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Lacunary Statistical Convergence for Double Sequences on $\mathscr{L}-$ Fuzzy Normed Space

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Abstract

On $\mathscr{L}-$ fuzzy normed spaces, which is the generalization of fuzzy spaces, the notion of lacunary statistical convergence for double sequences which is a generalization of statistical convergence, are studied and developed in this paper. In addition, the definitions of lacunary statistical Cauchy and completeness for double sequences and related theorems are given on $\mathscr{L}-$ fuzzy normed spaces. Also, the relationship of lacunary statistical Cauchyness and lacunary statistical boundedness for double sequences with respect to $\mathscr{L}-$ fuzzy norm is shown.

1. Introduction

After the fuzzy set theory was introduced to the world of mathematics by Zadeh [1], this theory was developed and generalized by many different mathematicians such as intuitionistic fuzzy sets, which was developed by Atanassov [2]. Different convergence studies of sequences on these proposed spaces have received and continue to be of great interest in the mathematical community. The concept of statistical convergence [3]- [9] which can be accepted as a generalization of convergence in the classical sense, is also very important in the field of functional analysis, and together with this concept, statistical limitation, statistical Cauchy and statistical bounded sequences have been examined.
Many studies have been carried out in the fields fuzzy metric spaces [10], [11] and intuitionistic fuzzy metric spaces [12]- [15].
$\mathscr{L}-$ fuzzy normed spaces [16]- [18] are natural generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces, in which important work has been done on the theory of summability in this space [19]- [21], based on some logical algebraic structures.
To date, the types of convergence have been studied by many mathematician [22]- [29]. In particular, the characteristics of convergence types have been introduced to the mathematical community by Dündar [30]- [36].
The goal of the present study is to examine on $\mathscr{L}-$ fuzzy normed spaces the lacunary statistical convergence, which was initially introduced by Fridy, John Albert, and Cihan Orhan [37], [38]. Next, we give some results regarding lacunary statistical convergence of double sequences and investigate the relationship between lacunary statistical convergent, lacunary statistical Cauchy and lacunary statistical bounded sequences, which will be newly introduced on $\mathscr{L}-$ fuzzy normed spaces. We propose a relevant characterisation for lacunary statistically convergent for double sequences. Furthermore, we show an example where our convergence approach outperforms more than the traditional convergence on $\mathscr{L}-$ fuzzy normed spaces.

2. Preliminaries

Preliminaries on $\mathscr{L}-$ fuzzy normed spaces are presented in this section.

Definition 2.1 ([39]). *Assume that* $K:[0,1]\times[0,1]\to[0,1]$ *is a function that satisfies the following*

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1. $K(a,b)=K(b,a)$,
2. $K(K(a,b),c)=K(a,K(b,c))$,
3. $K(a,1)=K(1,a)=x$,
4. $a\leq b, c\leq d$ then $K(a,c)\leq K(b,d)$,

is known as a $t-$ norm.

Example 2.2 ([39]). *K_1, K_2 and K_3 are the functions that given with,*

$$K_1(a,b)=\min\{a,b\},$$
$$K_2(a,b)=ab,$$
$$K_3(a,b)=\max\{a+b-1,0\}$$

are the samples, which are well known of $t-$ norms.

Definition 2.3 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice and let a set A be called the universe. An $L-$fuzzy set, on A is defined with a function*

$$X:A\rightarrow L.$$

On a set A, the family of all $L-$sets is denoted by L^A.

Two $L-$ sets on A intersect

$$(C\cap D)(x)=C(x)\wedge D(x)$$

for all $x\in A$. Similarly, union and intersection of a family $\{B_i : i\in I\}$ of $L-$ fuzzy sets is given by

$$\Big(\bigcup_{i\in I} B_i\Big)(x)=\bigvee_{i\in I} B_i(x)$$

and

$$\Big(\bigcap_{i\in I} B_i\Big)(x)=\bigwedge_{i\in I} B_i(x).$$

0_L and 1_L are the smallest and biggest elements of the full Lattice L, respectively. On a given lattice $(L,\preceq)$, we also employ the symbols $\succeq, \prec$, and $\succ$ in the obvious meanings.

