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Partially nonexpansive mappings

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Abstract

It is defined a class of generalized nonexpansive mappings, which properly contains those defined by Suzuki in 2008, and that preserves some of its fixed point results.

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1. Introduction

Nonexpansive mappings, (those which have Lipschitz constant equal to one), played an important role in many aspects of nonlinear functional analysis, with links to variational inequalities and the theory of monotone and accretive operators. They can be considered as a limiting case of the classical Banach contractions. The study of the existence of fixed points of nonexpansive mappings, and its asymptotic behavior, developed since the mid-sixties of the last century mainly in the setting of the closed convex subsets of Banach spaces, nowadays could be considered as a specific branch of the metric fixed point theory.

For many years, considerable activity in this field was focused to extend the Banach contraction principle, by relaxing or modifying the contractivity condition, obtaining new classes of mappings enjoying yet the property that each one of its members has a unique fixed point, and that this fixed point can always be found by using Picard iteration. For instance, Kannan (1969), Reich (1971), Hardy and Rogers (1973), and many others (See [5] for more details).

Similarly, nonexpansive mappings were extended over the last decades, getting larger families of mappings enjoying yet of nice fixed point properties as, for instance, was done by Gobel, Kirk and Shimi (1973), Jaggi

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(1982), Suzuki (2008), Aoyama and Kohsaka (2011) among many others. In particular, Suzuki approach (see [6]), is a relevant class which has been widely studied.

In this note, we introduce and compare a class of generalized nonexpansive mappings which properly contains those defined by Suzuki, and we give a fixed point theorem for them in the setting of Banach spaces enjoying normal structure.

2. Preliminaires

We suppose that $(X, \|\cdot\|)$ is a real Banach space, and 0_X its zero vector. From now on, C stands for a given nonempty, closed, convex and bounded subset of X. A mapping $T : C \to X$ is *nonexpansive* if for all $x, y \in C$, $\|T(x) - T(y)\| \leq \|x - y\|$.

Given a mapping $T : C \to C$, a sequence (x_n) in C is called an *almost fixed point sequence* (a.f.p.s. for short) *for* T whenever $x_n - T(x_n) \to 0_X$.

It is well known that if $T : C \to C$ is nonexpansive and $D \subset C$ is nonempty and T-invariant (i.e. $T(D) \subset D$), then T has a.f.p. sequences in D provided that D is closed and convex. This fact was coined as property (A) by Dhompongsa and Nanan in 2010 [1].

A mapping $T : C \to C$ is said to be *quasi-nonexpansive*, (QNE in short), provided that T has at least one fixed point $p \in C$, and for every fixed point p_0 of T and all $x \in C$, $\|p_0 - T(x)\| \leq \|p_0 - x\|$. Of course, nonexpansive selfmappings of C with some fixed point are QNE, but the converse does not hold.

We recall Suzuki's definition of generalized nonexpansive mappings.

Definition 2.1. *(See [6]) Let C be a nonempty subset of a Banach space X. We say that a mapping $T : C \to X$ satisfies* condition *(C) on C, (or that T is a C-type mapping), if for all $x, y \in C$,*

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\| \text{ implies } \|T(x) - T(y)\| \leq \|x - y\|. \tag{1}$$

This definition was a breakpoint in the development of the theory of generalized nonexpansive mappings. Of course, every nonexpansive mapping $T : C \to X$ satisfies condition (C) on C, but in [6, Example 1] a non continuous mapping satisfying condition (C) is given.

Every mapping $T : C \to C$ which satisfies condition (C) on C and has some fixed point, is QNE.

If C is a closed, convex, bounded, subset of X, then every C-type selfmapping of C has a.f.p. sequences, that is, C-type mappings share property (A) with nonexpansive mappings (see [6, Lemma 6]).

In 2011 (see [2]), the class of the Suzuki type mappings was in turn generalized as follows.

