On a minimal set of generators for the algebra $H^*(BE_6; \mathbb{F}_2)$ as a module over the Steenrod algebra and applications

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Abstract

Let $P_n \cong H^*(BE_n; \mathbb{F}_2)$ be the graded polynomial algebra over the prime field of two elements $\mathbb{F}_2$, where $E_n$ is an elementary abelian 2-group of rank $n$, and $BE_n$ is the classifying space of $E_n$. We study the hit problem, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra $P_n$, viewed as a module over the mod-2 Steenrod algebra $A$. This problem remains unsolvable for $n > 4$, even with the aid of computers in the case of $n = 5$. By considering $\mathbb{F}_2$ as a trivial $A$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathbb{F}_2$-graded vector space $\mathbb{F}_2 \otimes_A P_n$.

This paper aims to explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_2 \otimes_A P_n$ in the generic degree $n(2^r - 1) + 2 \cdot 2^r$, where $r$ is an arbitrary non-negative integer, and in the case of $n = 6$.

As an application of these results, we obtain the dimension results for the polynomial algebra $P_n$ in degrees $(n - 1) \cdot (2^{n+u-1} - 1) + \ell \cdot 2^{n+u}$, where $u$ is an arbitrary non-negative integer, $\ell = 13$, and $n = 7$.

Moreover, for any integer $r > 1$, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 2 \cdot 2^r$ is also discussed at the end of this paper. Here, the Singer algebraic transfer is a homomorphism from the homology of the Steenrod algebra to the subspace of $\mathbb{F}_2 \otimes_A P_n$ consisting of all the $GL_n(\mathbb{F}_2)$-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\text{Tor}_{n,n+1}^A(\mathbb{F}_2, \mathbb{F}_2)$.

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1. Introduction

Let $X$ be a topological space. Cohomology operations are generated by the natural transformations of degree $i$ which are so-called Steenrod squares

$$Sq^i : H^*(X, \mathbb{F}_2) \longrightarrow H^{*+i}(X, \mathbb{F}_2),$$

where $H^*(X, \mathbb{F}_2)$ is the singular cohomology of $X$ with coefficients in the two-element field $\mathbb{F}_2$, and $i$ is arbitrary non-negative integers. In 1952, Serre established the structure of the
set of all cohomology operations. Serre [18] proved that the Steenrod squares generate all stable cohomology operations with the usual addition and the composition of maps. The algebra of stable cohomology operations with coefficients in $\mathbb{F}_2$ is known as the mod 2 Steenrod algebra, $A$.

Furthermore, the Steenrod algebra is able to be defined algebraically as a quotient algebra of $\mathbb{F}_2$-free graded associative algebra generated by the symbols $Sq^i$ of degree $i$, where $i$ is a non-negative integer, by the two-sided ideal generated by the relation $Sq^0 = 1$ and the Adem’s relations

$$Sq^aSq^b = \sum_{j=0}^{[a/2]} \binom{b - 1 - j}{a - 2j} Sq^{a+b-j}Sq^j,$$

for $0 < a < 2b$ (see Chapter 1 of [21]).

Calculations of various homotopy groups of spheres by Jean-Pierre Serre were among the first Steenrod algebra applications. Milnor proved in [9] that the dual of the mod 2 Steenrod algebra is a polynomial algebra and that the mod 2 Steenrod algebra admits a Hopf algebra’s structure. As a result, an object that was previously considered intractable suddenly became considerably easier to control. Recently, the mod 2 Steenrod algebra and the mod 2 dual Steenrod algebra as a subalgebra of the mod 2 dual Leibniz-Hopf algebra have been studied by many authors (see Crossley-Turgay [4], Crossley [5], Turgay-Kaji [27] and others).

Let $E_n$ be an elementary abelian 2-group of rank $n$. We will denote by $BE_n$ the classifying space of $E_n$. It may be thought of as the product of $n$ copies of real project space $\mathbb{R}P^\infty$. Then, based on the Künneth formula for cohomology, one gets an isomorphism of $\mathbb{F}_2$-algebras

$$\mathcal{P}_n := H^*(BE_n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1] \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_n] \cong \mathbb{F}_2[x_1, x_2, \ldots, x_n],$$

where $x_i \in H^1(BE_n; \mathbb{F}_2)$ for every $i$.

As is well-known, $\mathcal{P}_n$ is a module over the mod-2 Steenrod algebra $A$. The action of $A$ on $\mathcal{P}_n$ is determined by the formula

$$Sq^k(x_j) = \begin{cases} x_j, & k = 0, \\ x_j^2, & k = 1, \\ 0, & k > 1, \end{cases}$$

and the Cartan formula $Sq^k(uv) = \sum_{i=0}^{k} Sq^i(u)Sq^{k-i}(v)$, where $u, v \in \mathcal{P}_n$ (see Steenrod-Epstein [21], and Turgay [26]).

The Peterson hit problem is to find a minimal generating set for $\mathcal{P}_n$ regarded as a module over the mod-2 Steenrod algebra. The hit problem is analogous to the problem of finding a basis for the $\mathbb{F}_2$-graded vector space $\mathbb{F}_2 \otimes_A \mathcal{P}_n$ if we treat $\mathbb{F}_2$ as a trivial $A$-module.

This issue has first been studied by Peterson [14], Singer [19], Wood [34], and Priddy [16], who shows its relationship to several classical problems in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of the classifying space of finite groups. Then, this issue and its applications were investigated by Silverman [20], Repka-Selick [17], Janfada-Wood [6], Nam [13], Sum [22, 24], Mothebe-Kaelo-Ramatebele [12], Sum-Tin [25], Walker-Wood [33], the present writer [29, 30] and others.

Let $\alpha(d)$ be the number of digits 1 in the binary expansion of a natural integer $d$. Consider the function $\mu : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ which is defined as follows:

$$\mu(0) = 0, \quad \text{and} \quad \mu(d) = \min \{m \in \mathbb{N} : \alpha(d + m) \leq m\}.$$

In [14], Peterson hypothesized that as a module over the Steenrod algebra $A$, $\mathcal{P}_n$ is generated by monomials of degree $d$ obeying the inequality $\alpha(d + n) \leq n$, and proved it.
for $n \leq 2$. And then, Wood [34] demonstrates this in general. This is a fantastic tool for figuring out $A$-generators for $\mathcal{P}_n$.

Kameko’s squaring operation

$$\hat{S}^0_n := (\hat{S}^0_n)_{(n; n+2d)} : (\mathbb{F}_2 \otimes A \mathcal{P}_n)_{n+2d} \to (\mathbb{F}_2 \otimes A \mathcal{P}_n)_d,$$

which is induced by an $\mathbb{F}_2$-linear map $S_n : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$S_n(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \ldots x_n y^2 \\ 0, & \text{otherwise} \end{cases}$$

for any monomial $x \in \mathcal{P}_n$, is one of the most important tools in the analysis of the hit problem. Clearly, $(\hat{S}^0_n)_{(n; n+2d)}$ is an $\mathbb{F}_2$-epimorphism.

From the results of Kameko [7], Sum [24], and Wood [34], the hit problem is reduced to the case of degree $d$ of the form $d = r(2^l - 1) + 2^m t$, where $r, m, t$ are non-negative integers such that $0 \leq \mu(m) < r \leq n$.

Now, the tensor product $A \mathcal{P}_n := \mathbb{F}_2 \otimes A \mathcal{P}_n$ was entirely calculated for $n \leq 4$ (see Peterson [14] for $n = 1, 2$, see Kameko [7] for $n = 3$, see Sum [24] for $n = 4$), but it remains unresolved for $n \geq 5$, even with the aid of computers in the case of $n = 5$. Recently, in the case of $n = 5$, and in some degrees, this problem was studied by many authors (see Mong-Sum [11], Phuc [15], Tin [31] and others).

In the present paper, we explicitly determine an admissible monomial basis of the $F_2$-vector space $A \mathcal{P}_6$ in the generic degree $6(2^s - 1) + 2 \cdot 2^s$, with $s$ an arbitrary non-negative integer. The MAGMA computer algebra [35] was used to double-check these results.

