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Asymptotic Expressions of Fourth Order Sturm-Liouville Operator with Conjugate Conditions

Olgun Cabri

ABSTRACT: In this paper, it is studied the asymptotic expression of fourth order differential operator with periodic boundary conditions. For this operator, it is also considered conjugate boundary conditions at x=0 which shows discontinuity. For this purpose, firstly asymptotic expression of solutions areobtained. Then by using the the asymptotic formulas of fundamental solutions, asymptotic expression of eigenvalues and eigenfunctions are presented. It is also dealt with the asymptotic expression of same operator with antiperiodic boundary conditions and conjugate conditions

Keywords: Periodic boundary conditions, fourth order problem, eigenvalues, eigenfunctions

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INTRODUCTION

The eigenvalue problem arises during the solution of many problems of mathematical physics. When Fourier method or another method is applied to boundary value problems, it is important to examine the spectral properties of problem such as the forms of eigenvalue and eigenfunctions, orthogonality of eigenfunctions, expansion properties according to eigenfunctions (Tikhonov and Samarski, 1963). In this study, it is analyzed similar properties for a fourth order differential equation.

Fourth order eigenvalue problem modelling the deformations of elastic beam is studied by many scholars with different boundary conditions (Agarwal, 1989; Yao, 2004; Bonanno and Bella, 2008; Gupta, 1988; Mamedov, 1996) and references there in.

Consider the differential operator

$$l(y)=\begin{cases} l_1(y), x\in(-1,0) \\ l_2(y), x\in(0,1)\end{cases} \quad (1)$$

generated by fourth order differential expression

$$l_1(y_1)=y_1^{(4)}+q_1(x)y_1, \qquad l_2(y_2)=y_2^{(4)}+q_2(x)y_2, \quad (2)$$

where $q_1(x)\in C^4[-1,0)$and $q_2(x)\in C^4(0,1]$ complex valued functions. We consider boundary conditions of the operator (1) are with periodic

$$U_k(y) := U_{k,-1}(y)+U_{k,+1}(y)=y^{(k)}(-1)-y^{(k)}(+1)=0, \quad k=0,1,2,3 \quad (3)$$

and conjugate boundary conditions at zero

$$V_k(y) := V_{k,0_-}(y)+U_{k,0_+}(y)=y^{(k)}(-0)-y^{(k)}(+0)=0, \quad k=0,1,2,3. \quad (4)$$

It is known that periodic (antiperiodic) boundary conditions (2) are not strongly regular boundary conditions. (see: Naimark, 1967). In general, spectral properties of second order boundary value problems with periodic and antiperiodic boundary conditions are investigated in the studies (Naimark, 1967, Dunford and Schwartz, 1970; Marchenko, 1977; Levitan and Sargsyan, 1991; Coskun, 2003; Nabiev, 2007; Gasymov et al, 1990). Spectral properties in nonsefadjoint case are given by authors (Makin, 2006; Djakov and Mityagin, 2006; Gesztesy and Tkachenko, 2012; Baskakov and Polyakov, 2017; Baranets'kyi et al, 2018). Basis properties studied in the works (Mamedov, 1996; Kerimov and Mamedov, 1998; Mamedov and Menken, 2008; Mamedov, 2010; Kurbanov, 2006; Menken, 2010; Jwamer and Hawsar, 2015).

Linear diferential operator order n with strongly regular boundary conditions and conjugate conditions is firstly investigated by (Muravei, 1967). For second order boundary value problem with periodic (antiperiodic) and conjugate conditions are studied in several works (Cabri, 2019, Cabri and Mamedov, 2020; Cabri and Mamedov, 2020).

The existence and uniqueness of solutions of the fourth order boundary value problems with periodic boundary conditions is studied by (Gupta, 1988). Asymptotic expressions of eigenvalues and eigenfunctions of fourth order differential operator obtained by (Menken, 2010). Spectral properties of differential operators with integrable coefficients and a constant weight function is given by (Mitrokhin, 2010).