Definition 2.2. *For $\mu \geq 1$ we say that a mapping $T : C \to X$ satisfy condition (E_μ) on C if there exists $\mu \geq 1$ such that for all $x, y \in C$,*

$$\|x - T(y)\| \leq \mu\|x - T(x)\| + \|x - y\|. \tag{2}$$

We say that T satisfies condition (E) on C whenever T satisfies (E_μ) for some $\mu \geq 1$.

It is obvious that if $T : C \to X$ is nonexpansive, then it satisfies condition (E_1). The converse is not true. If a mapping satisfies condition (E) and has a fixed point then it is QNE.

From Lemma 7 in [6] we know that if $T : C \to X$ satisfies condition (C) on C, then it satisfies condition (E_3). The converse is not true.

Again in 2011 it was introduced a class of mappings properly containing the Suzuki C-type mappings.

Definition 2.3. *(See [4]).*

We say that the mapping $T : C \to C$ is an L-type mapping, (or that T satisfies condition (L)) on C), provided that

a) If $D \subset C$ is nonempty, closed, convex and T invariant, then there exists an a.f.p.s. (x_n) for T in D.

b) For every a.f.p.s. (x_n) for T in C, and for each $x \in C$,

$$\limsup_n \|x_n - T(x)\| \le \limsup_n \|x_n - x\|.$$

The above assumption (a) is just Condition (A).

If $T : C \to C$ satisfies Suzuki's condition (C), then it satisfies the condition (L) on C (see [4, Proposition 3.4.]).

Recall that a bounded and convex subset K of a Banach space X is said to have *normal structure* if every convex subset H of K that contains more than one point contains a non diametral point $x_0 \in H$, that is a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \mathrm{diam}(H),$$

where $\mathrm{diam}(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H.

A Banach space $(X, \|\cdot\|)$ is said to have *normal structure* if every bounded and convex subset of X has normal structure. and it is said to have *weak normal structure* if each weakly compact convex subset K of X has normal structure. The normal structure is more general than the uniform convexity and that many other relevant geometrical properties of (the norm of) the Banach spaces.

3. Partially nonexpansive mappings

Definition 3.1. *A mapping $T : C \to C$ is said to be partially nonexpansive, (PNE in short), if for all $x \in C$,*

$$\|T\big(\frac{1}{2}(x + T(x))\big) - T(x)\| \le \frac{1}{2}\|x - T(x)\|. \tag{3}$$

Proposition 3.2. *It $T : C \to C$ satisfies Suzuki's condition (C), then T is partially nonexpansive.*

Proof. Since, for all $x \in C$,

$$\frac{1}{2}\|x - T(x)\| = \|\frac{1}{2}(x + T(x)) - x\|,$$

then, from condition (C),

$$\|T\big(\frac{1}{2}(x + T(x))\big) - T(x)\| \le \|\frac{1}{2}(x + T(x) - x\| = \frac{1}{2}\|x - T(x)\|.$$

□

Consequently, every nonexpansive mapping $T : C \to C$ is partially nonexpansive.

The converse of the above proposition is not true.

Example 1. *Let $T : B[0_X, 1] \to B[0_X, 1]$ be the mapping defined as*

$$T(x) = \begin{cases} \frac{1}{2}\frac{x}{\|x\|} & x \in B[0_X, 1] \backslash B[0_X, \frac{1}{2}] \\ 0_X & x \in B[0_X, \frac{1}{2}]. \end{cases}$$

Claim 3.3. *The mapping T is PNE.*

Proof. Indeed, for $x \in B[0_X, \frac{1}{2}]$,

$$\frac{1}{2}(x+T(x)) = \frac{1}{2}x \Rightarrow \|\frac{1}{2}(x+T(x))\| \leq \frac{1}{4} \Rightarrow T\big(\frac{1}{2}(x+T(x))\big) = 0_X,$$

and $T(x) = 0_X$. Thus,

$$\|T\big(\frac{1}{2}(x+T(x))\big) - T(x)\| = 0 \leq \frac{1}{2}\|x - T(x)\|.$$

For $x \in B[0_X, 1] \backslash B[0_X, \frac{1}{2}]$ we have that $T(x) = \frac{1}{2}\frac{x}{\|x\|}$, and therefore

$$\frac{1}{2}(x+T(x)) = \frac{1}{2}(x + \frac{1}{2}\frac{x}{\|x\|}) = \frac{2\|x\|+1}{4\|x\|}x.$$