As an application of the above results, we obtain the dimension results for the polynomial algebra $\mathcal{P}_n$ in degree $d = (n - 1) \cdot (2^{n+u-1} - 1) + 2^{n+u}$, where $u$ is an arbitrary non-negative integer, $\ell = 13$, and $n = 7$.

One of the primary applications of the hit problem is in surveying a homomorphism proposed by Singer [19], which is from the homology of the Steenrod algebra to the sub-space of $A \mathcal{P}_n$ consisting of all the $GL_n(\mathbb{F}_2)$-invariant classes. Here, $GL_n(\mathbb{F}_2)$ is the general linear group over the field $\mathbb{F}_2$.

Recall that the general linear group $GL_n(\mathbb{F}_2)$ acts naturally on $\mathcal{P}_n$ by matrix substitution. Due to the fact that the two actions of $A$ and $GL_n(\mathbb{F}_2)$ upon $\mathcal{P}_n$ commute with each other, there is an inherited action of $GL_n(\mathbb{F}_2)$ on $A \mathcal{P}_n$. At the conclusion of this article, the behavior of the sixth Singer algebraic transfer in degree $2s+5 - 6$ is also discussed.

Next, in Section 2, we recall some auxiliary information on admissible monomials in $\mathcal{P}_n$. The main results are presented in Section 3. Finally, in the appendix, we provide an algorithm in MAGMA [35] to verify the dimension result of the main results of this paper.

2. Preliminaries

In this section, we review some important facts from Kameko [7], Singer [19], and Sum [24], which will be used in the following section.

We will denote by $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and

$$X_J = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathbb{N}_n, \setminus \{j\}} x_j, \quad J = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_n,$$

In particular, $X_{\mathbb{N}_n} = 1$, $X_\emptyset = x_1 x_2 \ldots x_n$, $X_j = x_1 \ldots \hat{x}_j \ldots x_n$, $1 \leq j \leq n$, and $X := X_n \in \mathcal{P}_{n-1}$.

Let $\alpha_t(d)$ be the $t$-th coefficient in dyadic expansion of $d$. Then, $d = \sum_{t \geq 0} \alpha_t(d) \cdot 2^t$ where $\alpha_t(d) \in \{0, 1\}$. Let $x = x_1 \alpha_1 x_2 \alpha_2 \ldots x_n \alpha_n \in \mathcal{P}_n$. Denote $\nu_j(x) = a_j, 1 \leq j \leq n$. Set

$$J_t(x) = \{ j \in \mathbb{N}_n : \alpha_t(\nu_j(x)) = 0 \},$$

for $t \geq 0$. Then, we have $x = \prod_{t \geq 0} X_{J_t(x)}$. 
Definition 2.1 (Weight vector-Exponent vector). For a monomial $x$ in $\mathcal{P}_n$, define two sequences associated with $x$ by
\[
\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \text{and} \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_n(x)),
\]
where $\omega_i(x) = \sum_{1 \leq j \leq n} \alpha_{i-j}(\nu_j(x)) = \deg X_{i-j-1}(x), \ i \geq 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of $x$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence $\omega$ is called the weight vector if $\omega_i = 0$ for $i > 0$. Then, we define $\deg \omega = \sum_{i>0} 2^{i-1} \omega_i$.

Denote by $\mathcal{P}_n(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y$ such that $\deg y = \deg \omega$, $\omega(y) \leq \omega$, and by $\mathcal{P}_n^- (\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y \in \mathcal{P}_n(\omega)$ such that $\omega(y) < \omega$.

Definition 2.2 (Equivalence relations on $\mathcal{P}_n$). Let $\mathcal{A}^+$ be an ideal of $\mathcal{A}$ generated by all Steenrod squares of positive degrees, and $u$, $v$ two polynomials of the same degree in $\mathcal{P}_n$. We define the equivalence relations “$\equiv$” and “$\equiv_\omega$” on $\mathcal{P}_n$ by stating that
(i) $u \equiv v$ if and only if $u - v \in \mathcal{A}^+ \mathcal{P}_n$,
(ii) $u \equiv_\omega v$ if and only if $u, v \in \mathcal{P}_n(\omega)$ and $u - v \in (\mathcal{A}^+ \mathcal{P}_n \cap \mathcal{P}_n(\omega) + \mathcal{P}_n^- (\omega))$.

Then, we have an $\mathbb{F}_2$-quotient space of $\mathcal{P}_n$ by the equivalence relation “$\equiv_\omega$” as follows:
\[
\mathcal{A} \mathcal{P}_n(\omega) = \mathcal{P}_n(\omega) / ((\mathcal{A}^+ \mathcal{P}_n \cap \mathcal{P}_n(\omega)) + \mathcal{P}_n^- (\omega)).
\]

If a polynomial $u$ in $\mathcal{P}_n$ can be expressed as a finite sum $u = \sum_{i \geq 0} Sq^i(f_i)$ for suitable polynomials $f_i \in \mathcal{P}_n$, it is called it hit. That means $u$ belongs to $\mathcal{A}^+ \mathcal{P}_n$.

Let $u \in \mathcal{P}_n$, and let $\omega$ be a weight vector. We denote by $[u]$ the class in $\mathcal{A} \mathcal{P}_n$ represented by $u$. If $u$ belongs to $\mathcal{P}_n(\omega)$, then denote by $[u]_\omega$ the class in $\mathcal{A} \mathcal{P}_n(\omega)$ represented by $u$.

Definition 2.3 (Linear order on $\mathcal{P}_n$). Let $u$, $v$ be monomials of the same degree in $\mathcal{P}_n$. We say that $u < v$ if one of the following holds:
(i) $\omega(u) < \omega(v)$;
(ii) $\omega(u) = \omega(v)$, and $\sigma(u) < \sigma(v)$.

Definition 2.4 (Admissible monomial-Inadmissible monomial). Let $u$ be a monomial in $\mathcal{P}_n$. The monomial $u$ is said to be inadmissible if there exist monomials $u_1, u_2, \ldots, u_m$ such that $u_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{i=1}^m u_i \in \mathcal{A}^+ \mathcal{P}_n$. If $u$ is not inadmissible monomial, we say it is admissible.

For instance, since $x_1^2 x_2 = Sq^1(x_1 x_2) + x_1 x_2^2$, and $\sigma(x_1 x_2) < \sigma(x_1^2 x_2)$, it follows that $x_1^2 x_2$ is inadmissible in $\mathcal{P}_2$. Moreover, the monomials $x_1^3; x_2^3$ are admissible in $\mathcal{P}_2$.

It is crucial to note that the set of all admissible monomials of degree $d$ in $\mathcal{P}_n$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_n$ in degree $d$. And therefore, $(\mathcal{A} \mathcal{P}_n)_d$ is an $\mathbb{F}_2$-vector space with a basis consisting of all the classes represent by the elements in $(\mathcal{P}_n)_d$.

Definition 2.5 (Strictly inadmissible monomial). Let $u$ be a monomial in $\mathcal{P}_n$. We say $u$ is strictly inadmissible if there exist monomials $u_1, u_2, \ldots, u_m$ in $\mathcal{P}_n$ such that $u_i < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^m u_j + \sum_{i=1}^s Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}, f_i \in \mathcal{P}_n$.

Observe that if $u$ is strictly inadmissible monomial, then it is inadmissible monomial, as defined by the definitions 2.4, and 2.5. In general, the inverse is not true.

For example, the monomial $x_1 x_2 x_3 x_4 x_5 x_6$ is inadmissible, but it is not strictly inadmissible in $\mathcal{P}_6$.

Theorem 2.6 (Kameko [7], Sum [24]). Let $u, v, w$ be monomials in $\mathcal{P}_n$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_r(w) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,
(i) $uw^{2k}$ is inadmissible if $w$ is inadmissible.
(ii) $wv^{2r}$ is strictly inadmissible if $w$ is strictly inadmissible.

