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Research Article

Properties of the subtractive prime spectrum of a semimodule

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Abstract

For a top semimodule over a semiring with zero and nonzero identity, this paper studies the interplay between topological properties of the subtractive prime spectrum and algebraic properties of the semimodule. We prove that the subtractive prime spectrum of the subtractively finitely generated top semimodule is a compact space, and establish necessary and sufficient conditions for the top semimodule to be subtractively finitely generated. For a multiplication semimodule over a commutative semiring, we prove that the radical of a subtractive subsemimodule coincides with its subtractive radical, that every proper subtractive subsemimodule is contained in a subtractive prime subsemimodule, that the multiplication semimodule is subtractively finitely generated iff its subtractive prime spectrum is a compact space, that in the subtractive prime spectrum, the intersection of finitely many basic open sets is compact, and that the subtractive prime spectrum of the subtractively finitely generated multiplication semimodule is a spectral space.

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1. Introduction

The notion of semirings was introduced by Vandiver [28] in 1934. Semirings are algebraic systems that generalize both rings and distributive lattices and have many applications in diverse branches of mathematics and computer science [10, 15]. Semirings have two binary operations of addition and multiplication which are connected by the ring-like distributive laws. But, subtraction is not allowed in semirings that are not rings, so there are considerable differences between ring theory and semiring theory. In order to narrow the gap, Henriksen [16] defined k-ideals in semirings.

For a top semimodule over a semiring with zero and nonzero identity, this paper studies the interplay between topological properties of the subtractive prime spectrum, i.e. the set of all the subtractive prime subsemimodules, and algebraic properties of the semimodule.

It is well known that the prime spectrum of a commutative ring with nonzero identity is a spectral space and plays an important role in commutative algebra, algebraic geometry

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and lattice theory [6, 8, 26]. The prime spectrum of a module over a ring with nonzero identity has been investigated mainly in [1, 2, 13, 20–22, 27, 30, 32–34].

For a commutative semiring with zero and nonzero identity, Peña, Ruza, and Vielma [23] proved that the prime spectrum is a spectral space. Recently, Han, Pae, and Ho [14] defined a top semimodule over a semiring with zero and nonzero identity and proved that the prime spectrum is a compact space if the top semimodule is finitely generated. For a multiplication semimodule over a commutative semiring with zero and nonzero identity, they proved that the multiplication semimodule is finitely generated iff the prime spectrum is a compact space and that the prime spectrum is a spectral space if the multiplication semimodule is finitely generated.

The theme of this paper is originated from [4, 12, 19]. Lescot [19] proved that the set of all the saturated prime ideals with the Zariski topology is a spectral space for a B_1-algebra, where a B_1-algebra is an additively idempotent commutative semiring with zero and nonzero identity and saturated ideals are just subtractive ideals. For arbitrary commutative semiring with zero and nonzero identity, Han [12] studied topological properties of the subtractive prime spectrum in detail and proved that the subtractive prime spectrum is a spectral space. Atani, Atani, and Tekir [4] defined a very strong multiplication semimodule over a commutative semiring with zero and nonzero identity, studied properties of the subtractive prime spectrum, and proposed a question "Assume that M is a very strong multiplication semimodule over a commutative semiring with zero and nonzero identity and let the subtractive prime spectrum of M be compact. Is M finitely generated?" However, Han, Pae, and Ho [14] pointed that a very strong multiplication semimodule does not exist. Summing up, there are few previous researches about the subtractive prime spectrum of a semimodule over a semiring with zero and nonzero identity, as far as we know.

The rest of this paper is as follows. Section 2 is for preliminaries. In Section 3, we define the subtractive prime spectrum of a top semimodule over a semiring with zero and nonzero identity, prove that the subtractive prime spectrum is a compact space if the top semimodule is subtractively finitely generated (Theorem 3.16), and establish necessary and sufficient conditions for the top semimodule to be subtractively finitely generated (Theorem 3.17). In Section 4, for a multiplication semimodule over a commutative semiring with zero and nonzero identity, we prove that the radical of a subtractive subsemimodule coincides with its subtractive radical (Theorem 4.3), that every proper subtractive subsemimodule is contained in a subtractive prime subsemimodule (Theorem 4.6), that the multiplication semimodule is subtractively finitely generated iff its subtractive prime spectrum is a compact space (Theorem 4.7), that in the subtractive prime spectrum, the intersection of finitely many basic open sets is compact (Theorem 4.12), and that the subtractive prime spectrum is a spectral space if the multiplication semimodule is subtractively finitely generated (Theorem 4.13).

Note that some of the main results in this paper cannot be proved by any techniques similar to what were done in the theory of modules.

2. Preliminaries

In this section, we recall some known definitions and facts [10, 15].

A nonempty set R together with two binary operations of addition $+$ and multiplication $\cdot$ is called a *semiring* if $(R, +)$ is a commutative semigroup, $(R, \cdot)$ is a semigroup, and multiplication distributes over addition from either side. A semiring R is said to be *commutative* if $rs = sr$ for all $r, s \in R$. If a semiring R has an additively neutral element 0 and $0r = r0 = 0$ for all $r \in R$, then 0 is called a *zero* of R. If R has a multiplicatively neutral element 1, then 1 is called an *identity* of R.

A nonempty subset I of a semiring R is called an *ideal* of R if $a+b \in I$, $ra \in I$ and $ar \in I$ for all $a, b \in I$ and all $r \in R$. An ideal I of a semiring R is said to be *proper* if $I \neq R$. A proper ideal I of a semiring R is said to be *prime* in R if for any $a, b \in R$, $aRb \subseteq I$ implies that $a \in I$ or $b \in I$.

For a subsemigroup A of a commutative semigroup $(S, +)$, the set

$$\overline{A} = \{x \in S \mid x + a = b \text{ for some } a, b \in A\}$$

is called the *subtractive closure* or *k-closure* of A in S. Then $\overline{A}$ is a subsemigroup of $(S, +)$ and it holds that $A \subseteq \overline{A}$ and $\overline{(\overline{A})} = \overline{A}$. A is said to be *subtractively closed* or *k-closed* in S if $A = \overline{A}$, i.e. if $x + a \in A$ and $a \in A$ imply $x \in A$. For subsemigroups A and B of $(S, +)$, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

An ideal I of a semiring R is called a *subtractive ideal* or *k-ideal* of R if $I = \overline{I}$. If I is an ideal of a semiring R, then $\overline{I}$ is a k-ideal of R. R is a k-ideal of itself, and $\{0\}$ is also a k-ideal of R if R has a zero 0.

A *semimodule* over a semiring R or an R-semimodule for short is a commutative semigroup $(M, +)$ together with a scalar multiplication $R \times M \to M$; $(r, x) \mapsto rx$ such that $(r+s)x = rx + sx$, $r(x+y) = rx + ry$, and $r(sx) = (rs)x$ for all $r, s \in R$ and all $x, y \in M$. If an R-semimodule M has an additively neutral element 0_M satisfying $r0_M = 0_M$ for all $r \in R$, then 0_M is called a *zero* of M. An R-semimodule M is said to be *unitary* if the semiring R has a zero 0 and an identity 1, M has a zero 0_M, and $0x = 0_M$ and $1x = x$ hold for all $x \in M$. Throughout this paper, all semimodules are assumed to be unitary.