Definition 2.4 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice. Therefore, $t-$ norm is a function $\mathscr{K}: L\times L\rightarrow L$ that satisfies the following for all $a,b,c,d\in L$:*

1. $\mathscr{K}(a,b)=\mathscr{K}(b,a)$,
2. $\mathscr{K}(\mathscr{K}(a,b),c)=\mathscr{K}(a,\mathscr{K}(b,c))$,
3. $\mathscr{K}(a,1_L)=\mathscr{K}(1_L,a)=a$,
4. $a\preceq b$ and $c\preceq d$, then $\mathscr{K}(a,c)\preceq\mathscr{K}(b,d)$.

Definition 2.5 ([39]). *For sequences (a_n) and (b_n) on L such that $(a_n)\rightarrow a\in L$ and $(b_n)\rightarrow b\in L$, if the property that $\mathscr{K}(a_n,b_n)\rightarrow\mathscr{K}(a,b)$ satisfies on L, then a $k-$norm $\mathscr{K}$ on a complete lattice $\mathscr{L}=(L,\preceq)$ is called continuous.*

Definition 2.6 ([39]). *The function $\mathscr{N}: L\rightarrow L$ is defined as a negator on $\mathscr{L}=(L,\preceq)$ if,*

$N_1)$ $\mathscr{N}(0_L)=1_L$,

$N_2)$ $\mathscr{N}(1_L)=0_L$,

$N_3)$ $a\preceq b$ *implies* $\mathscr{N}(b)\preceq\mathscr{N}(a)$ *for all* $a,b\in L$.

If in addition,

$N_4)$ $\mathscr{N}(\mathscr{N}(a))=a$ *for all* $a\in L$.

Therefore, $\mathscr{N}$ is known as an involutive.

The mapping $\mathscr{N}_s:[0,1]\rightarrow[0,1]$, on the lattice $([0,1],\leq)$ defined as $\mathscr{N}_s(x)=1-x$ is a well known sample of an involutive negator. This type of negator are using in the notion of stansard fuzzy sets. In addition, with the order

$$(\mu_1,\nu_1)\preceq(\mu_2,\nu_2)\iff\mu_1\leq\mu_2\quad and\quad\nu_1\geq\nu_2$$

given the lattice $([0,1]^2,\preceq)$ with for all $i=1,2$, $(\mu_i,\nu_i)\in[0,1]^2$. Therefore, the function $\mathscr{N}_1:[0,1]^2\rightarrow[0,1]^2$,

$$\mathscr{N}_1(\mu,\nu)=(\nu,\mu)$$

in the sense of Atanassov, is known as a involutive negator. This type of negator are using in the notion of intuitionistic fuzzy sets.

Definition 2.7 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice and V be a real vector space. $\mathscr{K}$ be a continuous $t-$norm on $\mathscr{L}$ and μ be an $L-$set on $V\times(0,\infty)$ satisfying the following*

(a) $\mu(a,t)\succ 0_L$ *for all* $a\in V, t>0$,
(b) $\mu(a,t)=1_L$ *for all* $t>0$ *if and only if* $a=\theta$,
(c) $\mu(\alpha a,t)=\mu(a,\frac{t}{|\alpha|})$ *for all* $a\in V, t>0$ *and* $\alpha\in\mathbb{R}-\{0\}$,
(d) $\mathscr{K}(\mu(a,t),\mu(b,s))\preceq\nu(a+b,t+s)$, *for all* $a,b\in V$ *and* $t,s>0$,
(e) $\lim_{t\rightarrow\infty}\mu(a,t)=1_L$ *and* $\lim_{t\rightarrow 0}\mu(a,t)=0_L$ *for all* $a\in V-\{\theta\}$,

(f) *The functions* $f_a : (0,\infty) \to L$ *which is* $f(t) = \mu(a,t)$ *are continuous.*
The triple $(V,\mu,\mathscr{K})$ *is referred to as an* $\mathscr{L}-$ *fuzzy normed space or* $\mathscr{L}-$ *normed space in this context.*

Definition 2.8 ([39]). *A sequence* (a_n) *is said to be Cauchy sequence in a* $\mathscr{L}-$ *fuzzy normed space* $(V,\mu,\mathscr{K})$ *if, there exists* $n_0 \in \mathbb{N}$ *such that, for all* $m,n > n_0$

$$\mu(a_n - a_m, t) \succ \mathscr{N}(\varepsilon),$$

where $\mathscr{N}$ *is a negator on* $\mathscr{L}$*, for each* $\varepsilon \in L - \{0_L\}$ *and* $t > 0$.