**Definition 2.7** (Minimal spike monomial). Let $z = x_1^{d_1}x_2^{d_2} \ldots x_n^{d_n}$ in $\mathcal{P}_n$. The monomial $z$ is called a spike if $d_j = 2^{t_j} - 1$ for $t_j$ a non-negative integer and $j = 1, 2, \ldots, n$. Moreover, $z$ is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geq t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer’s criterion on the hit monomials in $\mathcal{P}_n$.

**Theorem 2.8** (Singer [19]). Assume that $z$ is the minimal spike of degree $d$ in $\mathcal{P}_n$, and $u \in (\mathcal{P}_n)_d$ satisfying the condition $\mu(d) \leq n$. If $\omega(u) < \omega(z)$, then $u$ is hit.

From now on, let us denote by $\mathcal{C}_6^\odot(d)$ the set of all admissible monomials of degree $d$ in $\mathcal{P}_n$. The cardinal of a set $U$ is denoted by $|U|$. The $A$-submodules of $\mathcal{P}_n$ that spanned all the monomials $x_1^{d_1}x_2^{d_2} \ldots x_n^{d_n}$ such that $d_1 \ldots d_n = 0$, and $d_1 \ldots d_n > 0$, respectively, will be denoted by $\mathcal{P}^0_n$ and $\mathcal{P}^+_n$. It is easy to check that $\mathcal{P}^0_n$ and $\mathcal{P}^+_n$ are $A$-submodules of $\mathcal{P}_n$.

Moreover, we have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces:

$$A\mathcal{P}_n = A\mathcal{P}_n^0 \oplus A\mathcal{P}_n^+$$

where $A\mathcal{P}_n^0 := \mathbb{F}_2 \otimes_A \mathcal{P}_n^0$, $A\mathcal{P}_n^+ := \mathbb{F}_2 \otimes_A \mathcal{P}_n^+$.

**3. Main Results**

First, we explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $A\mathcal{P}_6$ in the generic degree $m_5 := 6(2^4 - 1) + 2 \cdot 2^1$, with $s$ an arbitrary non-negative integer.

For $s = 0$, then $m_0 = 2$. It is easy to see that the set $\{ [x_i x_j] : 1 \leq i < j \leq 6 \}$ is a basis of $\mathbb{F}_2$-vector space $(A\mathcal{P}_6)_{m_0}$. Consequently, $|\mathcal{C}_6^\odot(2)| = 15$.

For $s = 1$, then $m_1 = 6(2^4 - 1) + 2 \cdot 2^1$. Set $\tilde{\omega}_1 := (2, 4)$, $\tilde{\omega}_2 := (4, 1, 1)$, and $\tilde{\omega}_3 := (4, 3)$.

Then, the $\mathbb{F}_2$-vector space $(A\mathcal{P}_6)_{6(2^4 - 1) + 2 \cdot 2^1}$ is determined as follows:

Since Kameko’s homomorphism $(\tilde{\mathcal{S}}^0_q)_{(6;10)}$ is an $\mathbb{F}_2$-epimorphism, it follows that

$$(A\mathcal{P}_6)_{m_1} \cong (A\mathcal{P}_6^0)_{m_1} \bigoplus (\ker(\tilde{\mathcal{S}}^0_q)_{(6;10)} \cap (A\mathcal{P}_6^0)_{m_1}) \bigoplus \operatorname{Im}(\tilde{\mathcal{S}}^0_q)_{(6;10)}$$

Consider the homomorphism $\mathcal{T}_l : \mathcal{P}_{n-1} \to \mathcal{P}_n$, for $1 \leq l \leq n$ by substituting:

$$\mathcal{T}_l(x_k) = \begin{cases} x_k, & \text{if } 1 \leq k \leq t - 1, \\ x_{k+1}, & \text{if } t \leq k < n. \end{cases}$$

We have the following theorem.

**Theorem 3.1.** Let us denote by $\mathcal{D}_{6}^\odot := \{ b : b \in \bigcup_{l=1}^6 \mathcal{T}_l(\mathcal{C}_5^\odot(6(2^4 - 1) + 2 \cdot 2^1)) \}$, and set $\mathcal{D}_{lm}^\odot := \{ [c] : c = \Phi_6(u), \text{ for all } u \in \mathcal{C}_5^\odot(6(2^0 - 1) + 2 \cdot 2^0) \}$, where the map $\Phi_6 : \mathcal{P}_n \to \mathcal{P}_n$ is a homomorphism determined by $\Phi_6(x) = \prod_{i=1}^n x_i x_2^2$ for $x \in \mathcal{P}_n$. Then

(i) We have $|\mathcal{D}_{6}^\odot| = 880$, and the set $\{ [b] : b \in \mathcal{D}_{6}^\odot \}$ is a basis of the $\mathbb{F}_2$-vector space $(A\mathcal{P}_6^0)_{6(2^4 - 1) + 2 \cdot 2^1}$. This implies that $(A\mathcal{P}_6^0)_{6(2^4 - 1) + 2 \cdot 2^1}$ has dimension 880.

(ii) The space $\operatorname{Im}(\tilde{\mathcal{S}}^0_q)_{(6;10)}$ is isomorphic to a subspace of $(A\mathcal{P}_6^0)_{10}$ generated by all the classes $[c]$ of $\mathcal{D}_{lm}^\odot$. Consequently, $|\mathcal{D}_{lm}^\odot| = \dim(\operatorname{Im}(\tilde{\mathcal{S}}^0_q)_{(6;10)}) = 15$.

**Proof.** We begin by proving Part (i) of the above theorem. Recall that Tin [29] showed that the space $(A\mathcal{P}_5^0)_{10}$ is an $\mathbb{F}_2$-vector space of dimension 280 with a basis consisting of all the classes represented by the monomials $a_j$, $1 \leq j \leq 280$. Consequently, $|\mathcal{C}_5^\odot(10)| = 280$.

Using the above result, an easy computation shows that

$$\left| \bigcup_{l=1}^6 \mathcal{T}_l(\mathcal{C}_5^\odot(10)) \right| = \left| \bigcup_{l=1}^6 \mathcal{T}_l(a_j) \right|, 1 \leq j \leq 280 = 880,$$
and the set \( \{ b : b \in \bigcup_{i=1}^6 \mathcal{P}_i(a_j), 1 \leq j \leq 280 \} \) is a minimal set of generators for \( \mathcal{A} \)-modules \( \mathcal{P}_6 \) in degree ten. Hence, \( \dim(\mathcal{A}[\mathcal{P}_6]_{10}) = 880 \).

The proof of Part (ii) of the above theorem is straightforward. Indeed, since \( \Phi_6 : \mathcal{P}_6 \to \mathcal{P}_6 \) is the homomorphism defined by \( \Phi_6(x) = \prod_{i=1}^6 x_i x_2 \) for \( x \in \mathcal{P}_6 \), one obtains

\[
\{(c : c = \Phi_6(x), \text{ for all } x \in \mathcal{C}_6^\circ(5(2^0 - 1) + 2 \cdot 2^0))\} = |\mathcal{D}_6^\circ| = 15.
\]

Moreover, combining the above results with the fact that \( \widetilde{S}_q^0_{(6;10)} \) is an epimorphism, the space \( \text{Im}(\widetilde{S}_q^0_{(6;10)}) \) turns out to be isomorphic to a subspace of \( (\mathcal{A}\mathcal{P}_6)_{10} \) generated by all the classes \( [\mathcal{C}] \) of \( \mathcal{D}_6^\circ \). The theorem is proved. \( \square \)

**Theorem 3.2.** Let \( \mathcal{C}_6^\circ(\omega) \) be the set of all admissible monomials in \( \mathcal{P}_6^+ \), and set \( \mathcal{A}\mathcal{P}_6^+(\omega) := \mathcal{A}\mathcal{P}_6(\omega) \cap \mathcal{A}\mathcal{P}_6^+ \). Then, the space \( \text{Ker}(\widetilde{S}_q^0_{(6;10)}) \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2^2} \) is 50-dimensional. Moreover, we have

\[
\text{Ker}(\widetilde{S}_q^0_{(6;10)}) \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2^2} \cong \bigoplus_{k=1}^3 \mathcal{A}\mathcal{P}_6^+(\omega_k),
\]

and

\[
\dim(\mathcal{A}\mathcal{P}_6^+(\omega_k)) = |\mathcal{C}_6^\circ(\omega_k)| = \begin{cases} 4, & \text{if } k = 1, \\ 10, & \text{if } k = 2, \\ 36, & \text{if } k = 3. \end{cases}
\]

**Proof.** The idea of the proof of the above theorem is to explicitly determine an admissible monomial basis of the \( \mathbb{F}_2 \)-vector space \( \text{Ker}(\widetilde{S}_q^0_{(6;10)}) \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2^2} \).

