhd ((b \rhd a) \rhd z) - d \rhd ([a,b] \rhd z) + d \rhd (c \rhd (a \rhd y)) \\
&- d \rhd (c \rhd (b \rhd x)) + d \rhd (c \rhd [x,y]) - d \rhd [a \rhd y, z] + d \rhd [b \rhd x, z] - d \rhd [[x,y],z] \\
&+ [\{a,b\} \rhd z, t] - [c \rhd (a \rhd y), t] + [c \rhd (b \rhd x), t] - [c \rhd [x,y], t] \\
&+ [[a \rhd y, z], t] - [[b \rhd x, z], t] + [[[x,y],z],t],
\end{aligned}
$$

$$
\begin{aligned}
&(b,y),(c,z),(d,t),(a,x) = \\
&\{\{\{b,c\},d\},a\},\ ((b \rhd c) \rhd d) \rhd x - ((c \rhd b) \rhd d) \rhd x - (d \rhd (b \rhd c)) \rhd x \\
&+ (d \rhd (c \rhd b)) \rhd x + [b \rhd c, d] \rhd x - [c \rhd b, d] \rhd x + \{[b,c],d\} \rhd x \\
&- a \rhd ((b \rhd c) \rhd t) + a \rhd ((c \rhd b) \rhd t) - a \rhd ([b,c] \rhd t) + a \rhd (d \rhd (b \rhd z)) \\
&- a \rhd (d \rhd (c \rhd y)) + a \rhd (d \rhd [y,z]) - a \rhd [b \rhd z, t] + a \rhd [c \rhd y, t] - a \rhd [[y,z],t] \\
&+ [\{b,c\} \rhd t, x] - [d \rhd (b \rhd z), x] + [d \rhd (c \rhd y), x] - [d \rhd [y,z], x] \\
&+ [[b \rhd z, t], x] - [[c \rhd y, t], x] + [[[y,z],t],x],
\end{aligned}
$$

$$
\begin{aligned}
&(c,z),(d,t),(a,x),(b,y) = \\
&\{\{\{c,d\},a\},b\},\ ((c \rhd d) \rhd a) \rhd y - ((d \rhd c) \rhd a) \rhd y - (a \rhd (c \rhd d)) \rhd y \\
&+ (a \rhd (d \rhd c)) \rhd y + [c \rhd d, a] \rhd y - [d \rhd c, a] \rhd y + \{[c,d],a\} \rhd y \\
&- b \rhd ((c \rhd d) \rhd x) + b \rhd ((d \rhd c) \rhd x) - b \rhd ([c,d] \rhd x) + b \rhd (a \rhd (c \rhd t)) \\
&- b \rhd (a \rhd (d \rhd z)) + b \rhd (a \rhd [z,t]) - b \rhd [c \rhd t, x] + b \rhd [d \rhd z, x] - b \rhd [[z,t],x] \\
&+ [\{c,d\} \rhd x, y] - [a \rhd (c \rhd t), y] + [a \rhd (d \rhd z), y] - [a \rhd [z,t], y] \\
&+ [[c \rhd t, x], y] - [[d \rhd z, x], y] + [[[z,t],x],y],
\end{aligned}
$$

$$
\begin{aligned}
&(d,t),(a,x),(b,y),(c,z) = \\
&\{\{\{d,a\},b\},c\},\ ((d \rhd a) \rhd b) \rhd z - ((a \rhd d) \rhd b) \rhd z - (b \rhd (d \rhd a)) \rhd z \\
&+ (b \rhd (a \rhd d)) \rhd z + [d \rhd a, b] \rhd z - [a \rhd d, b] \rhd z + \{[d,a],b\} \rhd z \\
&- c \rhd ((d \rhd a) \rhd y) + c \rhd ((a \rhd d) \rhd y) - c \rhd ([d,a] \rhd y) + c \rhd (b \rhd (d \rhd x)) \\
&- c \rhd (b \rhd (a \rhd t)) + c \rhd (b \rhd [t,x]) - c \rhd [d \rhd x, y] + c \rhd [a \rhd t, y] - c \rhd [[t,x],y] \\
&+ [\{d,a\} \rhd y, z] - [y \rhd (d \rhd x), z] + [y \rhd (a \rhd t), z] - [y \rhd [t,x], z] \\
&+ [[d \rhd x, y], z] - [[a \rhd t, y], z] + [[[t,x],y],z].
\end{aligned}
$$