Our goal is to examine the spectral properties of the problem $l(y)=\lambda y$ with periodic boundary conditions (3) and (4). Morever, we consider $l(y)=\lambda y$ with antiperiodic boundary conditions

$$U_k(y) := U_{k,-1}(y)+U_{k,+1}(y)=y^{(k)}(-1)+y^{(k)}(+1)=0, \quad k=0,1,2,3 \quad (5)$$

and conjugate boundary conditions for $k=0,1,2,3$

$$V_k(y) := V_{k,0_-}(y)+U_{k,0_+}(y)=y^{(k)}(-0)-y^{(k)}(+0)=0. \quad k=0,1,2,3 \quad (6)$$

Here $q_1(x)\in C^4[-1,0)$and $q_2(x)\in C^4(0,1]$complex valued functions.

Without loss of generality, we assume that

$$\int_{-1}^{0} q_1(x)dx = 0, \int_{0}^{1} q_2(x)dx = 0. \tag{7}$$

MATERIALS AND METHODS

Asymptotic Expression of Fundamental Solutions

It is known from (Naimark, 1967) that fourth order differential operator (2) have four linearly independent solutions $y_n(x;s), (n = 1,2,3,4)$ in the both interval $(-1,0)$ and $(0,1)$ by

$$y_n(x,s) = e^{w_j sx}\left(\sum_{m=0}^{7} \frac{u_{m,n}(x)}{s^m}\right), \tag{8}$$

where $u_{m,n}(x)$ satisfy diffferential equation

$$4u'_{m,k}(x) + 6w_k^3 u''_{m-1,k}(x) + 4w_k^2 u_{m-2,k}(x) + w_k u^{(4)}_{m-3,k}(x) + w_k q(x) u_{m-3,k}(x) = 0.$$

where the numbers ω_k are the fourth roots of unity.

By using Equation (8) and (Menken,2010) $u_{m,n}(x)$ are obtained as follows in both interval. In the interval $(-1,0)$, $u^1_{m,n}(x)$ functions of $y^1_n(x,s)$ become

$$\begin{aligned}
&u^1_{0,n}(x) = 1, \quad u^1_{1,n}(x) = u^1_{2,n}(x) = 0,\ u^1_{3,n}(x) = -\frac{1}{\omega_n^3}\int_0^{-1} q_1(t)dt,\\
&u^1_{4,n}(x) = \frac{3}{8}\big(q_1(x) - q_1(-0)\big),\quad u^1_{5,n}(x) = -\frac{5}{16}\big(q'_1(x) - q'_1(-0)\big),\\
&u^1_{6,n}(x) = \frac{3\omega_n^2}{8}\big(q''_1(x) - q''_1(-0)\big) + \frac{\omega_n^2}{32}\left(\int_0^{-1} q_1(t)dt\right)^2,\\
&u^1_{7,k}(x) = \frac{-\omega_n}{64}\big(q'''_1(x) - q'''_1(-0)\big) + \frac{-3\omega_n}{64}\big(q_1(x) - q_1(-0)\big)\int_0^{-1} q_1(t)dt\\
&\qquad - \frac{3\omega_n}{32}\left(\int_0^{-1} q_1^2(t)dt\right).
\end{aligned} \tag{9}$$

In the interval $(0,1)$, $u^2_{m,n}(x)$ functions of $y^2_n(x,s)$ become

$$\begin{aligned}
&u^2_{0,n}(x) = 1, u_{1,n}(x) = u^2_{2,n}(x) = 0, u^2_{3,n}(x) = -\frac{1}{\omega_n^3}\int_0^{1} q_2(t)dt,\\
&u^2_{4,n}(x) = \frac{3}{8}\big(q_2(x) - q_2(+0)\big),\qquad u_{5,n}(x) = -\frac{5}{16}\big(q'_2(x) - q'_2(+0)\big),\\
&u^2_{6,n}(x) = \frac{3\omega_n^2}{8}\big(q''_2(x) - q''_2(+0)\big) + \frac{\omega_n^2}{32}\left(\int_0^{1} q_2(t)dt\right)^2,\\
&u^2_{7,k}(x) = \frac{-\omega_n}{64}\big(q'''_2(x) - q'''_2(0)\big) + \frac{-3\omega_n}{64}\big(q_2(x) - q_2(+0)\big)\int_0^{1} q_2(t)dt\\
&\qquad - \frac{3\omega_n}{32}\left(\int_0^{1} q_2^2(t)dt\right).
\end{aligned} \tag{10}$$