For a weight vector \( \omega \) of degree \( d \). We set \( \mathcal{C}_6^\circ(\omega) := \mathcal{C}_6^\circ(\omega) \cap \mathcal{P}_6(\omega) \). It is easy to see that \( \mathcal{C}_6^\circ(\omega) = \bigcup_{\text{deg } \omega = d} \mathcal{C}_6^\circ(\omega) \). Putting

\[
Q\mathcal{P}_6^\omega := \{ \{ x \in \mathcal{A}\mathcal{P}_6 : x \text{ is admissible and } \omega(x) = \omega \} \}.
\]

It is straightforward to check that the map \( \mathcal{A}\mathcal{P}_6(\omega) \to Q\mathcal{P}_6^\omega, [x]_\omega \mapsto [x] \) is an isomorphism of \( \mathbb{F}_2 \)-vector spaces. And therefore, \( Q\mathcal{P}_6^\omega \subset \mathcal{A}\mathcal{P}_6 \) can be used to identify the vector space \( \mathcal{A}\mathcal{P}_6(\omega) \). From this, one obtains

\[
(\mathcal{A}\mathcal{P}_6)_d = \bigoplus_{\text{deg } \omega = d} Q\mathcal{P}_6^\omega \cong \bigoplus_{\text{deg } \omega = d} \mathcal{A}\mathcal{P}_6(\omega).
\]

From this, it follows that \( (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2^2} \cong \bigoplus_{\text{deg } \omega = 10} \mathcal{A}\mathcal{P}_6^+(\omega) \).

Suppose that \( x \) is an admissible monomial of degree ten in \( \mathcal{P}_6^+ \) such that [\( x \)] belongs to \( \text{Ker}(\widetilde{S}_q^0_{(6;10)}) \). Observe that \( z = x_1^2 x_2 \) is the minimal spike of degree 10 in \( \mathcal{P}_6 \) and \( \omega(z) = (2,2,1) \). Using Theorem 2.8, we obtain \( \omega_1(x) > \omega_1(z) = 2 \). Since the degree of \( x \) is even, one get \( \omega_1(x) = 2 \), or \( \omega_1(x) = 4 \), or \( \omega_1(x) = 6 \).

Since \( \omega_1(x) = 6 \), \( x = X_3 y^2 \) with \( y \) a monomial of degree two in \( \mathcal{P}_6 \). Since \( x \) is admissible, by Theorem 2.6, it shows that \( y \) is also admissible, and \( [y] \neq 0 \). Hence, \( [y] = (\widetilde{S}_q^0_{(6;10)}([x])) \neq 0 \). This contradicts the fact that \( [x] \in \text{Ker}(\widetilde{S}_q^0_{(6;10)}) \).

Since \( \omega_2(x) = 4 \), \( x = X_3 u^2 \), with \( u \) an admissible monomial of degree three in \( \mathcal{P}_6 \), and \( 1 \leq i < j \leq 6 \). It is easy to see that either \( \omega(u) = (3,0) \) or \( \omega(u) = (1,1) \). Hence, \( \omega(x) = (4,3,0) \) or \( \omega(x) = (4,1,1) \).

Since \( \omega_1(x) = 2, x = x_i x_j v^2 \), with \( v \) an admissible monomial of degree four in \( \mathcal{P}_6 \) and \( 1 \leq i < j \leq 6 \). By Theorem 2.6, and \( x \in \mathcal{P}_6^+ \) it implies that \( \omega(x) = (2,4) \).
From the above results, one obtains
\[(\text{Ker}(\overline{S}_0^*)_{(6,10)} \cap (AP^+_0)_{10}) = \bigoplus_{k=1}^3 AP^+_0 (\omega(k)).\]

Now, we explicitly determining all admissible monomials in $P^+_0 (\omega(k))$, where $k \in \{1, 2, 3\}$. The proof is divided into three parts:

**Case 1.** Consider the weight vector $\omega = \overline{\omega}_1$. Assume that $x$ is an admissible monomial in $P_6$ such that $\omega(x) = \overline{\omega}_1$, then $x = x_{i,j}y^2$, where $y \in \mathbb{C}_6^\times(4)$, and $1 \leq i < j \leq 6$. We set $\mathcal{D}_1 := \{x_{i,j}y^2 : \omega(y) = (2,4), 1 \leq i < j \leq 6\} \cap P^+_0$. It is easy to see that $\text{Span}\{\mathcal{D}_1\} = P^+_0 (\overline{\omega}_1)$, and if $u \in \mathcal{D}_1$ then $u = x_{i,j}x_kx_{l,m}x_n$, where $(i,j,k,l,m,n)$ is an arbitrary permutation of $(1,2,3,4,5,6)$. Using the Cartan formula, we have
\[y = x_{1,2}x_{2,3}x_4x_5x_6 = Sq^4(x_{1,2}x_3x_4x_5x_6) + Sq^1(y_1) + \text{smaller than},\]
where $y_1 = x_{1,2}^2x_{2,3}x_4x_5x_6 + x_{2,3}^2x_4^2x_5x_6 + x_{1,2}^2x_3x_4x_5x_6 + x_{1,2}^2x_3^2x_4x_5x_6$. From this, the monomial $y$ is inadmissible.

Clearly, every monomial $x_{1,2}x_{2,3}x_4x_5x_6$ is an inadmissible (more precisely by $Sq^1$), with $(j,k,l,m,n)$ an arbitrary permutation of $(2,3,4,5,6)$.

From the aforementioned results, $P^+_0 (\overline{\omega}_1)$ is generated by the following four elements:
\[c_1 = x_{1,2}^2x_{2,3}x_4x_5x_6, c_2 = x_{1,2}^2x_{2,3}^2x_4x_5x_6, c_3 = x_{1,2}^2x_3x_4^2x_5x_6, c_4 = x_{1,2}^2x_3^2x_4^2x_5x_6.\]

We then show that the vectors $[c_i], 1 \leq i \leq 4$, are linearly independent in $AP_6$. Assume that there is a linear relation
\[S_1 = \sum_{1 \leq i \leq 4} \gamma_ic_i \equiv 0, \text{ with } \gamma_i \in \mathbb{F}_2. \tag{3.1}\]

For $1 \leq k < j \leq 6$, consider the homomorphism $\varphi_{(k,j)} : P_6 \to P_6$ by substituting:
\[\varphi_{(k,j)}(x_m) = \begin{cases} x_m, & \text{if } 1 \leq m \leq k - 1, \\ x_{j-1}, & \text{if } m = k, \\ x_{m-1}, & \text{if } k < m \leq 6. \end{cases}\]

It is simple to figure out if these homomorphisms are $A$-modules homomorphisms. Using the result in Tin [29], acting the homomorphisms $\varphi_{(5,6)}$ on both sides of (3.1), one obtains $\gamma_2 = \gamma_4 = 0$.

So, the relation (3.1) becomes
\[S_1 = \gamma_1c_1 + \gamma_3c_3 \equiv 0. \tag{3.2}\]

Similarly, acting the homomorphisms $\varphi_{(2,5)}$ on both sides of (3.2), we obtains $\gamma_1 = \gamma_3 = 0$.