Hence, using (4.13) of Proposition 4.6 and Definition 4.3, we have

$$
\begin{aligned}
&(a,x),(c,z),(b,y),(d,t) - (a,x),(b,y),(c,z),(d,t) \\
&- (b,y),(c,z),(d,t),(a,x) - (c,z),(d,t),(a,x),(b,y) \\
&- (d,t),(a,x),(b,y),(c,z) = (0,0).
\end{aligned}
$$

□

The following results illustrate that a λ-weighted $\mathcal{O}$-operator induces a post-Malcev algebra structure.

Theorem 4.2. *Let $(A,[\cdot,\cdot]_A)$ be a Malcev algebra and $(V;[\cdot,\cdot]_V,\rho)$ an A-module Malcev algebra. Let $T : V \to A$ be a λ-weighted $\mathcal{O}$-operator associated to $(V;[\cdot,\cdot]_V,\rho)$.*

(i) *Define two new bilinear operations* $[\cdot,\cdot],\ \rhd : V\times V\to V$ *as follows, for all* $a,b\in V$,

$$[a,b]=\lambda[a,b]_V,\quad a\rhd b=\rho(T(a))b. \tag{4.17}$$

Then $(V,[\cdot,\cdot],\rhd)$ *is a post-Malcev algebra.*

(ii) T *is a Malcev algebra homomorphism from the sub-adjacent Malcev algebra* $(V,\{\cdot,\cdot\})$ *given in Proposition 4.6 to* $(A,[\cdot,\cdot]_A)$.

Proof. (i) We use (4.1)-(4.3) of representation of Malcev algebras on $\mathbb{K}$-algebra.

$$\begin{aligned}
&\{a,c\}\rhd[b,d]-a\rhd[c\rhd b,d]+[c\rhd(a\rhd b),d]+[a\rhd(c\rhd d),b]-c\rhd[a\rhd d,b]\\
&\quad=(\rho(T(a))c-\rho(T(c))a+\lambda[a,c]_V)\rhd\lambda[b,d]_V-\rho(T(a))[\rho(T(c))b,d]\\
&\qquad+[\rho(T(c))\rho(T(a))b,d]+[\rho(T(a))\rho(T(c))d,b]-\rho(T(c))[\rho(T(a))d,b]\\
&\quad=\lambda\Big(\rho(T(\rho(T(a))c)-T(\rho(T(c))a)+T(\lambda[a,c]_V))[b,d]_V-\rho(T(a))[\rho(T(c))b,d]_V\\
&\qquad+[\rho(T(c))\rho(T(a))b,d]_V+[\rho(T(a))\rho(T(c))d,b]_V-\rho(T(c))[\rho(T(a))d,b]_V\Big)=0,
\end{aligned}$$

$$\begin{aligned}
&[a\rhd c,b\rhd d]-[\{a,b\}\rhd c,d]+a\rhd[b\rhd c,d]-b\rhd(a\rhd[c,d])-[b\rhd(a\rhd d),c]\\
&\quad=[\rho(T(a))c,\rho(T(b))d]-\lambda[(\rho(T(a))b-\rho(T(b))a+\lambda[a,b]_V)\rhd c,d]_V\\
&\qquad+\rho(T(a))[\rho(T(b))c,d]-\rho(T(b))\rho(T(a))[c,d]-[\rho(T(b))\rho(T(a))d,c]\\
&\quad=\lambda\Big([\rho(T(a))c,\rho(T(b))d]_V-[\rho(T(\rho(T(a))b)-T(\rho(T(b))a)+T(\lambda[a,b]_V))c,d]_V\\
&\qquad+\rho(T(a))[\rho(T(b))c,d]_V-\rho(T(b))\rho(T(a))[c,d]_V-[\rho(T(b))\rho(T(a))d,c]_V\Big)=0,
\end{aligned}$$