Since $\|x\| > \frac{1}{2}$ then

$$\left\|\frac{1}{2}(x+T(x))\right\| = \frac{2\|x\|+1}{4} > \frac{1}{2},$$

which implies that

$$T\big(\frac{1}{2}(x+T(x))\big) = \frac{1}{2\frac{2\|x\|+1}{4}}\frac{1}{2}(x+T(x)) = \frac{1}{2\|x\|+1}(x+T(x)).$$

Consequently,

$$T\big(\frac{1}{2}(x+T(x))\big) - T(x) = \frac{x+T(x)}{2\|x\|+1} - T(x) = \frac{x+T(x) - 2\|x\|T(x) - T(x)}{2\|x\|+1} = 0_X,$$

which immediately yields that

$$\left\|T\big(\frac{1}{2}(x+T(x))\big) - T(x)\right\| = 0 \leq \frac{1}{2}\|x - T(x)\|.$$

In both cases x satisfies condition (3). □

Claim 3.4. *The mapping T fails to satisfy Suzuki's condition (C).*

Proof. Indeed, taking $x \in B[0_X, 1]$ with $\|x\| = \frac{1}{2}$ and $y := \frac{3}{2}x$, we have that $\|y\| = \frac{3}{4}$ and

$$\frac{1}{2}\|x - T(x)\| = \|\frac{1}{2}x\| = \frac{1}{4} = \|y - x\|,$$

while

$$\|T(y) - T(x)\| = \left\|\frac{1}{2\|y\|}y - 0_X\right\| = \|\frac{2}{3}y\| = \frac{1}{2} > \frac{1}{4} = \|y - x\|.$$

□

Claim 3.5. *The mapping T is QNE.*

Indeed, the unique fixed point of T is just 0_X. Then for every $x \in B[0_X, 1]$,

$$\|0_X - T(x)\| \leq \|x\| = \|0_X - x\|.$$

□

♢

Our next example shows that there exists PNE mappings failing to be QNE.

Example 2. *In the Banach space* $(\mathbb{R}^2, \|\cdot\|_\infty)$ *let* $T : B_\infty[0_X, 2] \to B_\infty[0_X, 2]$ *be the mapping defined as*

$$T(x) = \begin{cases} \frac{x}{\|x\|_\infty} & x \in B_\infty[0_X, 2] \backslash B_\infty[0_X, 1] \\ x & x \in B_\infty[0_X, 1]. \end{cases}$$

It is well known that this mapping is 2-Lipchitzcian. For $r \in (0, 1)$ small enough, taking $x := (1, 1)$ and $y_r := (1 - r, 1 + r)$ we have that x is a fixed point of T and

$$\|x - T(y_r)\|_\infty = \left\|(1,1) - (\frac{1-r}{1+r}, 1)\right\|_\infty = 1 - \frac{1-r}{1+r} = \frac{2r}{1+r},$$

while,

$$\|x - y_r\|_\infty = \|(r, -r)\|_\infty = r.$$

Therefore

$$\|x - T(y_r)\|_\infty = \frac{2}{1+r}\|x - y_r\|_\infty > \|x - y_r\|_\infty.$$

Thus, T fails to be QNE (w.r.t. the norm $\|\cdot\|_\infty$). Hence it fails to satisfy both conditions (C) and (L). However, we have the following.