A nonempty subset N of an R-semimodule M is called a *subsemimodule* of M if $x+y \in N$ and $rx \in N$ for all $x, y \in N$ and all $r \in R$. Given a nonempty subset S of an R-semimodule M, the intersection of all the subsemimodules of M including S is called the subsemimodule of M *generated by* S and denoted by $\langle S \rangle$. For elements $a_1, \dots, a_n \in M$, put $\langle a_1, \dots, a_n \rangle = \langle \{a_1, \dots, a_n\} \rangle$.

A subsemimodule N of an R-semimodule M is called a *subtractive subsemimodule* or a *k-subsemimodule* of M if $N = \overline{N}$. If N is a subsemimodule of an R-semimodule M, then $\overline{N}$ is a k-subsemimodule of M. M and $\{0_M\}$ are k-subsemimodules of M.

For a subsemimodule N of an R-semimodule M, put $(N : M) = \{r \in R | \, rM \subseteq N\}$. Then $(N : M)$ is an ideal of R, called the *associated ideal* of N.

A subsemimodule N of an R-semimodule M is said to be *proper* if $N \neq M$. A proper subsemimodule P of an R-semimodule M is said to be *maximal* in M if for each subsemimodule N of M, $P \subseteq N \subseteq M$ implies that $N = P$ or $N = M$ [11, 29]. A proper subsemimodule P of an R-semimodule M is said to be *prime* in M if $rRm \subseteq P$ with $r \in R$ and $m \in M$ implies that $m \in P$ or $r \in (P : M)$ [3, 24, 31]. The set of all prime subsemimodules of an R-semimodule M is denoted by $\mathrm{Spec}(M)$.

A proper k-subsemimodule P of an R-semimodule M is said to be *subtractively maximal* or *k-maximal* in M if for each k-subsemimodule N of M, $P \subseteq N \subseteq M$ implies that $N = P$ or $N = M$ [11, 29]. The set of all prime k-subsemimodules of an R-semimodule M is denoted by $\mathrm{Spec}_k(M)$.

For a subsemimodule N of an R-semimodule M, the *radical* $\sqrt{N}$ of N is defined to be the intersection of all prime subsemimodules of M containing N. In case there is no such prime subsemimodule, put $\sqrt{N} = M$ [5, 14, 34].

Given a subsemimodule N of an R-semimodule M, the *subtractive radical* or *k-radical* $\sqrt{N}^{(k)}$ of N is defined to be the intersection of all prime k-subsemimodules of M containing N. If there is no such prime k-subsemimodule, put $\sqrt{N}^{(k)} = M$ [4].

An R-semimodule M is said to be *multiplication semimodule* if for every subsemimodule N of M there exists an ideal I of R such that $N = IM$. Then $N = (N : M)M$ [7, 25, 31].

Let R be a commutative semiring with zero and nonzero identity and M a multiplication R-semimodule. If N and K are subsemimodules of M with $N = IM$ and $K = JM$ for some ideals I and J of R, then the *product* of N and K is defined by $N \cdot K = (IJ)M$, which is independent of the choice of ideals I and J [31]. If N, K and L are subsemimodules of M, then $(N \cdot K) \cdot L = N \cdot (K \cdot L)$ and $(N + K) \cdot L = N \cdot L + K \cdot L$. For elements m and m' in M, the product of $\langle m \rangle$ and $\langle m' \rangle$ is simply denoted by $m \cdot m'$.

A topological space is called a T_0-*space* if for each pair of distinct points, there is a neighborhood of one point to which the other does not belong. A topological space is called a T_1-*space* if each subset which consists of a single point is closed. A topological space is said to be *compact* if each open cover has a finite subcover. A subset A of a topological space is said to be *compact* if the subspace A is compact [18].

A nonempty topological space X is said to be *irreducible* if every pair of nonempty open subsets in X have the nonempty intersection. A subset Y of a topological space X is said to be *irreducible* if the subspace Y is irreducible. A subset Y of a topological space X is irreducible iff for any closed subsets Y_1 and Y_2 in X, $Y \subseteq Y_1 \bigcup Y_2$ implies that $Y \subseteq Y_1$ or $Y \subseteq Y_2$. If Y is a closed subset of a topological space X and there exists a y in X such that $Y = \overline{\{y\}}$, then y is called a *generic point* of Y. A topological space X is said to be *spectral* if it is a compact T_0-space, the compact open subsets form a base for the topology and are closed under finite intersection, and every irreducible closed subset has a generic point [17].

Let Λ be an arbitrary nonempty set of subscripts and $\mathbb{N}$ denote the set of all positive integers.

We end this section by describing the results from [14], which will be used in Sections 3 and 4, where R is a semiring with zero and nonzero identity, M is an R-semimodule, and it is assumed that $\mathrm{Spec}(M) \neq \emptyset$.

Remark 2.1. ([14]) If $P \in \mathrm{Spec}(M)$, then $(P : M)$ is a prime ideal of R.

For a nonempty subset S of M, put $V(S) = \{P \in \mathrm{Spec}(M) |\ S \subseteq P\}$.

Lemma 2.2. ([14]) *If M is an R-semimodule, then the following hold.*

(1) *If $\emptyset \neq S \subseteq T \subseteq M$, then $V(T) \subseteq V(S)$.*
(2) *If $\emptyset \neq S_\lambda \subseteq M$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} V(S_\lambda) = V(\bigcup_{\lambda\in\Lambda} S_\lambda)$.*
(3) *If $\emptyset \neq S \subseteq M$, then $V(S) = V(\langle S \rangle) = V(\sqrt{\langle S \rangle})$.*
(4) *If S and T are nonempty subsets of M, then $V(S) \bigcup V(T) \subseteq V(S \bigcap T)$.*
(5) *$V(\{0_M\}) = \mathrm{Spec}(M)$ and $V(M) = \emptyset$.*
(6) *If N_λ are subsemimodules of M for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} V(N_\lambda) = V(\sum_{\lambda\in\Lambda} N_\lambda)$.*
(7) *If N is a subsemimodule of M, then $V(N) = V(\sqrt{N})$.*
(8) *If N and K are subsemimodules of M with $V(N) \subseteq V(K)$, then $K \subseteq \sqrt{N}$.*

An R-semimodule M is called a *top semimodule* if for any subsemimodules N and L of M, there exists a subsemimodule T of M such that $V(N) \bigcup V(L) = V(T)$. For a top R-semimodule M, the collection $\{V(S) |\ \emptyset \neq S \subseteq M\}$ of subsets of $\mathrm{Spec}(M)$ satisfies all the axioms for closed sets on a topology. The resulting topology is called the *Zariski topology* on $\mathrm{Spec}(M)$ and then $\mathrm{Spec}(M)$ is called the *prime spectrum* of M. The open set $\mathrm{Spec}(M) \setminus V(S)$ of $\mathrm{Spec}(M)$ is denoted by $D(S)$. For every element $m \in M$, put $V(m) = V(\langle m \rangle)$ and $D(m) = D(\langle m \rangle)$. $D(m)$ is called a *basic open set* of $\mathrm{Spec}(M)$ [14].

Lemma 2.3. ([14]) *If M is a top R-semimodule, then the collection $\{D(m) |\ m \in M\}$ is a base for the Zariski topology on* $\mathrm{Spec}(M)$.