$$\begin{aligned}
&[a\rhd c,[b,d]]-[[a\rhd b,c],d]+a\rhd[[b,c],d]+[a\rhd[c,d],b]+[[a\rhd d,b],c]\\
&\quad=[\rho(T(a))c,[b,d]]-[[\rho(T(a))b,c],d]+\rho(T(a))[[b,c],d]\\
&\qquad+[\rho(T(a))[c,d],b]+[[\rho(T(a))d,b],c]\\
&\quad=\lambda^2\Big([\rho(T(a))c,[b,d]_V]_V-[[\rho(T(a))b,c]_V,d]_V+\rho(T(a))[[b,c]_V,d]_V\\
&\qquad+[\rho(T(a))[c,d]_V,b]_V+[[\rho(T(a))d,b]_V,c]_V\Big)=0.
\end{aligned}$$

Using the condition (2.1) of Definition 2.1, we check

$$\begin{aligned}
&\{\{a,b\},c\}\rhd d-a\rhd(b\rhd(c\rhd d))+c\rhd(a\rhd(b\rhd d))+b\rhd(\{a,c\}\rhd d)+\{b,c\}\rhd(a\rhd d)\\
&\quad=\rho\big(T(\rho(T(\rho(T(a))b))c)-T(\rho(T(\rho(T(b))a))c)-T(\rho(T(c))\rho(T(a))b)\\
&\qquad+T(\rho(T(c))\rho(T(b))a)+T(\rho(T(\lambda[a,b]_V))c)-T(\rho(T(c))\lambda[a,b]_V)+T(\lambda[\rho(T(a))b,c]_V)\\
&\qquad-T(\lambda[\rho(T(b))a,c]_V)+T(\lambda^2[[a,b]_V,c]_V)\big)d-\rho(T(a))\rho(T(b))\rho(T(c))d\\
&\qquad+\rho(T(c))\rho(T(a))\rho(T(b))d+\rho(T(b))\rho\big(T(\rho(T(a))c)-T(\rho(T(c))a)+T(\lambda[a,c]_V)\big)d\\
&\qquad+\rho\big(T(\rho(T(b))c)-T(\rho(T(c))b)+T(\lambda[b,c]_V)\big)\rho(T(a))d=0.
\end{aligned}$$

(ii) The Malcev bracket $\{\cdot,\cdot\}$ is defined for all $a,b\in V$ by

$$\{a,b\}=a\rhd b-b\rhd a+[a,b]=\rho(T(a))b-\rho(T(b))a+\lambda[a,b]_V.$$

Then the sub-adjacent Malcev algebra of the above post-Malcev algebra $(V,[\cdot,\cdot],\rhd)$ is exactly the Malcev algebra $(V,[\cdot,\cdot]_T)$ given in Corollary 4.1 Then the result follows. □

Proposition 4.9. *Let* $T,T':(V,[\cdot,\cdot]_V)\to(A,[\cdot,\cdot])$ *be two* λ*-weighted* $\mathcal{O}$*-operators with respect to an* A*-module Malcev algebra* $(V;[\cdot,\cdot]_V,\rho)$*. Let* $(V,\{\cdot,\cdot\},\rhd)$ *and* $(V,\{\cdot,\cdot\}',\rhd')$ *be the post-Malcev algebras given in Theorem 4.2 and* (ϕ,ψ) *be a homomorphism from* T' *to* T*. Then* ψ *is a homomorphism from the post-Malcev algebra* $(V,\{\cdot,\cdot\},\rhd)$ *to the post-Malcev algebra* $(V,\{\cdot,\cdot\}',\rhd')$.

Proof. For all $a, b \in V$, by (4.7),(4.8) and (4.17), we have

$$\begin{aligned}\psi(\{a,b\}) &= \psi(\lambda[a,b]_V) = \lambda[\psi(a),\psi(b)]_V = \{\psi(a),\psi(b)\}',\\ \psi(a \rhd b) &= \psi(\rho(T(a))b) = \rho(\phi(T(a)))(\psi(b)) = \rho(T'(\psi(a)))(\psi(b)) = \psi(a) \rhd' \psi(b),\end{aligned}$$

which implies that ψ is a homomorphism between the post-Malcev algebras in Theorem 4.2. □

Given a Malcev algebra, the following result gives a necessary and sufficient condition to have a compatible post-Malcev algebra structure.