In the interval $(-1,0)$, $u_{m,n}^{(k)}(x)$ of $y_n^1(x,s)$ obtained by

$$\begin{aligned}
&u_{0,n}^{(k)}(x)=1,\quad u_{1,n}^{(k)}(x)=u_{2,n}^{(k)}(x)=0,\quad u_{(3,n)}^{(k)}(x)=-\frac{1}{\omega_n^3}\int_0^{-1} q_1(t)dt,\\
&u_{4,n}^{(1)}(x)=\frac{1}{8}q_1(x)-\frac{3}{8}q_1(-0),\ u_{4,n}^{(2)}(x)=-\frac{1}{8}q_1(x)-\frac{3}{8}q_1(-0),\\
&u_{4,n}^{(3)}(x)=-\frac{3}{8}q_1(x)-\frac{3}{8}q_1(-0),\quad u_{5,n}^{(1)}(x)=\frac{\omega_n^3}{16}q_1'(x)+\frac{5\omega_n^3}{16}q_1'(-0),\\
&u_{5,n}^{(2)}(x)=\frac{3\omega_n^3}{16}q_1'(x)+\frac{5\omega_k^3}{16}q_1'(-0),\ u_{5,n}^{(3)}(x)=\frac{\omega_k^3}{16}q_1'(x)+\frac{5\omega_k^3}{16}q_1'^{0}(-0),\\
&u_{6,n}^{(1)}(x)=\frac{-5\omega_n^2}{32}\left(q_1''(x)+q_1''(-0)\right)+\frac{\omega_n^2}{32}\left(\int_0^{-1} q_1''(t)dt\right)^2,\\
&u_{6,n}^{(2)}(x)=\frac{-3\omega_n^2}{32}q_1''(x)-\frac{5\omega_n^2}{32}q_1''(-0)+\frac{\omega_n^2}{32}\left(\int_0^{-1} q_1(t)dt\right)^2,\\
&u_{6,n}^{(3)}(x)=\frac{3\omega_n^2}{32}q_1''(x)-\frac{5\omega_n^2}{32}q_1''(-0)+\frac{\omega_n^2}{32}\left(\int_0^{-1} q_1(t)dt\right)^2,\\
&u_{7,n}^{(1)}(x)=\frac{9\omega_n}{64}q_1'''(x)+\frac{\omega_n}{64}q_1'''(-0)+\frac{-\omega_n}{64}\left(q_1(x)-3q_1(-0)\right)\int_0^{-1} q_1(t)dt\\
&\qquad-\frac{3\omega_n}{32}\left(\int_0^{-1} q_1^2(t)dt\right),\\
&u_{7,n}^{(2)}(x)=-\frac{\omega_n}{64}q_1'''(x)+\frac{\omega_n}{64}q_1'''(-0)+\frac{-\omega_n}{64}\left(q_1(x)-3q_1(-0)\right)\int_0^{-1} q_1(t)dt\\
&\qquad-\frac{3\omega_n}{32}\left(\int_0^{-1} q_1^2(t)dt\right),\\
&u_{7,n}^{(3)}(x)=-\frac{7\omega_n}{64}q_1'''(x)+\frac{\omega_n}{64}q_1'''(-0)+\frac{-\omega_n}{64}\left(q_1(x)-3q_1(-0)\right)\int_0^{-1} q_1(t)dt\\
&\qquad-\frac{3\omega_n}{32}\left(\int_0^{-1} q_1^2(t)dt\right).
\end{aligned}\tag{11}$$