Lemma 2.4. ([14]) *If M is a top R-semimodule, then* $\mathrm{Spec}(M)$ *is a T_0-space.*

If $\emptyset \neq Y \subseteq \mathrm{Spec}(M)$, then the intersection of all prime subsemimodules belonging to Y is denoted by $\tau(Y)$ [14].

Lemma 2.5. ([14]) *If M is a top R-semimodule and $\emptyset \neq Y \subseteq$* $\mathrm{Spec}(M)$*, then $\overline{Y} = V(\tau(Y))$.*

Lemma 2.6. ([14]) *Let M be a top R-semimodule and $\emptyset \neq Y \subseteq \mathrm{Spec}(M)$. Then Y is an irreducible subset of $\mathrm{Spec}(M)$ iff $\tau(Y)$ is a prime subsemimodule of M.*

Lemma 2.7. ([14]) *Let R be a commutative semiring. If N and K are subsemimodules of a multiplication R-semimodule M, then $V(N)\bigcup V(K) = V(N \cdot K) = V(N \bigcap K)$.*

Lemma 2.8. ([14]) *Let R be a commutative semiring. If M is a multiplication R-semimodule and $f, g \in M$, then $f \cdot g$ is a finitely generated subsemimodule of M.*

Lemma 2.9. ([14]) *Let R be a commutative semiring. If N and K are finitely generated subsemimodules of a multiplication R-semimodule M, then $N \cdot K$ is a finitely generated subsemimodule of M.*

Lemma 2.10. ([14]) *Let R be a commutative semiring. If N is a subsemimodule of a multiplication R-semimodule M, then $\sqrt{N} = \{m \in M \mid m^k \subseteq N \text{ for some } k \in \mathbb{N}\}$.*

Lemma 2.11. ([14]) *Let R be a commutative semiring. If N is a subsemimodule of a multiplication R-semimodule M, then $\sqrt{N} = \sqrt{A}M$, where $A = (N : M)$.*

Lemma 2.12. ([14]) *Let R be a commutative semiring. A multiplication R-semimodule M is a top semimodule.*

3. k-prime spectrum of a top semimodule

Throughout this section, R is a semiring with zero and nonzero identity, M is an R-semimodule, and it is supposed that $\mathrm{Spec}_k(M) \neq \emptyset$. Then $M \neq \{0_M\}$.

Remark 3.1. If $P \in \mathrm{Spec}_k(M)$, then $(P : M)$ is a prime k-ideal of R. In fact, by Remark 2.1, it suffices to show that $(P : M)$ is k-closed in R. If $a + b \in (P : M)$ and $a \in (P : M)$, then for all $m \in M$, $am + bm = (a + b)m \in P$ and $am \in P$. Since P is k-closed in M, $bm \in P$ and thus $b \in (P : M)$.

For a nonempty subset S of M, put $V_k(S) = \{P \in \mathrm{Spec}_k(M) \mid S \subseteq P\}$. Then $V_k(S) = V(S) \bigcap \mathrm{Spec}_k(M)$.

Lemma 3.2 is easily verified by Lemma 2.2 and direct computation.

Lemma 3.2. *If M is an R-semimodule, then the following hold.*

(1) *If $\emptyset \neq S \subseteq T \subseteq M$, then $V_k(T) \subseteq V_k(S)$.*
(2) *If $\emptyset \neq S_\lambda \subseteq M$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} V_k(S_\lambda) = V_k(\bigcup_{\lambda\in\Lambda} S_\lambda)$.*
(3) *If $\emptyset \neq S \subseteq M$, then $V_k(S) = V_k(\langle S\rangle) = V_k(\sqrt{\langle S\rangle})$.*
(4) *If S and T are nonempty subsets of M, then $V_k(S) \bigcup V_k(T) \subseteq V_k(S \bigcap T)$.*
(5) *$V_k(\{0_M\}) = \mathrm{Spec}_k(M)$ and $V_k(M) = \emptyset$.*
(6) *If N_λ are subsemimodules of M for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} V_k(N_\lambda) = V_k(\sum_{\lambda\in\Lambda} N_\lambda)$.*
(7) *If N is a subsemimodule of M, then $V_k(\overline{N}) = V_k(N) = V_k(\sqrt{N}) = V_k(\sqrt{N}^{(k)})$.*
(8) *If N and K are subsemimodules of M with $V_k(N) \subseteq V_k(K)$, then $K \subseteq \sqrt{N}^{(k)}$.*

Definition 3.3. For a top R-semimodule M, the topological subspace $\mathrm{Spec}_k(M)$ of the prime spectrum $\mathrm{Spec}(M)$ is called the *subtractive prime spectrum* or *k-prime spectrum* of M.

Then $\{V_k(S) \mid \emptyset \neq S \subseteq M\}$ is the collection of all the closed subsets in $\mathrm{Spec}_k(M)$. The open subset $\mathrm{Spec}_k(M) \setminus V_k(S)$ in $\mathrm{Spec}_k(M)$ is denoted by $D_k(S)$. Then $D_k(S) = D(S) \bigcap \mathrm{Spec}_k(M)$. For every element $m \in M$, put $V_k(m) = V_k(\langle m\rangle)$ and $D_k(m) = D_k(\langle m\rangle)$. $D_k(m)$ is called a *basic open set* of $\mathrm{Spec}_k(M)$.

Lemma 3.4. *If M is a top R-semimodule, then $\{D_k(m) \mid m \in M\}$ is a base for the topology on $\mathrm{Spec}_k(M)$.*

Proof. It follows from Lemma 2.3 and the fact that $D_k(m) = D(m) \bigcap \text{Spec}_k(M)$ for all $m \in M$. $\square$

Theorem 3.5 generalizes Proposition 2.4 in [34].

Theorem 3.5. *If M is a top R-semimodule, then $\text{Spec}_k(M)$ is a T_0-space.*

Proof. It follows from Lemma 2.4. $\square$

Lemma 3.6. *If M is a top R-semimodule and $\emptyset \neq Y \subseteq \text{Spec}_k(M)$, then the closure of Y in $\text{Spec}_k(M)$ is $V_k(\tau(Y))$.*

Proof. By Lemma 2.5, the closure $\overline{Y}$ of Y in $\text{Spec}(M)$ is $V(\tau(Y))$. Hence, the closure of Y in $\text{Spec}_k(M)$ is $\overline{Y} \bigcap \text{Spec}_k(M) = V(\tau(Y)) \bigcap \text{Spec}_k(M) = V_k(\tau(Y))$. $\square$

Theorem 3.7 generalizes Theorem 4.9 in [12] and Theorem 3 in [30].

Theorem 3.7. *Let M be a top R-semimodule. Then $\text{Spec}_k(M)$ is a T_1-space iff every prime k-subsemimodule is not contained in the other prime k-subsemimodule in M.*

Proof. $(\Rightarrow)$ If $\text{Spec}_k(M)$ is a T_1-space, then for every $P \in \text{Spec}_k(M)$, $\{P\}$ is a closed subset in $\text{Spec}_k(M)$. By Lemma 3.6, the closure of $\{P\}$ in $\text{Spec}_k(M)$ is $V_k(\tau(\{P\})) = V_k(P)$, and so $\{P\} = V_k(P)$. Hence, the only prime k-subsemimodule containing P is P in M.