Proposition 4.10. *Let $(A, [\cdot,\cdot])$ be a Malcev algebra. Then there exists a compatible post-Malcev algebra structure on A if and only if there exists an A-module Malcev algebra $(V; [\cdot,\cdot]_V, \rho)$ and an invertible 1-weighted $\mathcal{O}$-operator $T : V \to A$.*

Proof. Let $(A, [\cdot,\cdot], \rhd)$ be a post-Malcev algebra and $(A, [\cdot,\cdot])$ be the associated Malcev algebra. Then the identity map $id : A \to A$ is an invertible 1-weighted $\mathcal{O}$-operator on $(A, [\cdot,\cdot])$ associated to $(A, [\cdot,\cdot], ad)$.

Conversely, suppose that there exists an invertible 1-weighted $\mathcal{O}$-operator of $(A, [\cdot,\cdot])$ associated to an A-module Malcev algebra $(V; [\cdot,\cdot]_V, \rho)$. Then, using Theorem 4.2, there is a post-Malcev algebra structure on $T(V) = A$ given by

$$\{x,y\} = \lambda T([T^{-1}(x), T^{-1}(y)]_V), \quad x \rhd y = T(\rho(x)T^{-1}(y)).$$

This is compatible post-Malcev algebra structure on $(A, [\cdot,\cdot])$. Indeed,

$$\begin{aligned}x \rhd y - y \rhd x + \{x,y\} &= T(\rho(x)T^{-1}(y) - \rho(y)T^{-1}(x) + [T^{-1}(x), T^{-1}(y)]_V)\\ &= [TT^{-1}(x), TT^{-1}(y)] = [x,y].\end{aligned}$$

□

An obvious consequence of Theorem 4.2 is the following construction of a post-Malcev algebra in terms of λ-weighted Rota-Baxter operator on a Malcev algebra.

Corollary 4.4. *Let $(A, [\cdot,\cdot])$ be a Malcev algebra and the linear map $\mathcal{R} : A \to A$ is a λ-weighted Rota-Baxter operator. Then, there exists a post-Malcev structure on A given, for all $x, y \in A$, by*

$$\{x,y\} = \lambda[x,y], \quad x \rhd y = [\mathcal{R}(x), y].$$

If in addition, $\mathcal{R}$ is invertible, then there is a compatible post-Malcev algebra structure on A given, for all $x, y \in A$, by

$$\{x,y\} = \mathcal{R}([\mathcal{R}^{-1}(x), \mathcal{R}^{-1}(y)], \quad x \rhd y = \mathcal{R}([x, \mathcal{R}^{-1}(y)]).$$

Example 4.4. In this example, we calculate (-1)-weighted Rota-Baxter operators on the Malcev algebra A and we give the corresponding post-Malcev algebras. Let A be the simple Malcev algebra over the field of complex numbers $\mathbb{C}$ [11, Example 3]. In this case A has a basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ with the following table of multiplication:

$[\cdot,\cdot]$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	$2e_2$	$-2e_3$	$2e_4$	$-2e_5$	$2e_6$	$-2e_7$
e_2	$-2e_2$	0	e_1	$2e_7$	0	$-2e_5$	0
e_3	$2e_3$	$-e_1$	0	0	$-2e_6$	0	$2e_4$
e_4	$-2e_4$	$-2e_7$	0	0	e_1	$2e_3$	0
e_5	$2e_5$	0	$2e_6$	$-e_1$	0	0	$-2e_2$
e_6	$-2e_6$	$2e_5$	0	$-2e_3$	0	0	e_1
e_7	$2e_7$	0	$-2e_4$	0	$2e_2$	$-e_1$	0