Definition 2.9. *A sequence* $a = (a_n)$ *is said to be bounded with respect to fuzzy norm in a* $\mathscr{L}-$ *fuzzy normed space* $(V,\mu,\mathscr{K})$ *, provided that, for each* $r \in L - \{0_L, 1_L\}$ *and* $t > 0$,

$$\mu(a_n,t) \succ \mathscr{N}(r),$$

for all $n \in \mathbb{N}$.

On $\mathscr{L}-$ fuzzy normed spaces, we'll look at statistical convergence. Before we continue, let's go through basic statistical convergence terms. If $K \subseteq \mathbb{N}$, the set of natural numbers, then $\delta\{A\}$ is the asymptotic density of A, is

$$\delta\{A\} := \lim_k \frac{1}{k}\left|\{n \leq k : n \in A\}\right|$$

the limit exists the cardinality of the set A is given by $|A|$.
If the set $K(\varepsilon) = \{n \leq k : |a_n - l| > \varepsilon\}$ has the asymptotic density zero, i.e.

$$\lim_k \frac{1}{k} \mid \{n \leq k : |a_n - l| > \varepsilon\} \mid = 0,$$

then the sequence $a = (a_n)$ is known as a statistically convergent to the number l. In this case, we will write $st - \lim a = l$.
Despite the fact that every convergent sequence is statistically convergent to the same limit, the opposite of this is not necessarily true.

Definition 2.10 ([40]). *A sequence* $a = (a_n)$ *is statistically convergent to* $l \in V$ *with respect to* μ *fuzzy norm in a* $\mathscr{L}-$ *fuzzy normed space* $(V,\mu,\mathscr{K})$ *if provided that, for each* $\varepsilon \in L - \{0_L\}$ *and* $t > 0$,

$$\delta\{n \in \mathbb{N} : \mu(a_n - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$$

or equivalently

$$\lim_m \frac{1}{m}\{j \leq m : \mu(a_n - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

In this case, we will write $st_{\mathscr{L}} - \lim a = l$.

Definition 2.11 ([40]). *A sequence* $a = (a_k)$ *is said to be statistically Cauchy with respect to fuzzy norm* μ *in a* $\mathscr{L}-$ *fuzzy normed space* $(V,\mu,\mathscr{K})$*, if provided that*

$$\delta\{k \in \mathbb{N} : \mu(a_k - a_m, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$$

for each $\varepsilon \in L - \{0_L\}$*,* $m \in \mathbb{N}$ *and* $t > 0$.

Definition 2.12 ([40]). *A sequence* $a = (a_k)$ *is said to be statistically bounded with respect to fuzzy norm* μ *in a* $\mathscr{L}-$ *fuzzy normed space* $(V,\mu,\mathscr{K})$ *if provided that there exists* $r \in L - \{0_L, 1_L\}$ *and* $t > 0$ *such that*

$$\delta\{k \in \mathbb{N} : \mu(a_k,t) \nsucc \mathscr{N}(r)\} = 0$$

for each positive integer k.

3. Lacunary Statistical Convergence for Double Sequences on $\mathscr{L}-$Fuzzy Normed Space

In this section we define and study lacunary statistical convergence for double sequences on $\mathscr{L}-$ fuzzy normed space.

Definition 3.1. *By a lacunary sequence we mean an increasing integer sequence* $\theta = (k_r)$ *such that* $k_0 = 0$ *and* $h_r := k_r - k_{r-1} \to \infty$ *as* $r \to \infty$*. The intervals determined by* θ *will be denoted by* $I_r := (k_{r-1}, k_r]$ *and the ratio* $\frac{k_r}{k_{r-1}}$ *will be abbreviated by* q_r.
For any set $N \subseteq \mathbb{N}$*, the number*

$$\delta_\theta(N) = \lim_{r\to\infty} \frac{1}{h_r}\left|\{k \in I_r : k \in N\}\right|$$

is called the θ *density of the set* N*, provided the limit exists.*
A sequence $a = (a_k)$ *is said to be lacunary statistically convergent or* S_θ *convergent to a number* ℓ *provided that for each* $\varepsilon > 0$,

$$\delta_\theta\{k \in \mathbb{N} : |a_k - \ell| \geq \varepsilon\}| = 0.$$

In other words, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ *has* $\theta-$ *density zero. In this case the number* ℓ *is called lacunary statistical limit of the sequence* $x = (x_k)$ *and we write* $S_\theta - \lim_{r\to\infty} x_k = \ell$ *or* $x_k \to \ell(S_\theta)$.