In the interval $(0,1)$, $u_{m,n}^{(k)}(x)$ of $y_n^2(x,s)$ are found by

$$\begin{aligned}
&u_{0,n}^{(k)}(x)=1,\quad u_{1,n}^{(k)}(x)=u_{2,n}^{(k)}(x)=0,\ u_{(3,n)}^{(k)}(x)=-\frac{1}{\omega_n^3}\int_0^{1} q_2(t)dt,\\
&u_{4,n}^{(1)}(x)=\frac{1}{8}q_2(x)-\frac{3}{8}q_2(+0),\quad u_{4,n}^{(2)}(x)=-\frac{1}{8}q_2(+0)-\frac{3}{8}q_2(+0),\\
&u_{4,n}^{(3)}(x)=-\frac{3}{8}q_2(+0)-\frac{3}{8}q_2(+0),\quad u_{5,n}^{(1)}(x)=\frac{\omega_n^3}{16}q_2'(x)+\frac{5\omega_n^3}{16}q_2'(+0),
\end{aligned}\tag{12}$$

$$u_{5,n}^{(2)}(x) = \frac{3\omega_n^3}{16} q_2'(x) + \frac{5\omega_k^3}{16} q_2'(+0), \qquad u_{5,n}^{(3)}(x) = \frac{\omega_k^3}{16} q_2'(x) + \frac{5\omega_k^3}{16} q_2'(+0),$$

$$u_{6,n}^{(1)}(x) = \frac{-5\omega_n^2}{32}\left(q_2''(x) + q_2''(+0)\right) + \frac{\omega_n^2}{32}\left(\int_0^1 q_2''(t)dt\right)^2,$$

$$u_{6,n}^{(2)}(x) = \frac{-3\omega_n^2}{32} q_2''(x) - \frac{5\omega_n^2}{32} q_2''(+0) + \frac{\omega_n^2}{32}\left(\int_0^1 q_2(t)dt\right)^2,$$

$$u_{6,n}^{(3)}(x) = \frac{3\omega_n^2}{32} q_2''(x) - \frac{5\omega_n^2}{32} q_2''(-0) + \frac{\omega_n^2}{32}\left(\int_0^1 q_2(t)dt\right)^2,$$

$$u_{7,n}^{(1)}(x) = \frac{9\omega_n}{64} q_2'''(x) + \frac{\omega_n}{64} q_2'''(+0) + \frac{-\omega_n}{64}\left(q_2(x) - 3q_2(+0)\right)\int_0^1 q_2(t)dt - \frac{3\omega_n}{32}\left(\int_0^1 q_2^2(t)dt\right),$$

$$u_{7,n}^{(2)}(x) = -\frac{\omega_n}{64} q_2'''(x) + \frac{\omega_n}{64} q_2'''(+0) + \frac{-\omega_n}{64} q_2(x) - 3q_2(+0)\int_0^1 q_2(t)dt - \frac{3\omega_n}{32}\left(\int_0^1 q_2^2(t)dt\right),$$

$$u_{7,n}^{(3)}(x) = -\frac{7\omega_n}{64} q_2'''(x) + \frac{\omega_n}{64} q_2'''(+0) + \frac{-\omega_n}{64}\left(q_2(x) - 3q_2(+0)\right)\int_0^1 q_2(t)dt - \frac{3\omega_n}{32}\left(\int_0^1 q_2^2(t)dt\right).$$

RESULTS AND DISCUSSION

Asymptotic Expression of Eigenvalues and EigenFunctions

Theorem 1: Let $q_1(x) \in C^4[-1,0]$ and $q_2(x) \in C^4[0,1]$. Then, the eigenvalues of the boundary-value problem Problem (1)-(3) has two infinite sequences $\lambda_{k,1}, \lambda_{k,2}$ $(k = N, N+1 \ldots)$ and have the following expressions

$$\lambda_{k,1} = (k\pi)^4 + \frac{3}{16}\frac{\int_{-1}^0 q_1^2(t)dt + \int_0^1 q_2^2(t)dt}{k^4} + O\left(\frac{1}{k^5}\right),$$

$$\lambda_{k,2} = (k\pi i)^4 + \frac{3}{16}\frac{\int_{-1}^0 q_1^2(t)dt + \int_0^1 q_2^2(t)dt}{k^4} + O\left(\frac{1}{k^5}\right).$$