Now, define the linear map $\mathcal{R} : A \to A$ by

$$\mathcal{R}(e_1) = \frac{1}{2}e_1 + 2\alpha e_2 + 2\beta e_5 + 2\gamma e_6, \quad \mathcal{R}(e_2) = 0, \quad \mathcal{R}(e_3) = e_3 - \alpha e_1 + \delta e_5 - 2\beta e_6,$$

$$\mathcal{R}(e_4) = e_4 - \beta e_1 - \delta e_2 + \mu e_6, \quad \mathcal{R}(e_5) = \mathcal{R}(e_6) = 0, \quad \mathcal{R}(e_7) = e_7 - \gamma e_1 + 2\beta e_2 - \mu e_5.$$

Then $\mathcal{R}$ is a (-1)-weighted Rota-Baxter operator on A. Using Corollary 4.4, we can construct a post-Malcev algebra on A given by

$\{\cdot,\cdot\}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	$2\lambda e_2$	$-2\lambda e_3$	$2\lambda e_4$	$-2\lambda e_5$	$2\lambda e_6$	$-2\lambda e_7$
e_2	$-2\lambda e_2$	0	λe_1	$2\lambda e_7$	0	$-2\lambda e_5$	0
e_3	$2\lambda e_3$	$-\lambda e_1$	0	0	$-2\lambda e_6$	0	$2\lambda e_4$
e_4	$-2\lambda e_4$	$-2\lambda e_7$	0	0	λe_1	$2\lambda e_3$	0
e_5	$2\lambda e_5$	0	$2\lambda e_6$	$-\lambda e_1$	0	0	$-2\lambda e_2$
e_6	$-2\lambda e_6$	$2\lambda e_5$	0	$-2\lambda e_3$	0	0	λe_1
e_7	$2\lambda e_7$	0	$-2\lambda e_4$	0	$2\lambda e_2$	$-\lambda e_1$	0

$\rhd$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$4\beta e_5 - 4\alpha e_2$ $-4\gamma e_6$	$e_2 + 4\gamma e_5$	$2\alpha e_1 - e_3$ $+4\beta e_6$	$e_4 + 4\alpha e_7$ $-2\beta e_1 - 4\gamma e_3$	$-e_5$	$e_6 - 4\alpha e_5$	$2\gamma e_1 - e_7$ $-4\beta e_2$
e_2	0	0	0	0	0	0	0
e_3	$2e_3 + 2\delta e_5$ $+4\beta e_6$	$-e_1 - 2\alpha e_2$ $-4\beta e_5$	$2\alpha e_3 + 2\delta e_6$	$4\beta e_3 - 2\alpha e_4$ $-\delta e_1$	$2\alpha e_5 - 2e_6$	$-2\alpha e_6$	$2e_4 + 2\alpha e_7$ $-2\delta e_2 - 2\beta e_1$
e_4	$2\delta e_2 - 2e_4$ $-2\mu e_6$	$2\mu e_5 - 2e_7$ $-2\beta e_2$	$2\beta e_3 - \delta e_1$	$-2\beta e_4 - 2\delta e_7$ $-2\mu e_3$	$e_1 + 2\beta e_5$	$2e_3 + 2\delta e_5$ $-2\beta e_6$	$2\beta e_7 + \mu e_1$
e_5	0	0	0	0	0	0	0
e_6	0	0	0	0	0	0	0
e_7	$2e_7 - 4\beta e_2$ $-2\mu e_5$	$-2\gamma e_2$	$2\beta e_1 + 2\gamma e_3$ $-2e_4 - 2\mu e_6$	$\mu e_1 - 2\gamma e_4$ $+4\beta e_7$	$2e_2 + 2\gamma e_5$	$-e_1 - 2\gamma e_6$ $-4\beta e_5$	$2\gamma e_7 + 2\mu e_2$

The following result establishes a close relation between a post-alternative algebra and a post-Malcev algebra.