Definition 3.2. *Let* $(V,\mu,\mathscr{K})$ *be a* $\mathscr{L}-$*fuzzy normed space. Then a sequence* $a = (a_k)$ *is lacunary statistically convergent to* $l \in V$ *with respect to* μ *fuzzy norm, provided that, for each* $\varepsilon \in L - \{0_L\}$ *and* $t > 0$,

$$\delta_\theta\{k \in \mathbb{N} : \mu(a_k - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

In this scenario, $S_\theta^{\mathscr{L}} - \lim x = l$.

[41] The double sequence $\theta = \{(k_r, l_s)\}$ is called double lacunary if there exist there exist two increasing integer sequence such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty, as \ \ r \to \infty$$

and

$$l_0 = 0, m_s = l_s - l_{s-1} \to \infty, as \ \ s \to \infty.$$

The intervals are determined by θ, $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < s \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r, l_{s-1} < s \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $u_s = \frac{l_s}{l_{s-1}}$.
Note that the double $\theta-$ density will be denoted by δ_{θ_2}.

Definition 3.3. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then a double sequence $a = (a_{mn})$ is lacunary statistically convergent to $l \in V$ with respect to ν fuzzy norm, provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

In this scenario, it is denoted by $S_{\theta_2}^{\mathscr{L}} - \lim a = l$.

Proposition 3.4. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\varepsilon \in L - \{0_L\}$ and $t > 0$:*

(a) $S_{\theta_2}^{\mathscr{L}} - \lim a = l$,
(b) $\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$,
(c) $\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \succ \mathscr{N}(\varepsilon)\} = 1$,
(d) $S_{\theta_2}^{\mathscr{L}} - \lim \mu(a_{mn} - l, t) = 1_L$.

Theorem 3.5. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a = (a_{mn})$ be a double sequence. If $\lim a = l$ in Pringsheim sense, then $S_{\theta_2}^{\mathscr{L}} - \lim a = l$.*

Proof. Let $\lim a = l$. Then, for every $\varepsilon \in L - \{0_L\}$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\mu(a_{mn} - l, t) \succ \mathscr{N}(\varepsilon),$$

for all $m, n \geq k_0$. Therefore,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\}$$

has at most finitely many terms. We can see right away that any finite subset of the natural numbers has double $\theta-$ density zero. Hence,

$$\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

□

As shown in the following case, the converse of the theorem is not true.

Example 3.6. *Let $V = \mathbb{R}$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\mu : \mathbb{R} \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with*

$$\mu(x,t) = \{r \in \mathbb{R}^+ : | x |< \frac{t}{r}\}.$$

Then, $(\mathbb{R}, \mu, \mathscr{P}(\mathbb{R}^+))$ is a $\mathscr{L}-$ fuzzy normed space. On this space, consider the sequence $a = (a_{mn})$ given by the rule

$$a_{mn} = \begin{cases} 1, & for\ m \in (k_r - \ln(h_r), k_r] \ \ and \ \ n \in (l_s - \ln(m_s), l_s], r, s \in \mathbb{N}, \\ 0, & otherwise. \end{cases}$$

Then,

$$\lim_{r \to \infty} \delta_{\theta_2} = 0$$

which means $S_{\theta_2}^{\mathscr{L}} - \lim a = l \in \mathbb{R}$, while the sequence itself is not convergent.