Hence, the set $\{[c_i] : 1 \leq i \leq 4\}$ is a basis of the $\mathbb{F}_2$-vector space $AP^+_0 (\overline{\omega}_1)$.

**Case 2.** Consider the weight vector $\omega = \overline{\omega}_2$. Based on similar arguments, we also see that $\text{Span}\{\mathcal{D}_2\} = P^+_0 (\overline{\omega}_2)$, where
\[\mathcal{D}_2 := \{x_{i,j}x_{k,x_{l,m}}x^2 : \omega(u) = (4,1,1), 1 \leq i < j < k < l \leq 6\} \cap P^+_6.\]

Observe, if $u \in \mathcal{D}_2$ then $u = x_{i,j}x_{k,x_{l,m}}x^2$, where $(i,j,k,l,m,n)$ is an arbitrary permutation of $(1,2,3,4,5,6)$. It is clear that the monomials $x_{1,2}^4x_{2,3}x_{4,x_{5,m}}x^2_n$ and $x_{1,2}^4x_{1,2}x_{4,x_{5,m}}x^2_n$ are inadmissible by $Sq^4$, with $(j,k,l,m,n)$ an arbitrary permutation of $(2,3,4,5,6)$.

On the one hand, for any $1 < m < n \leq 6$, we have
\[x_{i,j}x_{k,x_{l,m}}x^2_n = Sq^2(x_{i,j}x_{k,x_{l,m}}x^2_n) + Sq^1(x_{i,j}x_{k,x_{l,m}}x^2_n) + \text{smaller than},\]
where \((i, j, k, \ell, m, n)\) is a permutation of \((1, 2, 3, 4, 5, 6)\). It shows that the monomials \(x_1x_jx_kx_\ell x_m x_n^2\) are inadmissible for \(m < n\). According to the aforementioned results, \(\mathcal{P}_6^+\) is generated by 10 elements \(c_i\), with \(5 \leq i \leq 14\) as follows:

5. \(x_1 x_2 x_3 x_4 x_5^4 x_6^2\)
6. \(x_1 x_2 x_3 x_4^2 x_5 x_6^2\)
7. \(x_1 x_2 x_3 x_4^2 x_5^2 x_6^2\)
8. \(x_1 x_2 x_3 x_4^2 x_5 x_6^4\)
9. \(x_1 x_2 x_3 x_4^2 x_5 x_6^2\)
10. \(x_1 x_2 x_3 x_4^2 x_5 x_6^2\)
11. \(x_1 x_2 x_3 x_4 x_5 x_6^2\)
12. \(x_1 x_2 x_3 x_4 x_5 x_6^2\)
13. \(x_1 x_2 x_3 x_4 x_5 x_6^2\)

We then show that the vectors \([c_i]\), \(5 \leq i \leq 14\), are linearly independent in \(\mathcal{A}\mathcal{P}_6\). Assume that there is a linear relation

\[
S_2 = \sum_{1 \leq i \leq 10} \gamma_i c_{i+4} \equiv 0, \quad \text{with } \gamma_i \in \mathbb{F}_2. \tag{3.3}
\]

We recall from [29] the finding that \((\mathcal{A}\mathcal{P}_6^+)_{10}\) is an \(\mathbb{F}_2\)-vector space of dimension 50 with a basis consisting of all the classes represented by the following admissible monomials:

\[
s_1 = x_1 x_2 x_3 x_4 x_5^2 x_6^2, \quad s_2 = x_1 x_2 x_3 x_4 x_5^2 x_6^2, \quad s_3 = x_1 x_2 x_3 x_4 x_5^2 x_6^2, \quad s_4 = x_1 x_2 x_3 x_4 x_5^2 x_6^2.
\]

By direct computations, apply the homomorphism \(\varphi_{(3,5)}\) to (3.3), one obtains

\[
\varphi_{(3,5)}(S_2) \equiv \gamma_1 s_{13} + \gamma_2 s_{14} + \gamma_3 s_{15} + \gamma_4 s_{14} + \gamma_5 s_{15} + \gamma_6 (s_4 + s_7) + \gamma_7 s_3 + \gamma_8 s_{21} + \gamma_9 (s_1 + s_2 + s_3) + \gamma_{10} s_{21} \equiv 0.
\]

From this, it implies \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 0\).

Using the above result, acting the homomorphism \(\varphi_{(4,6)}\) on both sides of (3.3), one gets \(\gamma_8 = \gamma_{10} = 0\).

Hence, the set \([c_i]\), \(5 \leq i \leq 14\), is a basis of the \(\mathbb{F}_2\)-vector space \(\mathcal{A}\mathcal{P}_6^+(\omega_2)\).

Case 3. Consider the weight vector \(\omega = \omega_3\). Let us denote by

\[
\mathcal{D}_6^3 := \{x_1 x_j x_k x_\ell x_m x_n^2 : \omega(v) = (4, 3), \ 1 \leq i < j < k < \ell \leq 6\} \cap \mathcal{P}_6^+.
\]

We also have \(\text{Span}(\mathcal{D}_6^3) = \mathcal{P}_6^+(\omega_3)\), and if \(x \in \mathcal{D}_6^3\) then \(x = x_1 x_j x_k x_\ell x_m x_n^2\), with \((i, j, k, \ell, m, n)\) an arbitrary permutation of \((1, 2, 3, 4, 5, 6)\).

The monomials \(x_1 x_j x_k x_\ell x_m x_n^2\) and \(x_1 x_2 x_3 x_4 x_5 x_6^2\) are clearly inadmissible by \(S\). And therefore, \(\mathcal{P}_6^+(\omega_3)\) is generated by 36 elements \(c_i\), with \(15 \leq i \leq 50\) as follows:

15. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
16. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
17. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
18. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
19. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
20. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
21. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
22. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
23. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
24. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
25. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
26. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
27. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
28. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
29. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
30. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
31. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
32. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
33. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
34. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
35. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
36. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
37. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
38. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
39. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
40. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
41. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
42. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
43. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
44. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
45. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
46. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
47. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
48. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
49. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
50. \(x_1 x_2 x_3 x_4 x_5^2 x_6^2\)
Assume that there is a linear relation

$$S_3 = \sum_{1 \leq i \leq 36} \gamma_i c_{i+14} \equiv 0, \text{ with } \gamma_i \in \mathbb{F}_2. \quad (3.4)$$

Using the same method as above, we explicitly compute $\varphi_{(k;j)}(S_3)$ in terms of $s_i$, $1 \leq i \leq 50$ in $P_5(\text{mod}(A^+P_5))$.

From the relations $\varphi_{(k;j)}(S_3) \equiv 0$ with $1 \leq k < j \leq 6$, one gets $\gamma_i = 0$ for all $15 \leq i \leq 50$. That means, the vectors $[c_i]$, $15 \leq i \leq 50$, are linearly independent in $AP_6$.

Hence, $\{[c_i], 15 \leq i \leq 50\}$ is a basis of $AP_6^+(\omega_3)$. The theorem is proved. \(\square\)

From the above results, we obtain the following corollary.

**Corollary 3.3.** There exist exactly 945 admissible monomials of degree ten in $P_6$. This implies that the space $(AP_6)_6(21-1)+2.2.1$ has dimension 945.

It should be noted that Mothebe-Kaelo-Ramatebele [12] utilized a different method to verify the dimension result of the vector space $(AP_6)_6(21-1)+2.2.1$.

Next, we consider the degree $m_s := 6(2^s - 1) + 2 \cdot 2^s$, for all $s \geq 2$. For $s = 2$, we have $m_2 = 6(2^2 - 1) + 2 \cdot 2^2$. Since $(\tilde{S}q_s)_6(26)$ is a $\mathbb{F}_2$-epimorphism, it shows that

$$(AP_6)_6(26) \cong (AP_6^0)_6(26) \oplus (\text{Ker}(\tilde{S}q_s)_6(26) \cap (AP_6^+)_6)(m_2) \oplus \text{Im}(\tilde{S}q_s)_6(26)$$

Then, the $\mathbb{F}_2$-vector space $(AP_6)_6(22-1)+2.2.2$ is explicitly determined by the following theorem.