$(\Leftarrow)$ Let $P \in \text{Spec}_k(M)$. If P is the unique prime k-subsemimodule of M containing P, then by Lemma 3.6, the closure of $\{P\}$ in $\text{Spec}_k(M)$ is $V_k(\tau(\{P\})) = V_k(P) = \{P\}$, and so $\{P\}$ is a closed subset in $\text{Spec}_k(M)$. Hence, $\text{Spec}_k(M)$ is a T_1-space. $\square$

Theorem 3.8 generalizes Theorem 4.14 in [12], Proposition 3 in [30] and Theorem 2.6(1) in [34].

Theorem 3.8. *If M is a top R-semimodule and $\emptyset \neq Y \subseteq \text{Spec}_k(M)$, then the following are equivalent to one another.*

(1) *Y is an irreducible subset in $\text{Spec}_k(M)$.*
(2) *Y is an irreducible subset in $\text{Spec}(M)$.*
(3) *$\tau(Y)$ is a prime subsemimodule of M.*
(4) *$\tau(Y)$ is a prime k-subsemimodule of M.*

Proof. (1)$\Rightarrow$(2) Suppose that Y is irreducible in $\text{Spec}_k(M)$. If Y_1 and Y_2 are closed subsets of $\text{Spec}(M)$ with $Y \subseteq Y_1 \bigcup Y_2$, then

$$Y = Y \bigcap \text{Spec}_k(M) \subseteq (Y_1 \bigcap \text{Spec}_k(M)) \bigcup (Y_2 \bigcap \text{Spec}_k(M)).$$

Since $Y_1 \bigcap \text{Spec}_k(M)$ and $Y_2 \bigcap \text{Spec}_k(M)$ are closed subsets of $\text{Spec}_k(M)$, it follows that $Y \subseteq Y_1 \bigcap \text{Spec}_k(M)$ or $Y \subseteq Y_2 \bigcap \text{Spec}_k(M)$, which implies that $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Hence, Y is irreducible in $\text{Spec}(M)$.

(2)$\Rightarrow$(1) Let F_1 and F_2 be closed subsets of $\text{Spec}_k(M)$ with $Y \subseteq F_1 \bigcup F_2$. Then there exist closed subsets Y_1 and Y_2 of $\text{Spec}(M)$ such that $F_1 = Y_1 \bigcap \text{Spec}_k(M)$ and $F_2 = Y_2 \bigcap \text{Spec}_k(M)$, and so

$$Y \subseteq (Y_1 \bigcap \text{Spec}_k(M)) \bigcup (Y_2 \bigcap \text{Spec}_k(M)) \subseteq Y_1 \bigcup Y_2.$$

Suppose that Y is irreducible in $\text{Spec}(M)$. Then it follows that $Y \subseteq Y_1$ or $Y \subseteq Y_2$, which implies that either $Y \subseteq Y_1 \bigcap \text{Spec}_k(M) = F_1$ or $Y \subseteq Y_2 \bigcap \text{Spec}_k(M) = F_2$. Hence, Y is irreducible in $\text{Spec}_k(M)$.

(2)$\Leftrightarrow$(3) follows from Lemma 2.6.

(3)$\Rightarrow$(4) follows from the fact that any intersection of k-subsemimodules of M is a k-subsemimodule of M.

(4)$\Rightarrow$(3) is obvious. $\square$

Theorem 3.9 generalizes Corollary 1 in [30] and Theorem 2.6(2)(3) in [34].

Theorem 3.9. *If N is a subsemimodule of a top R-semimodule M, then the following are equivalent to one another.*

(1) *$V_k(N)$ is an irreducible subset in $\mathrm{Spec}_k(M)$.*

(2) *$\sqrt{N}^{(k)}$ is a prime k-subsemimodule of M.*

(3) *$\sqrt{N}^{(k)}$ is a generic point of $V_k(N)$ in $\mathrm{Spec}_k(M)$.*

Proof. (1)⇔(2) follows from Theorem 3.8.

(2)⇒(3) If $\sqrt{N}^{(k)} \in \mathrm{Spec}_k(M)$, then by Lemmas 3.6 and 3.2(7), the closure of $\{\sqrt{N}^{(k)}\}$ in $\mathrm{Spec}_k(M)$ is $V_k(\tau(\{\sqrt{N}^{(k)}\})) = V_k(\sqrt{N}^{(k)}) = V_k(N)$.

(3)⇒(2) is obvious. □

Corollary 3.10. *If M is a top R-semimodule, then every irreducible closed subset has a generic point in $\mathrm{Spec}_k(M)$.*

Proof. It follows from Lemma 3.2(3) and Theorem 3.9. □

Definition 3.11. An R-semimodule M is said to be *subtractively finitely generated* or *k-finitely generated* if there exists a nonempty finite subset F of M such that $M = \overline{\langle F\rangle}$.

Obviously, every finitely generated R-semimodule is k-finitely generated, but the converse is not true, as shown in Example 3.12.

Example 3.12. Let $B_1 = \{0, 1\}$ be a Boolean lattice and $M = \mathbb{N}\bigcup\{0, +\infty\}$. For $a, b \in M$ and $r \in B_1$, define

$$a + b = \max\{a, b\}, \qquad r \cdot a = \begin{cases} a, & \text{if } r = 1, \\ 0, & \text{if } r = 0. \end{cases}$$

Then M is a B_1-semimodule. Since $\langle a_1, \ldots, a_n\rangle = \{0, a_1, \ldots, a_n\}$ for arbitrary finitely many elements $a_1, \ldots, a_n \in M$, M is not finitely generated. But for every element $a \in M$, $a+(+\infty) = +\infty$ and so $a \in \overline{\langle +\infty\rangle}$. Hence, $M = \overline{\langle +\infty\rangle}$, which implies that M is k-finitely generated.

Lemma 3.13. ([11]) *Let M be a k-finitely generated R-semimodule. If N is a proper k-subsemimodule of M, then there exists a k-maximal subsemimodule of M containing N. Therefore, M has a k-maximal subsemimodule.*

Lemma 3.14 generalizes Proposition 2.12 in [34].

Lemma 3.14. *Every k-maximal subsemimodule K of an R-semimodule M is prime.*

Proof. Suppose that $rRm \subseteq K$ with $r \in R$ and $m \in M$. If $m \notin K$, then $K \subset K + \langle m\rangle \subsetneq \overline{K + \langle m\rangle}$ and so $M = \overline{K + \langle m\rangle}$. For every $x \in M$, there are $k_1, k_2 \in K$ and $r_1, r_2 \in R$ such that $x+k_1+r_1m = k_2+r_2m$, and so $rx+rk_1+rr_1m = rk_2+rr_2m$. Since $rk_1+rr_1m \in K$ and $rk_2 + rr_2m \in K$, we have $rx \in \overline{K} = K$. Hence, K is prime. □

Lemma 3.15 generalizes Lemma 2.8 in [34].