Theorem 3.7. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. If a double sequence $a = (a_{mn})$ is lacunary statistically convergent with respect to the $\mathscr{L}-$ fuzzy norm μ, then $S_{\theta_2}^{\mathscr{L}}-$ limit is unique.*

Proof. Suppose that $S_{\theta_2}^{\mathscr{L}} - \lim a = \ell_1$ and $S_{\theta_2}^{\mathscr{L}} - \lim a = \ell_2$, where $\ell_1 \neq \ell_2$. For any given $\varepsilon \in L - \{0_L\}$ and $t > 0$, we can choose a $r \in L - \{0_L\}$ such that

$$\mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

Define the following sets

$$K_1 = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - \ell_1, t)) \nsucc \mathscr{N}(r)\}$$

and

$$K_2 = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - \ell_2, t)) \nsucc \mathscr{N}(r)\}$$

for any $t > 0$. Since for elements of the set $K(\varepsilon, t) = K_1(\varepsilon, t) \cup K_2(\varepsilon, t)$ we have

$$\mu(\ell_1 - \ell_2, t) \succeq \mathscr{K}(\mu(a_{mn} - \ell_1, \frac{t}{2}), \mu(a_{mn} - \ell_2, \frac{t}{2})) \succ \mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

it can be concluded that $\ell_1 = \ell_2$. □

Theorem 3.8. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}$–fuzzy normed space. Then, $S_{\theta_2}^{\mathscr{L}}-\lim a=\ell$ if and only if there exists a subset $K\subset\mathbb{N}\times\mathbb{N}$ such that $\delta_{\theta_2}(K)=1$ and $\mathscr{L}-\lim_{k,l\to\infty}a_{kl}=\ell$.*

Proof. Suppose that $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. Let (ε_n) be a sequence in $L-\{0_L\}$ such that $\mathscr{N}(\varepsilon_n)\to 1_L$ in L increasingly, and for any $t>0$ and $j\in\mathbb{N}$, let

$$K(j)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(a_{kl}-l,t)\succ\mathscr{N}(\varepsilon_j)\}.$$

Then, observe that, for any $t>0$ and $j\in\mathbb{N}$,

$$K(j+1)\subset K(j).$$

Since $S_{\theta_2}^{\mathscr{L}}-\lim a=l$, it is obvious that

$$\delta_{\theta_2}\{K(j)\}=1,(j\in\mathbb{N}\text{ and }t>0).$$

Now, let (p_1,q_1) be an arbitrary number of $K(1)$. Then, there exist numbers $(p_2,q_2)\in K(2)$, $p_2>p_1,q_2>q_1$, such that for all $l>p_2,k>q_2$,

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_2)\}|>\frac{1}{2}.$$

Further, there is a number $(p_3,q_3)\in K(3),p_3>p_2,q_3>q_2$ such that for all $l>p_3,k>q_3$,

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_3)\}|>\frac{2}{3}$$

and so on. So, we can construct, by induction, an increasing index sequence increasing in both coordinates $(p_j,q_k)_{j,k\in\mathbb{N}}$ of the natural numbers such that $(q_j,q_j)\in K(j)$ and that the following statement holds for all $l>p_j,k>q_j$:

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_j)\}|>\frac{j-1}{j}.$$

Now, we construct an index sequence increasing in both coordinates as follows:

$$K:=\{(k,l)\in\mathbb{N}\times\mathbb{N}:1<l<p_1,1<k<q_1\}\cup\Big[\bigcup_{j\in\mathbb{N}}\{(k,l)\in K(j):p_j\le l<p_{j+1},q_j\le k<q_{j+1}\}\Big].$$

Hence, it follows that $\delta_{\theta_2}(K)=1$. Now, let $\varepsilon\succ 0_L$ and choose a positive integer j such that $\varepsilon_j\prec\varepsilon$. Such a number j always exists since $(\varepsilon_n)\to 0_L$. Assume that $l\ge p_j,k\ge q_j$ and $k,l\in K$. Then, by the definiton of K, there exists a number $d\ge j$ such that $p_d\le l<p_{d+1},q_d\le k<q_{d+1}$ and $(k,l)\in K(j)$. Hence, we have, for every $\varepsilon\succ 0_L$

$$\mu(a_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_k)\succ\mathscr{N}(\varepsilon)$$

for all $l\ge p_j,k\ge q_j$ and $(k,l)\in K$ and this means

$$\mathscr{L}-\lim_{k,l\in K}a_{kl}=\ell.$$

Conversely, suppose that there exists an increasing index sequence $K=(a_{kl})_{k,l\in\mathbb{N}}$ of pairs of natural numbers such that $\delta_{\theta_2}(K)=1$ and $\mathscr{L}-\lim\limits_{k,l\in K}a_{kl}=\ell$. Then, for every $\varepsilon\succ 0_L$ there is a number n_0 such that for each $k,l\ge n_0$ the inequality $\mu(a_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon)$ holds. Now, define

$$M(\varepsilon):=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(a_{kl}-\ell,t)\not\succ\mathscr{N}(\varepsilon)\}.$$