Claim 3.6. *The mapping* T *is PNE.*

Proof. Let us suppose that $\|x\|_\infty > 1$. Then, $T(x) = \frac{x}{\|x\|_\infty}$ and

$$\frac{1}{2}(x + T(x)) = \frac{(1 + \frac{1}{\|x\|_\infty})x}{2} = \frac{1 + \|x\|_\infty}{2\|x\|_\infty}x.$$

Then,

$$\|\frac{1}{2}(x + T(x))\| = \frac{1 + \|x\|_\infty}{2} > 1.$$

Consequently,

$$T\left(\frac{1}{2}(x + T(x))\right) = \frac{\frac{1+\|x\|_\infty}{2\|x\|_\infty}x}{\frac{1+\|x\|_\infty}{2}} = \frac{x}{\|x\|_\infty} = T(x).$$

It follows that

$$\left\|T\left(\frac{1}{2}(x + T(x))\right) - T(x)\right\|_\infty = 0 \le \frac{1}{2}\|x - T(x)\|_\infty.$$

If $\|x\|_\infty \le 1$, then, $T(x) = x$ and

$$\frac{1}{2}(x + T(x)) = x \Rightarrow T(\frac{1}{2}(x + T(x))) = T(x).$$

Again it follows that

$$\Rightarrow \left\|T\left(\frac{1}{2}(x + T(x))\right) - T(x)\right\|_\infty = 0 \le \frac{1}{2}\|x - T(x)\|_\infty.$$

□

Of course, this mapping T also fails to satisfy Suzuki's condition as well as to belong to many other classes of generalized nonexpansive mapping which very often are QNE. ♢

4. Fixed point results

The following example shows that PNE mappings can fail to have fixed points even when they are defined on compact convex sets.

Example 3. Consider the sequence (a_n) in $[0,1]$ defined inductively by $a_0 = 0$, $a_1 = \frac{a_0+1}{2} = \frac{1}{2}$, and, for $n \geq 2$ $a_{n+1} = \frac{a_n + \frac{3}{4}}{2}$. It is easy to see that, for every n, $a_n < \frac{3}{4}$. Then

$$2a_{n+1} - a_n = \frac{3}{4} \Rightarrow a_{n+1} - a_n = \frac{3}{4} - a_{n+1} > 0.$$

Let $T : [0,1] \to [0,1]$ defined as

$$T(x) := \begin{cases} 1 & \text{if } x = a_0, \\ \frac{3}{4} & \text{if } x \in \{a_1, a_2, \ldots\}, \\ 0 & \text{if } x \notin \{a_0, a_1, a_2, \ldots\}, x < 1, \\ \frac{1}{3} & \text{if } x = 1. \end{cases}$$

It is obvious that T is fixed-point free. According [6, Theorem 4.] and [4, Theorem 4.2.] this mapping cannot satisfy neither condition (C) nor condition (L).

For $x = a_0 = 0$, one has that

$$\left|T(\frac{0+T(0)}{2}) - T(0)\right| = \left|T(\frac{1}{2}) - 1\right| = \frac{1}{4} = \frac{1}{2}|0 - T(0)|.$$

For $x = a_n$, with $n \geq 1$, one has that

$$\left|T(\frac{a_n + T(a_n)}{2}) - T(a_n)\right| = |T(a_{n+1}) - T(a_n)| = 0 \leq \frac{1}{2}|a_n - T(a_n)|.$$

For $x \notin \{a_0, a_1, a_2, \ldots\}$, $0 < x < 1$ one has that $0 < \frac{x}{2} < \frac{1}{2}$ which implies that $\frac{x}{2} \notin \{a_0, a_1, a_2, \ldots\}$, and therefore, since $T(x) = 0$,

$$\left|T(\frac{x+T(x)}{2}) - T(x)\right| = \left|T(\frac{x}{2}) - 0\right| = 0 \leq \frac{1}{2}|x - T(x)|.$$

Finally, for $x = 1$, one has that

$$\left|T(\frac{1+T(1)}{2}) - T(1)\right| = \left|T(\frac{2}{3}) - \frac{1}{3}\right| = \frac{1}{3} = \frac{1}{2}|1 - T(1)|.$$

Thus, T is a fixed point-free PNE self-mapping of the compact convex set $[0,1]$.