Lemma 3.15. *Let M be a k-finitely generated R-semimodule. If $M = \overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle}$, then there exists a nonempty finite subset Γ of Λ such that $M = \overline{\sum_{\lambda\in\Gamma}\langle f_\lambda\rangle}$.*

Proof. Let $M = \overline{\langle m_1, \ldots, m_n\rangle}$. For every j with $1 \le j \le n$, $m_j \in M$ and so there are $n_j, t_j \in \mathbb{N}$ with $n_j < t_j$, $\{j_k|\ 1 \le k \le t_j\} \subseteq \Lambda$ and $\{r_{j_k}|\ 1 \le k \le t_j\} \subseteq R$ such that

$$m_j + \sum_{k=1}^{n_j} r_{j_k} f_{j_k} = \sum_{k=n_j+1}^{t_j} r_{j_k} f_{j_k},$$

which implies that $m_j \in \overline{\sum_{k=1}^{t_j}\langle r_{j_k} f_{j_k}\rangle} \subseteq \overline{\sum_{k=1}^{t_j}\langle f_{j_k}\rangle}$. Hence,

$$M = \overline{\sum_{j=1}^{n}\langle m_j\rangle} \subseteq \overline{\sum_{j=1}^{n}\sum_{k=1}^{t_j}\langle f_{j_k}\rangle}$$

and thus $M = \overline{\sum_{j=1}^{n}\sum_{k=1}^{t_j}\langle f_{j_k}\rangle}$. □

Theorem 3.16 generalizes Theorem 2.13 in [34].

Theorem 3.16. *If M is a k-finitely generated top R-semimodule, then* $\mathrm{Spec}_k(M)$ *is a compact space.*

Proof. By Lemma 3.4, it suffices to show that every cover of $\mathrm{Spec}_k(M)$ consisting of basic open sets has a finite subcover. Suppose that $\mathrm{Spec}_k(M) = \bigcup_{\lambda\in\Lambda} D_k(f_\lambda)$. Then by Lemma 3.2(6)(7),

$$\emptyset = \mathrm{Spec}_k(M) \setminus \bigcup_{\lambda\in\Lambda} D_k(f_\lambda) = \bigcap_{\lambda\in\Lambda} V_k(\langle f_\lambda\rangle) = V_k(\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle) = V_k(\overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle}).$$

Now show that $M = \overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle}$. In fact, if $\overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle} \neq M$, then by Lemma 3.13, there exists a k-maximal subsemimodule K of M containing $\overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle}$. By Lemma 3.14, $K \in \mathrm{Spec}_k(M)$, and so $V_k(\overline{\sum_{\lambda\in\Lambda}\langle f_\lambda\rangle}) \neq \emptyset$, a contradiction.

Since M is k-finitely generated, by Lemma 3.15, there exists a nonempty finite subset Γ of Λ such that $M = \overline{\sum_{\lambda\in\Gamma}\langle f_\lambda\rangle}$. By Lemma 3.2(5), $V_k(\overline{\sum_{\lambda\in\Gamma}\langle f_\lambda\rangle}) = V_k(M) = \emptyset$, and by Lemma 3.2(6)(7),

$$\begin{aligned}\textstyle\bigcup_{\lambda\in\Gamma} D_k(f_\lambda) &= \mathrm{Spec}_k(M) \setminus \textstyle\bigcap_{\lambda\in\Gamma} V_k(\langle f_\lambda\rangle) = \mathrm{Spec}_k(M) \setminus V_k(\textstyle\sum_{\lambda\in\Gamma}\langle f_\lambda\rangle)\\ &= \mathrm{Spec}_k(M) \setminus V_k(\overline{\textstyle\sum_{\lambda\in\Gamma}\langle f_\lambda\rangle}) = \mathrm{Spec}_k(M),\end{aligned}$$

which completes the proof. □

Theorem 3.17. *If M is a top R-semimodule, then the following are equivalent to one another.*

(1) *M is k-finitely generated.*

(2) $\mathrm{Spec}_k(M)$ *is a compact space and for every proper k-subsemimodule N of M, there exists a k-maximal subsemimodule of M containing N.*

(3) $\mathrm{Spec}_k(M)$ *is a compact space and for every proper k-subsemimodule N of M, there exists a prime k-subsemimodule of M containing N.*

Proof. (1)⇒(2) follows from Lemma 3.13 and Theorem 3.16.

(2)⇒(3) follows from Lemma 3.14.

(3)⇒(1) By Lemma 3.4, there is a nonempty subset $\{f_i \in M \mid i \in \Lambda\}$ of M such that $\mathrm{Spec}_k(M) = \bigcup_{i\in\Lambda} D_k(f_i)$. Since $\mathrm{Spec}_k(M)$ is compact, there is a nonempty finite subset Γ of Λ such that $\mathrm{Spec}_k(M) = \bigcup_{i\in\Gamma} D_k(f_i)$. By Lemma 3.2(6)(7), $\emptyset = \bigcap_{i\in\Gamma} V_k(f_i) = V_k(\sum_{i\in\Gamma}\langle f_i\rangle) = V_k(\overline{\sum_{i\in\Gamma}\langle f_i\rangle})$. If $\overline{\sum_{i\in\Gamma}\langle f_i\rangle} \neq M$, then there exists a prime k-subsemimodule of M containing $\overline{\sum_{i\in\Gamma}\langle f_i\rangle}$, and so $V_k(\overline{\sum_{i\in\Gamma}\langle f_i\rangle}) \neq \emptyset$, a contradiction. Therefore, $M = \overline{\sum_{i\in\Gamma}\langle f_i\rangle}$. □

4. k-prime spectrum of a multiplication semimodule

Throughout this section, R is a commutative semiring with zero and nonzero identity, and M is a nonzero multiplication R-semimodule.

Lemma 4.1. [31] *For a proper subsemimodule P of a multiplication R-semimodule M, the following are equivalent to one another.*

(1) *P is prime in M.*

(2) *For all subsemimodules U and V of M, $U \cdot V \subseteq P$ implies that $U \subseteq P$ or $V \subseteq P$.*

(3) *For all elements m and m' in M, $m \cdot m' \subseteq P$ implies that $m \in P$ or $m' \in P$.*

Lemma 4.2. *If N and K are subsemimodules of a multiplication R-semimodule M, then $V_k(N) \bigcup V_k(K) = V_k(N \cdot K) = V_k(N \bigcap K)$.*

Proof. It follows from Lemma 2.7. $\square$

Theorem 4.3 generalizes Lemma 4.1 in [12].

Theorem 4.3. *If N is a k-subsemimodule of a multiplication R-semimodule M, then $\sqrt{N} = \sqrt{N}^{(k)}$.*

Proof. It suffices to show that $\sqrt{N}^{(k)} \subseteq \sqrt{N}$. Assume that $\sqrt{N}^{(k)} \nsubseteq \sqrt{N}$. Then there is an $m \in \sqrt{N}^{(k)}$ such that $m \notin \sqrt{N}$, and by Lemma 2.10, $m^k \nsubseteq N$ for all $k \in \mathbb{N}$.

Let Σ be the family of all the k-subsemimodules T of M such that $N \subseteq T$ and $m^k \nsubseteq T$ for all $k \in \mathbb{N}$. Since $N \in \Sigma$, Σ is nonempty and a partially ordered set under set-inclusion.