Then, there exists an $n_0\in\mathbb{N}$ such that

$$M(\varepsilon)\subseteq\mathbb{N}\times\mathbb{N}-(K-\{(a_k,a_l):k,l\le n_0\}).$$

Since $\delta_{\theta_2}(K)=1$, we get $\delta_{\theta_2}\{(\mathbb{N}\times\mathbb{N})-(K-\{(a_k,a_l):k,l\le n_0\})\}=0$, which yields that $\delta_{\theta_2}\{M(\varepsilon)\}=0$. In other words, $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. □

4. The Relationship Between Lacunary Statistical Double Cauchy and Lacunary Statistical Double Bounded Sequences

In this section, the notion of lacunary statistically double Cauchy and lacunary statistically double bounded sequences will be defined and relationship between them will be given.

Definition 4.1. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then, a sequence $a=(a_{mn})$ is said to be lacunary statistically double Cauchy with respect to $\mathscr{L}-$ fuzzy norm μ, if for every $\varepsilon\in L-\{0_L\}$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that for all $m,k\ge N$ and $n,l\ge M$ provided that*

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\not\succ\mathscr{N}(\varepsilon)\}=0.$$

Theorem 4.2. *Every lacunary statistically convergent double sequence is lacunary statistically double Cauchy.*

Proof. Let $a=(a_{mn})$ be a double sequence such that lacunary statistical convergent to ℓ with respect to $\mathscr{L}-$ fuzzy norm μ, in other saying $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. For a given $\varepsilon>0$, choose $r>0$ such that,

$$\mathscr{K}(\mathscr{N}(r),\mathscr{N}(r))\succ\mathscr{N}(\varepsilon).$$

For $t>0$ we can write,

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-\ell,\frac{t}{2})\succ\mathscr{N}(r)\}.$$

Take $(p,q)\in A$. Obviously, $\mu(a_{pq}-\ell,\frac{t}{2})\succ\mathscr{N}(r)$. Also since,

$$\mu(\ell-a_{pq},\frac{t}{2})=\mu(a_{pq}-\ell,\frac{\frac{t}{2}}{|-1|})=\mu(a_{pq}-\ell,\frac{t}{2})\succ\mathscr{N}(\varepsilon)$$

we have

$$\begin{aligned}\mu(a_{mn}-x_{pq},t)&=\mu\big((a_{mn}-\ell)+(\ell-a_{pq}),\frac{t}{2}+\frac{t}{2}\big)\\&\succ\mathscr{K}\big(\mu(a_{mn}-\ell,\frac{t}{2}),(\nu(\ell-a_{pq},\frac{t}{2})\big)\\&\succ\mathscr{K}\big(\mathscr{N}(r),\mathscr{N}(r)\big)\\&\succ\mathscr{N}(\varepsilon).\end{aligned}$$

If we define a set $B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{pq},t)\succ\mathscr{N}(\varepsilon)\}$, then $A\subseteq B$. Since $\delta_{\theta_2}(A)=1$, $\delta_{\theta_2}(B)=1$. Thus, the double theta density of complement of B equals to zero,i.e. $\delta_{\theta_2}(B^c)=0$, which means $a=(a_{mn})$ is lacunary statistical double Cauchy. □

Definition 4.3. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a=(a_{mn})$ be a double sequence. Then, $a=(a_{mn})$ is said to be lacunary statistically double bounded with respect to $\mathscr{L}-$ fuzzy norm μ, provided that there exists $r\in L-\{0_L,1_L\}$ and $t>0$ such that*

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=0$$

for each positive integer m,n.

Theorem 4.4. *Every double bounded sequence on a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$, is lacunary statistically double bounded.*

Proof. Let (a_{mn}) be a double bounded sequence on $(V,\mu,\mathscr{K})$. Then, there exist $t>0$ and $r\in L-\{0_L,1_L\}$ such that $\mu(a_{mn},t)\succ\mathscr{N}(r)$. In that case we have,

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=\emptyset$$

which yields

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=0.$$

Thus, (a_{mn}) is lacunary statistically bounded. □

However the converse of this theorem does not hold in general as seen in the example below.