$\diamondsuit$

In [2, Example 1] an example of a closed convex bounded subset of a Banach space, and a fixed-point free selfmapping of it satisfying condition (E_1) are given. Even more, the following example shows that a mapping defined on a compact convex set and satisfying some condition (E_μ) can fail to have fixed points.

Example 4. *Let $T : [0,1] \to [0,1]$ be the mapping defined as*

$$T(x) = \begin{cases} x + \frac{1}{2} & x \in [0, \frac{1}{2}] \\ x - \frac{1}{2} & x \in (\frac{1}{2}, 1]. \end{cases}$$

It is obvious that, for every $x \in [0,1]$, $|x - T(x)| = \frac{1}{2}$ and therefore, for every $y \in [0,1]$

$$|x - T(y)| \leq 1 \leq 2|x - T(x)| + |x - y|.$$

Thus, T satisfies condition (E_2) in $[0,1]$. Of course T is fixed point free. According [6, Theorem 4] and [4, Theorem 4.4.] it fails to satisfy conditions (C) and (L) too.

The previous examples show that, neither condition PNE nor condition (E) imply to have fixed points for a mapping. However, next we will see that both properties together, (PNE along with (E)), guarantee fixed points in Banach spaces whose norm enjoy of with a suitable geometrical property.

Theorem 4.1. *Suppose that the space $(X, \|\cdot\|)$ enjoys normal structure. Let C be a nonempty weakly compact convex subset of X and let $T : C \to C$ be a mapping such that.*

a) T is PNE.

b) T satisfies a condition (E_μ) for some $\mu \geq 1$.

Then T has a fixed point.

Proof. First, we will see that the mapping T admits a.f.p. sequences in each nonempty closed convex T-invariant subset D of C. Take $x_0 \in D$ and define, for $n \geq 1$

$$x_{n+1} = \frac{x_n + T(x_n)}{2} \tag{4}$$

It is obvious that this algorithm generates a sequence (x_n) in D. From the definition of x_{n+1}, bearing in mind that T is PNE, we have that,

$$\begin{aligned} \|T(x_{n+1}) - T(x_n)\| &= \left\|T\left(\tfrac{x_n+T(x_n)}{2}\right) - T(x_n)\right\| \\ &\overset{(3)}{\leq} \tfrac{1}{2}\|x_n - T(x_n)\| = \|x_{n+1} - x_n\|. \end{aligned}$$

Therefore, from [6, Lemma 3], (see also [3]), we have that $\|x_n - T(x_n)\| \to 0$. In other words, we have seen that the mapping T under consideration satisfies Property (A). If $x \in C$ and (x_n) is an a.f.p.s for T, then from (b),

$$\limsup \|x_n - T(x)\| \overset{(b)}{\leq} \limsup(\mu\|x_n - T(x_n)\| + \|x_n - x\|) = \limsup \|x_n - x\|.$$

Thus, T is an L-type mapping and, from [4, Theorem 4.4.], it follows that T has a fixed point in C. □

Remark 4.2. In this proof we have seen that PNE mappings satisfy property (A).

In the same proof we have seen too that conditions (a) and (b) together imply condition (L). However, the class of the PNE mappings neither contains nor is contained in the class of the L-type mappings. The mapping of the above Example (2) is PNE but it fails to be QNE, and consequently cannot satisfy condition (L). On the other hand, the following example shows an (L) type mapping failing to be PNE.

Example 5. *Let $T : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ be the mapping defined as $T(x) = x^2$.*

Claim 4.3. *The mapping T satisfies conditions (L) and (E) but it fails to be PNE.*

Proof. In [4, Example 3.7]) is seen that T satisfies condition (L).