**Theorem 3.4.** Let us denote by $AP_6^+(\omega) = AP_n(\omega) \cap AP_6^+$, $\tilde{\omega}$ := $(4, 3, 2, 1), \, \tilde{\omega} := (4, 5, 3), \, \tilde{\omega} := (4, 5, 1, 1)$. Then

(i) The space $\text{Im}(\tilde{S}q_s)_6(26)$ is isomorphic to a subspace of $(AP_6)_6(22-1)+2.2.2$ generated by all the classes represented by the admissible monomials of the form $\Phi_6(u)$ for all $u \in \mathcal{C}_6^0(6(2^1 - 1) + 2 \cdot 2^1)$. Consequently, $\dim(\text{Im}(\tilde{S}q_s)_6(26)) = 945$.

(ii) The set $\{[e_i] : e_i \in \mathcal{U}_{\tilde{\omega}}(\mathcal{C}_6^0(26), 1 \leq i \leq 5184\}$ is a basis of the $\mathbb{F}_2$-vector space $(AP_6^0)_6(22-1)+2.2.2$. This implies that $(AP_6^0)_6(22-1)+2.2.2$ has dimension 5184.

(iii) We have $(\text{Ker}(\tilde{S}q_s)_6(26) \cap (AP_6^+)_6(22-1)+2.2.2) \cong \bigoplus_{i=1}^4 AP_6^+(\omega_i)$. Moreover, the vector space $\text{Ker}(\tilde{S}q_s)_6(26) \cap (AP_6^+)_6(22-1)+2.2.2$ is 3636-dimensional.

**Proof.** Since Kameko’s homomorphism

$$(\tilde{S}q_s)_6(26) : (AP_6)_6(22-1)+2.2.2 \rightarrow (AP_6)_6(22-1)+2.2.2$$

is a $\mathbb{F}_2$-epimorphism, it shows that the proof of Part (i) of the above theorem is straightforward. It is an immediate consequence of Corollary 3.3. More specifically, we see that the space $\text{Im}(\tilde{S}q_s)_6(26)$ turns out to be isomorphic to a subspace of $(AP_6)_6$ generated by all the classes $[e]$ of $\mathcal{E}_{1,m}^6$, where

$$\mathcal{E}_{1,m}^6 = \{[e] : e = \Phi_6(u), \text{ for all } u \in \mathcal{C}_5^0(5(2^1 - 1) + 2 \cdot 2^1)\}.$$

Hence, $\dim(\text{Im}(\tilde{S}q_s)_6(26)) = \mathcal{E}_{1,m}^6 | = 945$.

Next, we prove Part (ii) of the above theorem. Recall that, Wood-Walker [33] proved that the space $AP_n$ has dimension $2\binom{n}{2}$ in degree $d(n) = 2^n - n - 1$. For $n = 5$, we have $d(5) = 2^5 - 5 - 1 = 26$, and therefore, $\dim(AP_5)_6 = 2\binom{5}{2} = 1024.$
Assume that the set $C^\circ_5(26) = \{u_i \in (P_5)_{26} : 1 \leq i \leq 1024\}$. A simple calculation reveals that

\[
\left\{ \sum_{t=1}^{6} T_t(u_i) : 1 \leq i \leq 1024 \right\} = 5184,
\]

and the set

\[
\left\{ v : v \in \sum_{t=1}^{6} T_t(u_i) \right\}, 1 \leq j \leq 1024
\]

is a minimal set of generators for $A$-modules $P_n^d$ in degree twenty-six.

Therefore, $\dim(\mathcal{A}P_n^d)_{6(22-1)+22} = 5184$. The second part has been established.

Finally, we prove Part (iii) of the theorem. For a weight vector $\omega$ of degree $d$, we set $C_n^\circ(\omega) := C_n^\circ(d) \cap P_n(\omega)$. Then, one has $C_n^\circ(d) = \bigcup_{\deg \omega = d} C_n^\circ(\omega)$. Denote by $Q\mathcal{P}_n^\omega$ the subspace of $\mathcal{A}P_n$ spanned by all the classes represented by the admissible monomials of weight vector $\omega$ in $P_n$.

By the same arguments as in the proof of the previous theorem, the map $\mathcal{A}P_n(\omega) \rightarrow Q\mathcal{P}_n^\omega$, $[u]\omega \rightarrow [u]$ is an isomorphism of $F_2$-vector spaces. And therefore, $Q\mathcal{P}_n^\omega \subset \mathcal{A}P_n$ can be used to identify the vector space $\mathcal{A}P_n(\omega)$. From this, one obtains

\[
\mathcal{A}P_n(d) = \bigoplus_{\deg \omega = d} Q \mathcal{P}_n^\omega \cong \bigoplus_{\deg \omega = d} \mathcal{A}P_n(\omega).
\]

Hence, it shows that $(\mathcal{A}P_6^d)_{m_2} = \bigoplus_{\deg \omega = m_2} \mathcal{A}P_6^+(\omega)$.

Suppose that $x$ is an admissible monomial of degree twenty-six in $P_6$ such that $[x]$ belongs to Ker$(\tilde{S}q_x^0)_{(6;26)}$. Observe that $z = x_1^5 x_2^2 x_3^3 x_4$ is the minimal spike of degree twenty-six in $P_6$ and $\omega(z) = \omega[2]$. By Theorem 2.8, we obtain $\omega_1(x) \geq \omega_1(z) = 4$. Since the degree of $(x)$ is even, one gets either $\omega_1(x) = 4$, or $\omega_1(x) = 6$.

If $\omega_1(x) = 6$ then $x = X_6 y^2$, with $y$ a monomial of degree ten in $P_6$. Since $x$ is admissible, by Theorem 2.6, it shows that $y$ is also admissible, and $[y] \neq 0$. Hence, $[y] = (\tilde{S}q_x^0)_{(6;26)}([x]) \neq 0$ which contradicts the fact that $[x]$ belongs to Ker$(\tilde{S}q_x^0)_{(6;26)}$.

Hence, $\omega_1(x) = 4$ then $x = X_{(i,j)} u^2$, with $u$ an admissible monomial of degree eleven in $P_6$, and $1 \leq i < j \leq 6$. Using the result in [12], one has $\omega(u) = (3,4)$ or $\omega(u) = (3,2,1)$, or $\omega(u) = (5,3)$ or $\omega(u) = (5,1,1)$. Hence, $\omega(x) = \omega[u]$ for all $i = 1, 2, 3, 4$.

From the above results, one obtains

\[
(\text{Ker}(\tilde{S}q_x^0)_{(6;26)} \cap (\mathcal{A}P_6^d)_{6(22-1)+22}) = \bigoplus_{i=1}^{4} \mathcal{A}P_6^+(\omega[i]).
\]

By the same arguments as in the proof of the previous theorem, we determine the space Ker$(\tilde{S}q_x^0)_{(6;26)} \cap (\mathcal{A}P_6^d)_{m_2}$ by explicitly determining all admissible monomials in $P_6^+(\omega[i])$, where $i \in \{1, 2, 3, 4\}$. However, to list all the elements of the admissible monomial basis of these subspaces is far too long and computationally very technical. The following is a sketch of its proof with the aid of computers. At the same time, we provide an algorithm in MAGMA [35] to verify the dimension result of the vector space Ker$(\tilde{S}q_x^0)_{(6;26)} \cap (\mathcal{A}P_6^d)_{6(22-1)+22}$ which is presented in the appendix of this article.

We will denote by $D_{6(22-1)+22}^\circ(\omega)$ the set of classes represented by the admissible monomials of the vector space Ker$(\tilde{S}q_x^0)_{(6;26)} \cap (\mathcal{A}P_6^d)_{6(22-1)+22}$. Consider the set

\[
B_{26}^\circ := \{ X_{(i,j)} \cdot f^2 : f \in C^\circ_6(11), 1 \leq i < j \leq 6 \} \cap P_6^+.
\]

Using Theorem 2.6, it shows that if $u$ is an admissible monomial of degree twenty-six in $P_6^+$ such that $(\tilde{S}q_x^0)_{(6;26)}([u])$ does not belong to Im$(\tilde{S}q_x^0)_{(6;26)}$, then $u \in B_{26}^\circ$. 