Now show that for every chain $C = \{T_i | \ i \in \Lambda\}$ in Σ, $\bigcup_{i \in \Lambda} T_i$ is an upper bound of C in Σ. Obviously, $\bigcup_{i \in \Lambda} T_i$ is a k-subsemimodule of M and $N \subseteq \bigcup_{i \in \Lambda} T_i$. If $m^k \subseteq \bigcup_{i \in \Lambda} T_i$ for some $k \in \mathbb{N}$, then by Lemma 2.9, m^k is a finitely generated subsemimodule of M, so there are $f_1, \dots, f_s \in M$ such that $\sum_{t=1}^{s} \langle f_t \rangle = m^k \subseteq \bigcup_{i \in \Lambda} T_i$. For every t with $1 \leq t \leq s$, there is an $i_t \in \Lambda$ such that $f_t \in T_{i_t}$. Since C is a chain, there is a $T_j \in C$ such that $\{f_t | \ 1 \leq t \leq s\} \subseteq T_j$, and so $m^k = \sum_{t=1}^{s} \langle f_t \rangle \subseteq T_j$, which contradicts that $T_j \in \Sigma$. Thus $\bigcup_{i \in \Lambda} T_i \in \Sigma$.

Therefore, by Zorn's lemma, Σ has a maximal element P.

Now show that P is prime in M. If P is not prime in M, then there are $r \notin (P : M)$ and $x \notin P$ such that $rx \in P$. By the maximality of P, $\overline{P + \langle x \rangle} \notin \Sigma$ and $\overline{P + rM} \notin \Sigma$, so there are $l_1, l_2 \in \mathbb{N}$ such that

$$m^{l_1} \subseteq \overline{P + \langle x \rangle}, \tag{4.1}$$

$$m^{l_2} \subseteq \overline{P + rM}. \tag{4.2}$$

By Eq.(4.1), for every $y \in m^{l_1}$, there are $p_1, p_1' \in P$ and $r_1, r_1' \in R$ such that $y + p_1 + r_1 x = p_1' + r_1' x$, and so $ry + rp_1 + r_1 rx = rp_1' + r_1' rx$. Since $rx \in P$, we have $ry \in \overline{P} = P$. Thus $rm^{l_1} \subseteq P$. Since $\langle m \rangle = IM$ for some ideal I of R and $m^{l_1} = I^{l_1} M$, $rI^{l_1} M = rm^{l_1} \subseteq P$, which implies that $rI^{l_1} \subseteq (P : M)$.

By Eq.(4.2), for every $z \in m^{l_2}$, there are $p_2, p_2' \in P$ and $n, n' \in M$ such that $z + p_2 + rn = p_2' + rn'$, and so for every $s \in I^{l_1}$, $sz + sp_2 + rsn = sp_2' + rsn'$. Since $rs \in rI^{l_1} \subseteq (P : M)$, we have $rsn, rsn' \in P$, and so $sz \in \overline{P} = P$. Thus $I^{l_1} m^{l_2} \subseteq P$.

Therefore, $m^{l_1 + l_2} = I^{l_1} I^{l_2} M = I^{l_1} m^{l_2} \subseteq P$, which contradicts that $P \in \Sigma$.

Since $P \in \Sigma$, $P \in V_k(N)$ and $m \notin P$, so $m \notin \sqrt{N}^{(k)}$, a contradiction. $\square$

Remark 4.4. Example 4.3 in [12] shows that the assumption "N is k-closed" cannot be omitted in Theorem 4.3.

Corollary 4.5 generalizes Theorem 2.12 in [9].

Corollary 4.5. *If N is a k-subsemimodule of a multiplication R-semimodule M, then $\sqrt{N}^{(k)} = \sqrt{(N : M)}M$.*

Proof. It follows from Theorem 4.3 and Lemma 2.11. $\square$

Theorem 4.6 generalizes Theorem 3.3 in [19].

Theorem 4.6. *Let M be a multiplication R-semimodule. If N is a proper k-subsemimodule of M, then there is a prime k-subsemimodule of M containing N. Therefore,* $\mathrm{Spec}_k(M) \neq \emptyset$.

Proof. Suppose that there are no prime k-subsemimodules of M containing N. Then $\sqrt{N}^{(k)} = M$. By Corollary 4.5, $\sqrt{N}^{(k)} = \sqrt{A}M$ and so $M = \sqrt{A}M$, where $A = (N : M)$. If $m \in M$, then $\langle m \rangle = IM$ for some ideal I of R, thus $\langle m \rangle = I(\sqrt{A}M) = (I\sqrt{A})M = (\sqrt{A}I)M = \sqrt{A}(IM) = \sqrt{A}\langle m \rangle$. Hence, $m = rm$ for some $r \in \sqrt{A}$. Since $r^k \in A$ for some $k \in \mathbb{N}$, we have $m = rm = r^2 m = \ldots = r^k m \in AM = N$. Therefore, $M = N$, a contradiction. □

Theorem 4.7 generalizes Proposition 3.9 in [2].

Theorem 4.7. *A multiplication R-semimodule M is k-finitely generated iff* $\mathrm{Spec}_k(M)$ *is a compact space.*

Proof. It follows from Lemma 2.12, Theorem 3.17 and Theorem 4.6. □

Theorem 4.8 generalizes Theorem 4.6 in [12] and Theorem 3.7 in [2].

Theorem 4.8. *If M is a multiplication R-semimodule, then every basic open set of* $\mathrm{Spec}_k(M)$ *is compact.*

Proof. By Lemma 3.4, it suffices to consider every cover consisting of the basic open sets. Let $D_k(f)$ be a basic open set of $\mathrm{Spec}_k(M)$. Suppose that $D_k(f) \subseteq \bigcup_{t\in\Lambda} D_k(f_t)$ and put $N = \overline{\langle\{f_t|\ t \in \Lambda\}\rangle}$. By Lemma 3.2(6)(7), $V_k(f) \supseteq \bigcap_{t\in\Lambda} V_k(f_t) = V_k(\sum_{t\in\Lambda}\langle f_t\rangle) = V_k(\overline{\sum_{t\in\Lambda}\langle f_t\rangle}) = V_k(N)$. By Lemma 3.2(8), $\langle f \rangle \subseteq \sqrt{N}^{(k)}$ and by Corollary 4.5, $\sqrt{N}^{(k)} = \sqrt{A}M$, where $A = (N : M)$. Thus $f \in \langle f \rangle \subseteq \sqrt{A}M$, and so $f = \sum_{i=1}^{s} r_i m_i$ for some $r_1, \ldots, r_s \in \sqrt{A}$ and some $m_1, \ldots, m_s \in M$. For every i with $1 \le i \le s$, $r_i^{l_i} \in A$ for some $l_i \in \mathbb{N}$, and $\langle m_i \rangle = I_i M$ for some ideal I_i of R. Now put $l = \sum_{i=1}^{s} l_i$. Then

$$f = \sum_{i=1}^{s} r_i m_i \in \sum_{i=1}^{s} r_i\langle m_i\rangle = \sum_{i=1}^{s} r_i I_i M = (\sum_{i=1}^{s} r_i I_i)M$$

and

$$\begin{aligned} f^l &\subseteq (\sum_{i=1}^{s} r_i I_i)^l M \subseteq (\sum_{i=1}^{s} r_i^{l_i} I_i^{l_i})M \subseteq (\sum_{i=1}^{s} r_i^{l_i} I_i)M \\ &= \sum_{i=1}^{s} r_i^{l_i} I_i M = \sum_{i=1}^{s} r_i^{l_i}\langle m_i\rangle = \sum_{i=1}^{s}\langle r_i^{l_i} m_i\rangle. \end{aligned}$$