To see that it fails to be PNE, take $x = \frac{2}{3}$. $T(\frac{2}{3}) = \frac{4}{9}$. Then,

$$\left|T\Big(\frac{\frac{2}{3} + T(\frac{2}{3})}{2}\Big) - T\Big(\frac{2}{3}\Big)\right| = \left|\left(\frac{\frac{2}{3} + (\frac{2}{3})^2}{2}\right)^2 - \Big(\frac{2}{3}\Big)^2\right| = \left|\frac{25}{81} - \frac{4}{9}\right| = \frac{11}{81},$$

while

$$\frac{1}{2}\left|\frac{2}{3} - T\Big(\frac{2}{3}\Big)\right| = \frac{1}{2}\left|\frac{2}{3} - \frac{4}{9}\right| = \frac{1}{9} < \frac{11}{81}.$$

Now we will see that satisfies the condition (E_μ) for $\mu \geq \frac{9}{8}$.

Case 1. $0 \leq x^2 \leq x \leq y^2 \leq y \leq \frac{2}{3}$

Then, for $\mu \geq 1$,

$$\begin{aligned} y^2 - y \leq 0 \leq (x - x^2) \quad &\Rightarrow y^2 - x \leq (x - x^2) + y - x \\ &\Rightarrow |y^2 - x| \leq |x - x^2| + |y - x| \\ &\Rightarrow |x - T(y)| \leq \mu|x - T(x)| + |y - x|. \end{aligned}$$

Case 2. $0 \leq x^2 \leq y^2 \leq x \leq y \leq \frac{2}{3}$

Then, since $\mu \geq \frac{9}{8} > 1$,

$$\begin{aligned} x - y^2 \leq x - x^2 \quad &\Rightarrow x - y^2 \leq (x - x^2) + y - x \\ &\Rightarrow |x - y^2| \leq |x - x^2| + |y - x| \\ &\Rightarrow |x - T(y)| \leq \mu|x - T(x)| + |y - x|. \end{aligned}$$

Case 3. $x \geq y$ and $x \in (\frac{1}{2}, \frac{2}{3}]$. Then, since $t - t^2 \leq \frac{1}{4}$ for every $t \in [0, 1]$,

$$y - y^2 \leq \frac{1}{4} \leq \mu(\frac{2}{3} - (\frac{2}{3})^2) \Rightarrow x - y^2 \leq \mu(\frac{2}{3} - (\frac{2}{3})^2) + x - y$$

Notice that on the interval $[\frac{1}{2}, \frac{2}{3}]$ the function $t \mapsto t - t^2$ decreases and hence $x - x^2 \geq \frac{2}{3} - \left(\frac{2}{3}\right)^2$ whenever $x \in [\frac{1}{2}, \frac{2}{3}]$. Therefore, for such x,

$$x - y^2 \leq \mu(\frac{2}{3} - (\frac{2}{3})^2) + x - y \leq \mu(x - x^2) + x - y,$$

that is,

$$|x - T(y)| \leq \mu|x - T(x)| + |x - y|.$$

Case 4. $x \geq y$ and $x \leq \frac{1}{2}$.

Then,

$$\begin{aligned} |x - y^2| \leq |x - x^2| + |x^2 - y^2| \quad &\Rightarrow |x - y^2| \leq |x - x^2| + |x - y||x + y| \\ &\Rightarrow |x - y^2| \leq |x - x^2| + |x - y|(|x| + |y|) \\ &\Rightarrow |x - y^2| \leq |x - x^2| + |x - y| \\ &\Rightarrow |x - T(y)| \leq \mu|x - T(x)| + |x - y|. \end{aligned}$$

□

♢

5. Conclusions

We have defined a large class of nonlinear mappings which properly contains the Suzuki C-type mappings and hence the nonexpansive mappings. To check that a mapping satisfy Condition (C) sometimes is not an easy task. However, is often easier to check inequality (3) for each $x \in C$.

Notice that (PNE) are mappings for which the nonexpansivity is asked only for pairs $(x, y) \in C \times C$ such that y is the result of applying in x the Karnoselskii algorithm (4). This opens a way to define other families of generalized nonexpansive mappings satisfying condition (A).

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