By observing that each monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6}$ corresponds to a series of numbers of the type $(a_1; a_2; a_3; a_4; a_5; a_6)$, we set up an algorithm implemented in Microsoft Excel software to eliminate the inadmissible monomials in $B_{26}^6$. By direct calculations, using Theorem 2.6, we get $|T_{6}^{\otimes 2} \otimes_{6} (2^{2r-1}+2^{2s})(\omega)| = 3636$.

So, we have $\dim \ker (\Phi (6;26)) \cap (A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})} = 3636$. The theorem has been established. \( \square \)

From the above results, we get the following corollary.

**Corollary 3.5.** There exist exactly $9765$ admissible monomials of degree twenty-six in $\mathcal{P}_{6}$. That means, $(A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})}$ has dimension $9765$.

Consider the degree $m_s := 6(2^s - 1) + 2 \cdot 2^s$, for any $s > 2$. Then, we have the following theorem.

**Theorem 3.6.** The set $\{ [x] : x \in \Phi (6;26) \}$ is a basis of the $\mathcal{F}_{2}$-vector space $(A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})}$ for any $s > 2$. Consequently, $|\Phi (6;26)| = 9765$, for each integer $s \geq 2$.

**Proof.** We will begin the proof of the theorem by calling a result in [28] as follows:

Let $d$ be an arbitrary non-negative integer, and let $\zeta (u)$ be the greatest integer $v$ such that $u$ is divisible by $2^v$. That means $u = 2^v m$, with $m$ an odd integer. Put

$$\lambda (n, d) = \max \{ 0, n - \alpha (d + n) - \zeta (d + n) \}.$$  

Then, Tin-Sum showed in [28] that the map

$$(\Phi (6;26))_{6(2^{2r-1}+2^{2s})} : (A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})} \rightarrow (A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})}$$

is an isomorphism of $GL_n(\mathcal{F}_{2})$-modules for every $r \geq t$ if and only if $t \geq \lambda (n, d)$.

For $n = 6, m_s = 6(2^s - 1) + 2 \cdot 2^s$, and $d = 2$ then $\alpha (d + n) = \alpha (8) = 1$, and one has $\zeta (d + n) = \zeta (2^3 \cdot 1) = 3$. And therefore $\lambda (n, d) = 2$. Using the above result, we get an isomorphism of $\mathcal{F}_{2}$-vector space:

$$(A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})} \cong (A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})}$$

for all $r \geq 2$.

Hence, the set $\{ [x] : x \in \Phi (6;26) \}$ is a basis of the $\mathcal{F}_{2}$-vector space $(A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})}$ for any $s > 2$.

Moreover, $\dim (A_{P_{6}}^{+})_{6(2^{2r-1}+2^{2s})} = |\Phi (6;26)| = 9765$, for any $s \geq 2$. The theorem has been established. \( \square \)

Now, we describe the dimension result for the polynomial algebra by studying the $\mathcal{F}_{2}$-vector space $A_{P_{n}}$ in the generic degree $(n-1)(2^n+u-1)-1$, where $u$ is an arbitrary non-negative integer, $\ell = 13$, and $n = 7$.

As is well known, after explicitly determining $A_{P_{4}}$, Sum [24] has established an inductive formula by $n$ for the dimension of the vector space $(A_{P_{n}})$, where $d$ is of general degree (see Theorem 1.3 in [24]). As a result of combining this result in Sum [24] with the results above, we get the following.

**Theorem 3.7.** For any integer $r > 0$, there exist exactly $1240155$ admissible monomials of degree $d_r = 6(2^{r+5} - 1) + 13 \cdot 2^{2s+6}$ in $\mathcal{P}_{7}$. This implies that the space $(A_{P_{7}})$, has dimension $1240155$, for all $r \geq 1$.

**Proof.** Consider the degree $d = (n-1)(2^s - 1) + 2^t$, where $s$ and $t$ are positive integers such that $1 \leq n - 3 \leq \mu (t) \leq n - 2$. If $s \geq n - 1$, then we have an inductive formula by $n$ for the dimension of the vector space $(A_{P_{n}})$, which was shown in Sum [24] as follows:
\[ \dim(A \mathcal{P}_n)_d = (2^n - 1) \dim(A \mathcal{P}_{n-1})_t. \]

We can easily observe that for \( n = 6 \), and \( t = 26 \), then we have
\[ \mu(t) = \mu(26) = 4 = \alpha(t + \mu(t)) = \alpha(70) = n - 2. \]

Hence, using the above result, one gets
\[ |c_6^0(6(2^s - 1) + 13 \cdot 2^{s+1})| = (2^7 - 1) \cdot |c_6^0(26)| = 1240155, \]
for any integer \( s \geq 6 \).

So, there exist exactly 1240155 admissible monomials of degree \( 6(2^s - 1) + 13 \cdot 2^{s+1} \)
in \( \mathcal{P}_7 \), for any integer \( r > 5 \). Consequently, \( \dim(A \mathcal{P}_7)_{6(2^r + 5) + 13 \cdot 2^{r+1}} = 1240155 \), for all \( r \geq 1 \). The theorem is proved.

One of the key applications of the hit problem is in surveying a homomorphism proposed by Singer. It is a useful tool for describing the cohomology groups of the Steenrod algebra, \( \Ext_A^n(\mathbb{F}_2, \mathbb{F}_2) \). We will start by recalling the definition of the Singer algebraic transfer, which was shown in [19] as follows:

Consider the graded polynomial ring \( \mathcal{P}_1 = \mathbb{F}_2[x_1] \) with \( \deg(x_1) = 1 \). Thus, the canonical \( \mathcal{A} \)-action on \( \mathcal{P}_1 \) is extended to an \( \mathcal{A} \)-action on the ring of finite Laurent series \( \mathbb{F}_2[x_1, x_1^{-1}] \).

Hence, there exists an \( \mathcal{A} \)-submodule \( \mathcal{P}_1 \) of \( \mathbb{F}_2[x_1, x_1^{-1}] \), spanned by all powers \( x_1^i \) with \( i \geq -1 \). The inclusion \( \mathcal{P}_1 \subset \mathcal{P}_1 \) gives rise to a short exact sequence of \( \mathcal{A} \)-modules:

\[ 0 \to \mathcal{P}_1 \to \mathcal{P} \to \sum_{i=-1}^{\infty} \mathbb{F}_2 \to 0, \]

where \( \pi \) is the inclusion, and \( \sigma \) is given by \( \sigma(x_1^i) = 0 \) if \( i \neq -1 \) and \( \sigma(x_1^{-1}) = 1 \).

Writing \( e_1 \) for the corresponding element in \( \Ext_A^n(\sum_{i=-1}^{\infty} \mathbb{F}_2, \mathcal{P}_1) \). Based on the cross, the Yoneda and the cap products, Singer set

\[ e_n = (e_1 \times \mathcal{P}_{n-1}) \circ (e_1 \times \mathcal{P}_{n-2}) \circ \cdots \circ (e_1 \times \mathcal{P}_1) \circ e_1 \in \Ext_A^n(\sum_{i=-1}^{\infty} \mathbb{F}_2, \mathcal{P}_n). \]

Then, Singer defined the following homomorphism

\[ \widetilde{\varphi}_n : \Tor_n^A(\mathbb{F}_2, \sum_{i=-1}^{\infty} \mathbb{F}_2) \to \Tor_0^A(\mathbb{F}_2, \mathcal{P}_n) = A \mathcal{P}_n \]

by \( \widetilde{\varphi}_n(z) = e_n \cap z \). Its image is a submodule of \( (A \mathcal{P}_n)^{GL_n(\mathbb{F}_2)} \).