For each i with $1 \le i \le s$, since

$$r_i^{l_i} m_i \in AM = N = \overline{\sum_{t\in\Lambda}\langle f_t\rangle},$$

we can see that

$$\begin{gathered} r_1^{l_1} m_1 + r_{11} f_{11} + \cdots + r_{1k_1} f_{1k_1} = r_{1k_1+1} f_{1k_1+1} + \cdots + r_{1n_1} f_{1n_1} \\ \vdots \\ r_s^{l_s} m_s + r_{s1} f_{s1} + \cdots + r_{sk_s} f_{sk_s} = r_{sk_s+1} f_{sk_s+1} + \cdots + r_{sn_s} f_{sn_s} \end{gathered}$$

for some $k_i, n_i \in \mathbb{N}$ with $k_i < n_i$, $r_{ij_i} \in R$ and $f_{ij_i} \in \{f_t|\ t \in \Lambda\}$, where $1 \le i \le s$ and $1 \le j_i \le n_i$. Then there exists a nonempty finite subset Γ of Λ such that

$$\{f_{ij_i}|\ 1 \le i \le s,\ 1 \le j_i \le n_i\} = \{f_t|\ t \in \Gamma\}.$$

Thus for all i with $1 \le i \le s$, $r_i^{l_i} m_i \in \overline{\langle\{f_t|\ t \in \Gamma\}\rangle}$, which implies that

$$f^l \subseteq \sum_{i=1}^{s}\langle r_i^{l_i} m_i\rangle \subseteq \overline{\langle\{f_t|\ t \in \Gamma\}\rangle}.$$

By Lemmas 4.1 and 3.2(6)(7),

$$V_k(f) = V_k(f^l) \supseteq V_k(\overline{\langle\{f_t|\ t \in \Gamma\}\rangle}) = V_k(\overline{\sum_{t\in\Gamma}\langle f_t\rangle}) = V_k(\sum_{t\in\Gamma}\langle f_t\rangle) = \bigcap_{t\in\Gamma} V_k(f_t),$$

and so $D_k(f) \subseteq \bigcup_{t\in\Gamma} D_k(f_t)$. Therefore, $D_k(f)$ is compact. □

Corollary 4.9 generalizes Corollary 4.7 in [12] and Corollary 3.8 in [2].

Corollary 4.9. *Let M be a multiplication R-semimodule. An open set of* $\mathrm{Spec}_k(M)$ *is compact iff it is a union of a finite number of basic open sets.*

Proof. Let $D_k(S)$ be an open set of $\mathrm{Spec}_k(M)$.

($\Rightarrow$) By Lemma 3.4, $D_k(S) = \bigcup_{i\in\Lambda} D_k(f_i)$ for some subset $\{f_i|\ i \in \Lambda\}$ of M. If $D_k(S)$ is compact, then there exists a nonempty finite subset Γ of Λ such that $D_k(S) = \bigcup_{i\in\Gamma} D_k(f_i)$.

($\Leftarrow$) follows immediately from Theorem 4.8. □

Corollary 4.10. *If N is a k-finitely generated subsemimodule of a multiplication R-semimodule M, then $D_k(N)$ is compact in* $\mathrm{Spec}_k(M)$.

Proof. Let $N = \overline{\langle f_1, \ldots, f_n\rangle}$. By Lemma 3.2(6)(7),

$$V_k(N) = V_k(\langle f_1, \ldots, f_n\rangle) = \bigcap_{i=1}^{n} V_k(f_i),$$

and so $D_k(N) = \bigcup_{i=1}^{n} D_k(f_i)$, which is compact by Corollary 4.9. □

Lemma 4.11. *If M is a multiplication R-semimodule and $f, g \in M$, then the following hold for* $\mathrm{Spec}_k(M)$.

(1) $D_k(f) \bigcap D_k(g) = D_k(f \cdot g)$.

(2) $D_k(f \cdot g)$ *is compact.*

Proof. (1) follows from Lemma 4.2.

(2) By Lemma 2.8, $f \cdot g$ is a finitely generated subsemimodule of M. By Lemma 3.2(6), $D_k(f \cdot g)$ is expressed as a union of a finite number of basic open sets. So the conclusion follows from Corollary 4.9. □

Theorem 4.12. *If M is a multiplication R-semimodule, then the intersection of finitely many basic open sets is compact in* $\mathrm{Spec}_k(M)$.

Proof. Use induction on the number n of basic open sets. In case $n = 2$, the statement is true by Lemma 4.11. Assume that the statement is true in case $n = k$ and consider the case $n = k + 1$. Let $f_1, \ldots, f_{k+1} \in M$. By inductive hypothesis, $\bigcap_{i=1}^{k} D_k(f_i)$ is compact and by Corollary 4.9, it equals the union $\bigcup_{j=1}^{l} D_k(g_j)$ of a finite number of basic open sets. Then

$$\begin{aligned} \bigcap_{i=1}^{k+1} D_k(f_i) &= \bigcap_{i=1}^{k} D_k(f_i) \bigcap D_k(f_{k+1}) \\ &= (\bigcup_{j=1}^{l} D_k(g_j)) \bigcap D_k(f_{k+1}) = \bigcup_{j=1}^{l} (D_k(g_j) \bigcap D_k(f_{k+1})). \end{aligned}$$

By Lemma 4.11, $D_k(g_j) \bigcap D_k(f_{k+1})$ are all compact, so is $\bigcap_{i=1}^{k+1} D_k(f_i)$. □

Theorem 4.13 generalizes Theorem 4.18 in [12] and the part about saturated prime ideals in Corollary 6.3 in [19].

Theorem 4.13. *If M is a k-finitely generated multiplication R-semimodule, then* $\mathrm{Spec}_k(M)$ *is a spectral space.*

Proof. By Theorem 3.16, $\mathrm{Spec}_k(M)$ is a compact space. By Theorem 3.5, $\mathrm{Spec}_k(M)$ is a T_0-space. By Lemma 3.4 and Theorem 4.8, the compact open subsets form a base for the Zariski topology in $\mathrm{Spec}_k(M)$. By Corollary 4.9 and Theorem 4.12, the intersection of finitely many compact open subsets is also a compact open subset in $\mathrm{Spec}_k(M)$. By Corollary 3.10, every irreducible closed subset has a generic point in $\mathrm{Spec}_k(M)$. Hence, $\mathrm{Spec}_k(M)$ is a spectral space. □

We give an example of a k-finitely generated multiplication semimodule that is not finitely generated.

Example 4.14. Let $R = \mathbb{Z}\bigcup\{a, b, +\infty\}$, where a and b are two distinct elements, not belonging to $\mathbb{Z}\bigcup\{+\infty\}$. Extend the natural ordering on $\mathbb{Z}$ onto R by defining $a < b < x < +\infty$ for all $x \in \mathbb{Z}$ so that R is a linearly ordered set. For $x, y \in R$, define

$$x + y = \max\{x, y\}, \qquad x \cdot y = \begin{cases} \max\{x, y\}, & \text{if } x \neq a \text{ and } y \neq a, \\ a, & \text{if } x = a \text{ or } y = a. \end{cases}$$

Then R is an additively idempotent and multiplicatively idempotent commutative semiring with zero a and identity b, and the nonzero proper ideals of the semiring R are $M = R\backslash\{b\}$ and $I_k = \langle k\rangle = \{a\}\bigcup\{x \in R|\ x \geq k\}$, where $k \in \mathbb{Z}\bigcup\{+\infty\}$. Obviously, M is a unique maximal ideal of the semiring R and is an R-semimodule.