Example 4.5. *Let $V=\mathbb{R}$ and $\mathscr{L}=(L,\leq)$ where L is the set of non-negative extended real numbers, that is $L=[0,\infty]$. Then, $0_L=0,1_L=\infty$. Define a $\mathscr{L}-$fuzzy norm ν on V by $\mu(x,t)=\frac{t}{|x|}$ for $x\neq0$ and $\nu(0,t)=\infty$ for each $t\in(0,\infty)$. Consider the $t-$ norm $\mathscr{K}(a,b)=\min\{a,b\}$ on $\mathscr{L}$. Given the sequence,*

$$x_{mn}=\begin{cases}m+n, & \text{if m+n is a prime number},\\ \frac{1}{\tau(m+n)-2}, & \text{otherwise}\end{cases}$$

where, $\tau(m+n)$ denotes the number of positive divisors of $m+n$. Note that (x_{mn}) is not bounded since for each $t>0$ and $r\in L-\{0,\infty\}$, for any prime number $m+n$ such that $rt\leq m+n$ we have

$$\mu(x_{mn},t)=\mu(m+n,t)=\frac{t}{|\,m+n\,|}=\frac{t}{m+n}\not\succ\frac{1}{r}=\mathscr{N}(r).$$

However for $t=1$ and any non-prime integer $m+n$, $r=2$ satisfies

$$\mu(x_{mn},1)=\mu(\frac{1}{\tau(m+n)-2},1)=\frac{1}{|\frac{1}{\tau(m+n)-2}|}=|\tau(m+n)-2|>\frac{1}{2}=\mathscr{N}(r)$$

since $\tau(m+n)\neq2$ for any non-prime $m+n$, and since the density of prime numbers converges zero by Prime Number Theorem we have,

$$\delta_{\theta_2}\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk},1)\not\succ\mathscr{N}(2)\}=0$$

suggesting that (x_{mn}) is lacunary statistically double bounded.

Theorem 4.6. *Every lacunary statistically double Cauchy sequence on a $\mathscr{L}-$fuzzy normed space $(V,\mu,\mathscr{K})$ is lacunary statistically double bounded.*

Proof. Let (a_{mn}) be a lacunary statistically double Cauchy on $(V,\mu,\mathscr{K})$. Then, for every $\varepsilon \in L-\{0_L\}$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that for all $m,k\geq N$ *and* $n,l\geq M$ provided that

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\nsucc\mathscr{N}(\varepsilon)\}=0.$$

Then,

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\succ\mathscr{N}(\varepsilon)\}=1.$$

Consider a number $(m,n)\in\mathbb{N}\times\mathbb{N}$ such that $\mu(a_{mn}-a_{kl},1)\succ\mathscr{N}(\varepsilon)$. Then, for $t=2$

$$\mu(a_{mn},2)=\mu(a_{mn}-a_{kl}+a_{kl},2)\succ\mathscr{K}(\mu(a_{mn}-a_{kl},1),\mu(a_{kl},1))\succ\mathscr{K}(\mathscr{N}(\varepsilon),\nu(x_{kl},1)).$$

Say $r:=\mathscr{N}(\mathscr{K}(\mathscr{N}(\varepsilon),\mu(a_{kl},1)))$. Then,

$$\mu(a_{mn},2)\succ\mathscr{K}(\mathscr{N}(\varepsilon),\mu(a_{kl},1))=\mathscr{N}(r),$$

which implies

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},2)\succ\mathscr{N}(r)\}=1$$

or equivalently

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},2)\nsucc\mathscr{N}(r)\}=0$$

giving lacunary statistically double boundedness of (a_{mn}). □

5. Conclusion

In this study, the properties of Lacunary statistical convergence for double sequences, which is a generalization of statistical convergence, are defined on L fuzzy spaces, which are a generalization of fuzzy spaces, and their properties are examined. Some characteristics of the lacunary statistical convergence of sequences within the context of the current investigation are examined on L-fuzzy normed spaces, a structure that provides a flexible frame- work that generalizes other structures like normed spaces, fuzzy normed spaces, and IF-normed spaces. As a result of this research, the concept of norm was emphasized on a broader concept, the topological vector space, by combining the lattice structure and the norm structure.

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