Theorem 4.3. *Let $T : V \to A$ be a λ-weighted $\mathcal{O}$-operator of alternative algebra $(A, \cdot)$ with respect to $(V, \cdot_V, \mathfrak{l}, \mathfrak{r})$ and $(V, \circ, \prec, \succ)$ be the associated post-alternative algebra given in Theorem 3.2. Then T is a λ-weighted $\mathcal{O}$-operator on the Malcev admissible algebra $(A, [\cdot,\cdot])$ with respect to an A-module Malcev algebra $(V; [\cdot,\cdot]_V, \mathfrak{l} - \mathfrak{r})$.*

Moreover, if $(V, \{\cdot,\cdot\}, \rhd)$ be a post-Malcev algebra associated to the Malcev admissible algebra $(A, [\cdot,\cdot])$ on $(V; [\cdot,\cdot]_V, \mathfrak{l} - \mathfrak{r})$. Then, the products $(\{\cdot,\cdot\}, \rhd)$ are related with $(\circ, \prec, \succ)$ as follow, for all $a, b \in V$,

$$\{a, b\} = a \circ b - b \circ a, \quad a \rhd b = a \succ b - b \prec a. \tag{4.18}$$

Proof. Using the condition of λ-weighted $\mathcal{O}$-operator in (3.28) and Proposition 4.2, for $a, b \in A$,

$$\begin{aligned}
&[T(a), T(b)] - T\big(\rho(T(a))b - \rho(T(b))a + \lambda[a, b]_V\big) \\
&= T(a) \cdot T(b) - T(b) \cdot T(a) - T\big((\mathfrak{l} - \mathfrak{r})(T(a))b - (\mathfrak{l} - \mathfrak{r})(T(b))a + \lambda(a \cdot_V b - b \cdot_V a\big) = 0.
\end{aligned}$$

Then T is a λ-weighted $\mathcal{O}$-operator on the Malcev admissible algebra $(A, [\cdot,\cdot])$ with respect to an A-module Malcev algebra $(V; [\cdot,\cdot]_V, \mathfrak{l} - \mathfrak{r})$.

On the other hand, from (3.30) of Theorem 3.2 and (4.17) of Theorem 4.2 that

$$\begin{aligned}
&\{a, b\} = \lambda[a, b]_V = \lambda a \cdot_V b - \lambda b \cdot_V a = a \circ b - b \circ a, \\
&a \rhd b = (\mathfrak{l} - \mathfrak{r})(T(a))b = \mathfrak{l}(T(a))b - \mathfrak{r}(T(a))b = a \succ b - b \prec a.
\end{aligned}$$

□

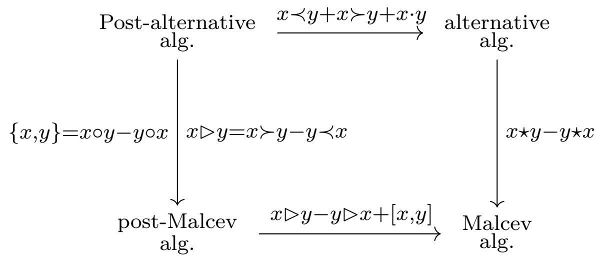
Corollary 4.5. *Let $(A, \circ, \prec, \succ)$ be a post-alternative algebra given in Corollary 3.1, $(A, \{\cdot,\cdot\}, \rhd)$ be a post-Malcev algebra associated to the Malcev algebras $(A, [\cdot,\cdot])$ and let $\mathcal{R}$*

be a λ-weighted Rota-Baxter operator of $(A, \cdot)$. Then, the operations

$$\{x, y\} = x \circ y - y \circ x, \quad x \rhd y = x \succ y - y \prec x, \tag{4.19}$$

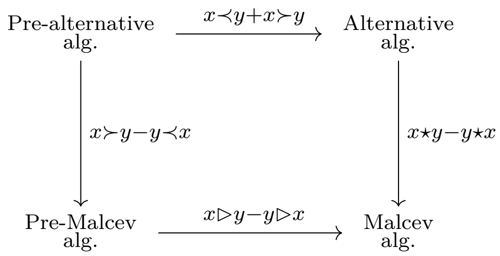
define a post-Malcev structure in A.

It is easy to see that (4.13) and (4.19) fit into the commutative diagram



(4.20)

When the operation $\cdot$ of the post-alternative algebra and the bracket $[\cdot,\cdot]$ of the post-Malcev algebra are both trivial, we obtain the following commutative diagram.



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