Proof: By using asymptotic expression of fundamental solution (9)-(12), characteristic determinant is easily obtained as

$$\Delta(s)=\begin{vmatrix} y_{1,1}(-1) & y_{2,1}(-1) & \ldots & y_{4,1}(-1) & y_{1,2}(-1) & y_{2,2}(-1) & \ldots & y_{4,2}(-1) \\ y_{1,1}^{(1)}(-1) & y_{2,1}^{(1)}(-1) & \ldots & y_{4,1}^{(1)}(-1) & y_{1,2}^{(1)}(-1) & y_{2,2}^{(1)}(-1) & \ldots & y_{4,2}^{(1)}(-1) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y_{1,1}^{(3)}(-1) & y_{2,1}^{(3)}(-1) & \ldots & y_{4,1}^{(3)}(-1) & y_{1,2}^{(1)}(-1) & y_{2,2}^{(1)}(-1) & \ldots & y_{4,2}^{(3)}(-1) \\ y_{1,1}(-0) & y_{2,1}(-0) & \ldots & y_{4,1}(-0) & y_{1,2}(+0) & y_{2,2}(+0) & \ldots & y_{4,2}(+0) \\ y_{1,1}^{(1)}(-0) & y_{2,1}^{(1)}(-0) & \ldots & y_{4,1}^{(1)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \ldots & y_{4,2}^{(1)}(+0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y_{1,1}^{(3)}(-0) & y_{2,1}^{(3)}(-0) & \ldots & y_{4,1}^{(3)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \ldots & y_{4,2}^{(3)}(+0) \end{vmatrix}.$$

For simplicity let us denote C and M by

$$C=q_1(-0)+q_2(+0),\qquad M=\int_{-0}^{-1} q_1^2(t)dt-\int_{+0}^{1} q_2^2(t)dt.$$

Chacteristic determinant can be simplify as follows

$$\frac{-\Delta(s)}{256s^9}=\left(\left(e^{-2is}+e^{2is}\right)\left[1+\frac{3iM}{32s^7}+O\left(\frac{1}{s^8}\right)\right]-2\left[1+O\left(\frac{1}{s^8}\right)\right]\right)\times$$
$$\times \tag{13}$$
$$\left(\left(e^{-2s}+e^{2s}\right)\left[1-\frac{3C}{2s^4}+\frac{3M}{32s^7}+O\left(\frac{1}{s^8}\right)\right]-2\left[1-\frac{3C}{2s^4}+O\left(\frac{1}{s^5}\right)\right]\right).$$

Multiplying equation (13) by

$$e^{2is}\left[1-\frac{3iM}{32s^7}+O\left(\frac{1}{s^8}\right)\right]\times e^{2s}\left[1+\frac{3C}{2s^4}-\frac{3M}{32s^7}+O\left(\frac{1}{s^8}\right)\right], \tag{14}$$

we get

$$\frac{\Delta(s)}{256s^9}=\left(e^{2s}-\left[1-\frac{3M}{32s^7}+O\left(\frac{1}{s^8}\right)\right]\right)^2\times\left(e^{2is}-\left[1-\frac{3iM}{32s^7}+O\left(\frac{1}{s^8}\right)\right]\right)^2. \tag{15}$$

Therefore, for sufficiently large $|s|$, roots of $\Delta(s)=0$ satisfy

$$e^{2is}=1+\frac{3i}{32}\frac{\int_{-1}^{0} q_1^2(t)dt+\int_0^1 q_2^2(t)dt}{s^7}+O\left(\frac{1}{s^8}\right),$$
$$e^{2s}=1+\frac{3}{32}\frac{\int_{-1}^{0} q_1^2(t)dt+\int_0^1 q_2^2(t)dt}{s^7}+O\left(\frac{1}{s^8}\right). \tag{16}$$