So, \( \widetilde{\varphi}_n \) induces the homomorphism

\[ \varphi_n : \Tor_n^A(\mathbb{F}_2, \sum_{i=-1}^{\infty} \mathbb{F}_2) \to (A \mathcal{P}_n)^{GL_n(\mathbb{F}_2)}. \]

Let \( (A \mathcal{P}_n)^{GL_n(\mathbb{F}_2)}_m \) be the subspace of \( (A \mathcal{P}_n)_m \) consisting of all the \( GL_n(\mathbb{F}_2) \)-invariant classes of degree \( m \), and let the space \( \mathbb{F}_2^{\otimes GL_n(\mathbb{F}_2)} \cdot PH_m((\mathbb{R}^\infty)^n) \) be the dual to \( (A \mathcal{P}_n)^{GL_n(\mathbb{F}_2)}_m \).

Then, the dual of \( \varphi_n \) : \( \psi_n := (\varphi_n)^* : \mathbb{F}_2^{\otimes GL_n(\mathbb{F}_2)} \cdot PH_n((\mathbb{R}^\infty)^n) \to \Ext_A^m(\mathbb{F}_2, \mathcal{P}_2) \)
is also called the \( n \)-th Singer algebraic transfer.

Singer illustrated the importance of the algebraic transfer by proving that \( \psi_n \) is an isomorphism with \( n = 1, 2 \) and at other degrees with \( n = 3, 4 \), but he refuted this for \( \psi_5 \) at degree 9, and then proposed the following conjecture.

**Conjecture 3.8 (Singer [19]).** For any \( n \geq 0 \), the algebraic transfer \( \psi_n \) is a monomorphism.
The hit problem’s importance and relevance may be found throughout Singer’s work. Moreover, using the modular representation theory of linear groups, Boardman confirmed in \([1]\) that \(\psi_3\) is also an isomorphism. Many authors have studied the Singer algebraic transfer for \(n \geq 4\) (see Minami \([10]\), Bruner-Ha-Hung \([2]\), Phuc \([15]\), Sum-Tin \([23]\), Tin \([29]\) and others). However, Singer’s conjecture remains open for \(n \geq 4\).

Based on the results for the hit problem, combining the computation of the cohomology groups \(\text{Ext}^{5,5+}_A(F_2, F_2)\) by Lin \([8]\) and Chen \([3]\), we studied and verified the Singer’s conjecture for the algebraic transfer in degrees \(5(2^s-1) + 2^s m, \) where \(m \in \{1, 2, 3\}\). We have the following.

**Theorem 3.9** (see \([23]\) and \([29]\)). Let \(s\) be an arbitrary positive integer. Singer’s conjecture is true for \(n = 5\) and the generic degrees \(d_s = 5(2^s-1) + 2^s m, \) where \(m \in \{1, 2, 3\}\). In the current article, using the result in Tin-Sum \([28]\) (see Theorem 1.3), we also see that \((\text{AP}_6)^{GL_6(F_2)}_{6(2^s-1)+2^s m} \approx (\text{AP}_6)^{GL_6(F_2)}_{6(2^2-1)+2^2}, \) for all \(s > 1\).

By passing to the dual, we obtain the following theorem.

**Theorem 3.10.** We have an isomorphism of \(\mathbb{F}_2\)-vector spaces:

\[
\mathbb{F}_2 \otimes^{GL_6(F_2)} PH_{6(2^2-1)+2^2}((\mathbb{R}P^\infty)^6) \cong \mathbb{F}_2 \otimes^{GL_6(F_2)} PH_{6(2^3-1)+2^3}((\mathbb{R}P^\infty)^6),
\]

for any positive integers \(s \geq 2\).

Remarkability, from the result of the above theorem, we only need to study the vector space \(\mathbb{F}_2 \otimes^{GL_6(F_2)} PH_{6(2^2-1)+2^2}((\mathbb{R}P^\infty)^6)\) for \(s \leq 2\).

### 4. Conclusions and Future work

In this article, we study the hit problem for the polynomial algebra as a module over the Steenrod algebra in some generic degrees and its application to the sixth algebraic transfer of Singer. In the future, we will verify the Singer’s conjecture for the sixth algebraic transfer in degree \(6(2^s-1) + 2^s, \) with \(s\) an arbitrary positive integer, by combining the computations of the cohomology groups of the Steenrod algebra in these cases.

### 5. Appendix

In the appendix, we provide an algorithm in MAGMA \([35]\) to verify the dimension result of the vector space \((\text{AP}_6)^{GL_6(F_2)}_{6(2^2-1)+2^2}\). And then, we obtain the dimension result of the vector space \(\text{Ker}(\tilde{S}q_5)_{(6,26)} \cap (\text{AP}_6)^{GL_6(F_2)}_{6(2^2-1)+2^2}\).

\[
N := 26;
\]

\[
R<x1, x2, x3, x4, x5, x6> := \text{PolynomialRing}(\text{GF}(2), 6);
\]

\[
P<t> := \text{PolynomialRing}(R);
\]

\[
\text{Sq}hom := \text{hom} < R \rightarrow P | [x + t * x^2 : x \in [x1, x2, x3, x4, x5, x6]] >;
\]

function \(\text{Sq}(j, x)\)

\[
c := \text{Coefficients}(\text{Sq}hom(x));
\]

if \(j+1 \gt |c|\) then

return \(R!0\);

else

return \(c[j+1]\);

end if;

end function;

\[
M := \text{MonomialsOfDegree}(R, N);
\]

\[
V := \text{VectorSpace}(\text{GL}(2), M);
\]
function MtoV(m)
If IsZero(m) then
return V ! 0;
else
    cv := [Index(M, mm) : mm in Terms(m)];
    return CharacteristicVector(V, cv);
end if;
end function;
M1 := MonomialsOfDegree(R, N-1);
M2 := MonomialsOfDegree(R, N-2);
M4 := MonomialsOfDegree(R, N-4);
M8 := MonomialsOfDegree(R, N-8);
M16 := MonomialsOfDegree(R, N-16);
print [ ♯ m : m in [M1, M2, M4, M8, M16] ];
S1 := [Sq(1, x) : x in M1];
print "S1";
H1 := sub <V | [M to V(x) : x in S1]>;
print Dimension(H1);
S2 := [Sq(2, x) : x in M2];
print "S2";
H2 := sub <V | [M to V(x) : x in S2]>;
print Dimension(H2);
H := H1 + H2;
print Dimension(H);
S4 := [Sq(4, x) : x in M4];
print "S4";
H4 := sub <V | [M to V(x) : x in S4]>;
print Dimension(H4);
H := H + H4;
print Dimension(H);
S8 := [Sq(8, x) : x in M8];
print "S8";
H8 := sub <V | [M to V(x) : x in S8]>;
print Dimension(H8);
H := H + H8;
print Dimension(H);
S16 := [Sq(16, x) : x in M16];
print "S16";
H16 := sub <V | [M to V(x) : x in S16]>;
print Dimension(H16);
H := H + H16;
print Dimension(H);
print "\n", Dimension(V);
print Dimension(H);
print Dimension(V) - Dimension(H) = 9765.
According to the above calculations, we get
\[
\dim (A\mathcal{P}_6)^{(2 \cdot 2^2 - 1)} = \text{Dimension}(V) - \text{Dimension}(H) = 9765.
\]
Hence, we obtain
\[
\dim (\text{Ker}(\tilde{Sq}^0_{26}) \cap (A\mathcal{P}_6^+)_{26}) = 9765 - \dim(A\mathcal{P}_6^0)_{26} - \dim(\text{Im}(\tilde{Sq}^0_{26})_{26}) = 3636.
\]
On a minimal set of generators for the algebra $H^*(BE_6; F_2)$

Note that the result on the dimension of $\ker(\tilde{Sq}^0_{\ast})_{(6:26)} \cap (A_{\ast}^{+6})_{(2^2-1)}+2 \cdot 2^2$ can also be verified by using a computer calculation program in SAGE (Software for Algebra and Geometry Experimentation), see Viet [32], and [36].

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