Show that M is a multiplication R-semimodule. In fact, if N is a subsemimodule of M, then N is an ideal of the semiring R and so $NM \subseteq NR = N$. Since R is multiplicatively idempotent, $N = N^2 \subseteq NM$ and hence $N = NM$.

Show that this multiplication R-semimodule M is k-finitely generated, but not finitely generated. In fact, for every $x \in M$, $x + (+\infty) = +\infty$ and so $x \in \overline{\langle +\infty\rangle}$, which implies $M = \overline{\langle +\infty\rangle}$. Assume that there exists a nonempty finite subset S of M such that $M = \langle S\rangle$. Then $S \neq \{a\}$ and for the least element s in $S \setminus \{a\}$, $s \in \mathbb{Z}\bigcup\{+\infty\}$ and so $S \subseteq \langle s\rangle = I_s$, which implies $M = I_s$, a contradiction.

In addition, it is easy to see that this multiplication R-semimodule M has no maximal subsemimodule.

Given below is an example of a multiplication semimodule without k-maximal subsemimodule.

Example 4.15. Let $R = \mathbb{Z}\bigcup\{-\infty, +\infty\}$. Extend the natural ordering on $\mathbb{Z}$ onto R by defining $-\infty < x < +\infty$ for all $x \in \mathbb{Z}$ so that R is a linearly ordered set. For $x, y \in R$, define $x + y = \max\{x, y\}$ and $x \cdot y = \min\{x, y\}$. Then R is an additively idempotent and multiplicatively idempotent commutative semiring with zero $-\infty$ and identity $+\infty$, and the nonzero proper ideals of the semiring R are $M = R \setminus \{+\infty\}$ and $J_m = \langle m\rangle = \{x \in R|\ x \leq m\}$ which are all k-ideals, where $m \in \mathbb{Z}$. Obviously, M is a unique maximal ideal of the semiring R and turns out to be a multiplication R-semimodule in the same way as in Example 4.14.

Show that this multiplication R-semimodule M has no k-maximal subsemimodule. In fact, assume that K is a k-maximal subsemimodule of M. Then K is a proper ideal of the semiring R, neither M nor $\{-\infty\}$, and so $K = J_m$ for some $m \in \mathbb{Z}$. But for $n \in \mathbb{Z}$ with $n > m$, J_n is a proper k-subsemimodule of M containing K properly, a contradiction.

References

[1] J.Y. Abuhlail, *A Zariski topology for modules*, Comm. Algebra **39** (11), 4163-4182, 2011.

[2] R. Ameri, *Some properties of Zariski topology of multiplication modules*, Houston J. Math. **36** (2), 337-344, 2010.

[3] R. E. Atani, *Prime subsemimodules of semimodules*, Int. J. Algebra **4** (26), 1299-1306, 2010.

[4] S. E. Atani, R. E. Atani and U. Tekir, *A Zariski topology for semimodules*, Eur. J. Pure Appl. Math. **4** (3), 251-265, 2011.

[5] S. E. Atani and M. S. Kohan, *A note on finitely generated multiplication semimodules over commutative semirings*, Int. J. Algebra **4** (8), 389-396, 2010.

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison Wesley, Massachusetts, 1969.

[7] A. Barnard, *Multiplication modules*, J. Algebra **71**, 174-178, 1981.

[8] F. Çallialp, G. Ulucak and U. Tekir, *On the Zariski topology over an L-module M*, Turkish J. Math. **41**, 326-336, 2017.

[9] Z. A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Algebra **16** (4), 755-779, 1988.

[10] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] S. C. Han, *Maximal and prime k-subsemimodules in semimodules over semirings*, J. Algebra Appl. **16** (7), 1750130(11 pages), 2017.

[12] S. C. Han, *k-Congruences and the Zariski topology in semirings*, Hacet. J. Math. Stat. **50** (3), 699-709, 2021.

[13] S. C. Han and W. S. Pae, *Note on "Some properties of Zariski topology of multiplication modules": Proof of compactness of basic open sets*, Houston J. Math. **45** (4), 995-998, 2019.

[14] S. C. Han, W. S. Pae and J. N. Ho, *Topological properties of the prime spectrum of a semimodule*, J. Algebra **566**, 205-221, 2021.

[15] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in Computer Science*, World Scientific, Singapore, 1998.

[16] M. Henriksen, *Ideals in semirings with commutative addition*, Notices Amer. Math. Soc. **5**, 321, 1958.

[17] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142**, 43-60, 1969.

[18] J. L. Kelley, *General Topology*, Springer, New York, 1975.

[19] P. Lescot, *Absolute algebra III – the saturated spectrum*, J. Pure Appl. Algebra **216**, 1004-1015, 2012.

[20] C. P. Lu, *The Zariski topology on the prime spectrum of a module*, Houston J. Math. **25** (3), 417-432, 1999.

[21] R. L. McCasland, M. E. Moore and P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra **25** (1), 79-103, 1997.

[22] O. Öneş and M. Alkan, *Multiplication modules with prime spectrum*, Turkish J. Math. **43**, 2000-2009, 2019.

[23] A. Peña, L. M. Ruza and J. Vielma, *Separation axioms and the prime spectrum of commutative semirings*, Rev. Notas Mat. **5** (2), 66-82, 2009.

[24] Y. Tiraş, A. Harmanci and P. F. Smith, *A characterization of prime submodules*, J. Algebra **212**, 743-752, 1999.

[25] A. A. Tuganbaev, *Multiplication modules over noncommutative rings*, Sb. Math. **194** (12), 1837-1864, 2003.

[26] G. Ulucak, U. Tekir and K. H. Oral, *Separation axioms between T_0 and T_1 on lattices and lattice modules*, Ital. J. Pure Appl. Math. **36**, 245-256, 2016.

[27] G. Ulucak, U. Tekir and K. P. Shum, *On Spec(M) and separation axioms between T_0 and T_1*, Southeast Asian Bull. Math. **39**, 717-725, 2015.

[28] H. S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40**, 916-920, 1934.

[29] G. Yeşilot, *On prime and maximal k-subsemimodules of semimodules*, Hacet. J. Math. Stat. **39**, 305-312, 2010.

[30] G. Yeşilot, *On the prime spectrum of a module over noncommutative rings*, Int. J. Algebra **5** (11), 523-528, 2011.

[31] G. Yeşilot, K. H. Oral and U. Tekir, *On prime subsemimodules of semimodules*, Int. J. Algebra **4** (1), 53-60, 2010.

[32] G. Zhang, *Multiplication modules over any rings*, J. Nanjing Univ. Math. Biquarterly **23** (1), 51-61, 2006.

[33] G. Zhang, *Properties of top modules*, Int. J. Pure Appl. Math. **31** (3), 297-306, 2006.

[34] G. Zhang and W. Tong, *Spectral spaces of top right R-modules*, J. Nanjing Univ. Math. Biquarterly **17** (1), 15-20, 2000.