Using Rouche's theorem in (16) by writing $s=k\pi+\varepsilon_n$ and $s=k\pi i+\varepsilon_n$ asymptotic expression of eigenvalues are obtained by

$$\lambda_{k_1}=(k\pi)^4+\frac{3}{16}\frac{\int_{-1}^{0} q_1^2(t)dt+\int_0^1 q_2^2(t)dt}{k^4}+O\left(\frac{1}{k^5}\right),$$
$$\lambda_{k_2}=(k\pi i)^4+\frac{3}{16}\frac{\int_{-1}^{0} q_1^2(t)dt+\int_0^1 q_2^2(t)dt}{k^4}+O\left(\frac{1}{k^5}\right).$$

This ends proof.

Theorem 2: Asymptotic expression of eigenfunctions of boundary value problem (1)-(4) are

$$y_{k_1}(x)=sin(k\pi x)+O\left(\frac{1}{k}\right),\qquad x\in[-1,0)\cup(0,1],$$
$$y_{k_2}(x)=cos(k\pi x)+O\left(\frac{1}{k}\right),\qquad x\in[-1,0)\cup(0,1]. \tag{17}$$

Proof: If $U_j(y_k)$ and $V_j\left(y_k\left(x,s_{k,1}\right)\right)$ $(j=0,1,2,3)$ are calculated up to order $O(s^{-5})$, then first part $y_{1,k}^1(x)$ of $y_{k,1}(x)$ is obtained by

$$y_{k,1}^1(x)=\begin{vmatrix} y_{k,1}(x) & y_{k,2}(x) & \dots & y_{k,4}(x) & 0 & 0 & \dots & 0 \\ y_{1,1}(-1) & y_{2,1}(-1) & \dots & y_{4,1}(-1) & y_{1,2}(+1) & y_{2,2}(+1) & \dots & y_{4,2}(+1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_{1,1}^{(3)}(-1) & y_{2,1}^{(3)}(-1) & \dots & y_{4,1}^{(3)}(-1) & y_{1,2}^{(2)}(+1) & y_{2,2}^{(2)}(+1) & \dots & y_{4,2}^{(2)}(+1) \\ y_{1,1}(-0) & y_{2,1}(-0) & \dots & y_{4,1}(-0) & y_{1,2}(+0) & y_{2,2}(+0) & \dots & y_{4,2}(+0) \\ y_{1,1}^{(1)}(-0) & y_{2,1}^{(1)}(-0) & \dots & y_{4,1}^{(1)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \dots & y_{4,2}^{(1)}(+0) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_{1,1}^{(3)}(-0) & y_{2,1}^{(3)}(-0) & \dots & y_{4,1}^{(3)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \dots & y_{4,2}^{(3)}(+0) \end{vmatrix}$$

This determinant yields

$$y_{k,1}^1(x)=\frac{16\big(q_1(-1)-q_1(-0)\big)-\big(q_2(1)-q_2(+0)\big)(1-sinhs)}{s^5}\left(\frac{e^{isx}-e^{isx}}{2i}\right)+O\left(\frac{1}{s^9}\right). \tag{18}$$

By normalizing the result (15), we can write eigenfunction corresponding to λ_{k_1}

$$y_{k,1}^1(x)=sin(k\pi x)+O\left(\frac{1}{k}\right), x\in[-1,0), \tag{19}$$

Similarly, second part $y_{k,1}^2(x)$ of $y_{k,1}(x)$ is found by

$$y_{k,1}^2(x)=sin(k\pi x)+O\left(\frac{1}{k}\right), x\in(0,1]. \tag{20}$$

In same way first part $y_{(k,2)}^1(x)$ of $y_{k,2}^1(x)$ is obtained by

$$y_{k,1}^2(x)=\begin{vmatrix} y_{k,1}(x) & y_{k,2}(x) & \dots & y_{k,4}(x) & 0 & 0 & \dots & 0 \\ y_{1,1}^{(1)}(-1) & y_{2,1}^{(1)}(-1) & \dots & y_{4,1}^{(1)}(-1) & y_{1,2}^{(1)}(-1) & y_{2,2}^{(1)}(-1) & \dots & y_{4,2}^{(1)}(-1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_{1,1}^{(3)}(-1) & y_{2,1}^{(3)}(-1) & \dots & y_{4,1}^{(3)}(-1) & y_{1,2}^{(1)}(-1) & y_{2,2}^{(1)}(-1) & \dots & y_{4,2}^{(3)}(-1) \\ y_{1,1}(-0) & y_{2,1}(-0) & \dots & y_{4,1}(-0) & y_{1,2}(+0) & y_{2,2}(+0) & \dots & y_{4,2}(+0) \\ y_{1,1}^{(1)}(-0) & y_{2,1}^{(1)}(-0) & \dots & y_{4,1}^{(1)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \dots & y_{4,2}^{(1)}(+0) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_{1,1}^{(3)}(-0) & y_{2,1}^{(3)}(-0) & \dots & y_{4,1}^{(3)}(-0) & y_{1,2}^{(1)}(+0) & y_{2,2}^{(1)}(+0) & \dots & y_{4,2}^{(3)}(+0) \end{vmatrix}$$

This determinant gives us

$$y_{k,2}^1(x)=\frac{16\big(q_1(-1)-q_1(-0)\big)-\big(q_2(1)-q_2(+0)\big)(1-sinhs)}{s^8}\left(\frac{e^{isx}+e^{isx}}{2}\right)+O\left(\frac{1}{s^9}\right), \tag{21}$$

Then we can get eigenfunction corresponding to λ_{k_2}

$$y_{k,2}^1(x)=cos(k\pi x)+O\left(\frac{1}{k}\right), \qquad x\in[-1,0) \tag{22}$$

Second part $y_{k,2}^2(x)$of $y_{k,2}(x)$is found by

$$y_{k,2}^2(x)=cos(k\pi x)+O\left(\frac{1}{k}\right), \qquad x\in(0,1]. \tag{23}$$

Thus from equations (19)-(20) and (22)-(23), proof is complete.

Theoem 3: Let $q_1(x) \in C^4[-1,0]$ and $q_2(x) \in C^4[0,1]$. Then, the eigenvalues of the boundary-value problem (1), (2) with antiperiodic boundary conditions (5)-(6) has two infinite sequences $\lambda_{k,1}, \lambda_{k,2}$ $(k = N, N+1 \ldots)$ and have the following expression

$$\lambda_{k_1} = \left(\frac{k\pi}{2}\right)^4 + 96\frac{\int_{-1}^{0} q_1^2(t)dt + \int_{0}^{1} q_2^2(t)dt}{k^4} + O\left(\frac{1}{k^5}\right)$$

$$\lambda_{k_2} = \left(\frac{k\pi i}{2}\right)^4 + 96\frac{\int_{-1}^{0} q_1^2(t)dt + \int_{0}^{1} q_2^2(t)dt}{k^4} + O\left(\frac{1}{k^5}\right).$$

and asymptotic expression of eigenfunctions of boundary value problem are

$$y_{k_1}(x) = sin(k\pi x) + O\left(\frac{1}{k}\right), \qquad x \in [-1,0) \cup (0,1],$$

$$y_{k_2}(x) = cos(k\pi x) + O\left(\frac{1}{k}\right), \qquad x \in [-1,0) \cup (0,1]$$

Proof: Proofs run as before.

CONCLUSION

In this work, it is considered fourth order problem with perodic boundary conditions which is not strongly regular which differs from (Muravei, 1967). (Menken, 2010) obtained the asymptotic expression of eigenvalues and eigenfunctions of fourth order differential operator in continuous case. In this study, by using fundemental solutions of problem, asymptotic expression of eigenvalues and eigenfunctions are acquired in discontinuous case.

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